Contents

Chapter 1

Math 31A

Week 1 - Limits and Continuity

Limits

Limit
The limit of a function is what the function **apprroaches** as x **approaches** some value c.
More technically, we say that the limit of $f(x)$ as x approaches c is equal to L if $|f(x) - L|$ can be made arbitrarily small by taking x sufficiently close (but not equal) to c. This is denoted by

$$\lim_{x \to c} f(x) = L$$

Without any other machinery, there are two main ways to investigate limits.

Graphically: Plot the function $f(x)$. As x gets close to c, what does $f(x)$ get close to?

Numerically: Make a table of $f(x)$ for values $x < c$ where $|x - c|$ is small and of values $x > c$ where $|x - c|$ is small. If both of these tables approach the same value, then the limit exists.

As defined, the limit is a **two-sided** idea. It matters not only what $f(x)$ does for $x > c$ but also what $f(x)$ does for $x < c$. We can investigate these ideas separately using the left and right handed limits, which is the same as the limit but restricted to $x < c$ and $x > c$ respectively. The limit exists if and only if the left and right handed limits exist and are equal.

Basic Limit Laws

Theorem. If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist, then

(i) **Sum Law :** $\lim_{x\to c}[f(x)+g(x)]$ exists and

$$\lim_{x\to c}[f(x)+g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$$

(ii) **Constant Multiple Law :** For any $k \in \mathbb{R}$, $\lim_{x\to c} kf(x)$ exists and

$$\lim_{x\to c} kf(x) = k\left(\lim_{x\to c} f(x)\right)$$

(iii) **Product Law :** $\lim_{x\to c} f(x)g(x)$ exists and

$$\lim_{x\to c} f(x)g(x) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right)$$

(iv) **Quotient Law :** If $\lim_{x\to c} g(x) \neq 0$ then $\lim_{x\to c} \frac{f(x)}{g(x)}$ exists and

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$$

(v) **Powers and Roots :** If p, q are integers with $q \neq 0$ then $\lim_{x\to c}[f(x)]^{p/q}$ exists and

$$\lim_{x\to c}[f(x)]^{p/q} = \left(\lim_{x\to c} f(x)\right)^{p/q}$$

Provided that $\lim_{x\to c} f(x) \geq 0$ if q is even and that $\lim_{x\to c} f(x) \neq 0$ if $p/q < 0$.

Note that this is easily applied to $[f(x)]^n$ and $[f(x)]^{1/n}$ for integer n.

Continuity

Continuous at a Point

Assume that $f(x)$ is defined on an open interval containing $x = c$. Then f is **continuous** at $x = c$ if

$$\lim_{x\to c} f(x) = f(c)$$

If the limit does not exist or is not equal to $f(c)$, then we say that $f(x)$ is **discontinuous** at $x = c$.

So to show that a function $f(x)$ is continuous at $x = c$, we must show that

(1) $f(c)$ is defined

(2) $\lim_{x\to c} f(x)$ is defined

(3) $\lim_{x\to c} f(x) = f(c)$

If a function is continuous at all points in its domain, we simply say that it is **continuous**.

Theorem. Laws of Continuity : Sums, products, multiples, inverses and composites of continuous functions are continuous. The same holds for a quotient $f(x)/g(x)$ at points where $g(x) \neq 0$.

Intuitively, we can think of continuous functions as having a graph that doesn't have any disconnections. That is, we can think of continuous functions as having graphs that could be drawn without your pencil leaving the paper.

Indeterminate Forms

Indeterminate Form

If the formula for $f(c)$ yields an undefined expression of the form

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \infty * 0, \quad \text{or} \quad \infty - \infty$$

Then we say that $f(x)$ has an **indeterminate form** at $x = c$ or that $f(x)$ is **indeterminate** at $x = c$.

Note that if $f(x)$ is indeterminate at $x = c$ then this does not imply that $\lim_{x \to c} f(x)$ does not exist. Rather, it only implies that this limit is more difficult to find than by simply checking $f(c)$.

If $f(x)$ is indeterminate at $x = c$, then we currently have two main methods to determine $\lim_{x \to c} f(x)$

(1) (Best option for now.) Algebraically transform $f(x)$ into a new expression that is defined and continuous at $x = c$ and then evaluate by plugging c in. This is usually accomplished by factoring the top and bottom and then cancelling. More complicated, but equally useful, is multiplying the top and bottom by a conjugate of either the top or bottom.

(2) (Worst option for now.) Examine the function graphically or numerically.

In the future, when we have derivatives and other machinery, we can evaluate these in more elegant ways.

Practice Problems

Exercise 1. Use the basic limit laws to complete the following.

(a) Find
$$\lim_{t\to -1} \frac{t^2+1}{(t^3+2)(t^4+1)}$$

(b) Can you apply the quotient rule to
$$\frac{\sin(x)}{x}$$
Why or why not?

Exercise 2. Give an example where $\lim_{x\to\infty}(f(x)+g(x))$ exists but neither $\lim_{x\to 0} f(x)$ nor $\lim_{x\to 0} g(x)$ exist.

Exercise 3. Draw the graph of a function $f : [0,5] \to \mathbb{R}$ such that f is right but not left continuous at $x = 1$, left but not right continuous at $x = 2$ and neither right nor left continuous at $x = 3$.

Exercise 4. *Challenge :* Give an example of functions f, g such that $f(g(x))$ is continuous but g has at least one discontinuity.

Exercise 5. *Challenge :* Show that the following function is only continuous at $x = 0$,
$$f(x) = \begin{cases} x, & x \text{ is rational} \\ -x, & x \text{ is irrational} \end{cases}$$

Week 2 - Squeeze Theorem and Limits at Infinity

Squeeze Theorem

Theorem. **(Squeeze Theorem) :** If there exists an open interval I containing c such that for all $x \in I \setminus \{c\}$ (I excluding c)

$$\ell(x) \leq f(x) \leq u(x) \qquad \text{and} \qquad \lim_{x\to c} \ell(x) = \lim_{x\to c} u(x) = L$$

Then $\lim_{x\to c} f(x)$ exists and $\lim_{x\to c} f(x) = L$.
Intuition : Near, but not necessarily at, c, f is bounded above by $u(x)$ and below by $\ell(x)$. So the limit of $f(x)$ must be between the limits of $\ell(x)$ and $u(x)$.

Significant Trigonomoetric Limits

Using the squeeze test and some cleverness, we can show that

Theorem.

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

Theorem.

$$\lim_{x\to 0} \frac{1-\cos(x)}{x} = 0$$

Limits at Infinity

Intuitively, limits at infinity behave exactly the same as limits at any other point. We say that $\lim_{x\to\infty} f(x) = L$ if we can make $|f(x) - L|$ arbitrarily small if we choose x sufficiently large.

For exponents, we know that following rules,

Theorem. For all $n > 0$,

$$\lim_{x\to\infty} x^n = \infty \qquad \text{and} \qquad \lim_{x\to\infty} x^{-n} = 0$$

If n is a positive whole number,

$$\lim_{x\to-\infty} x^n = \begin{cases} \infty, & \text{if } n \text{ is even} \\ -\infty, & \text{if } n \text{ is odd} \end{cases} \qquad \text{and} \qquad \lim_{x\to-\infty} x^{-n} = 0$$

Theorem. (Limits of a Rational Function) : The asymptotic behavior of a rational function depends only on the leading terms of its numerator and denominator. If $a_n, b_m \neq 0$ then

$$\lim_{x\to\pm\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0} = \frac{a_n}{b_m} \lim_{x\to\pm\infty} x^{n-m}$$

Practice Problems

Exercise 6. Draw the graph of a function $f : [0,5] \to \mathbb{R}$ such that f is right but not left continuous at $x = 1$, left but not right continuous at $x = 2$ and neither right nor left continuous at $x = 3$.

Exercise 7. Evaluate the following limits using the identity

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

(a)

$$\lim_{x\to 1} \frac{x^2 - 5x + 4}{x^3 - 1}$$

(b)

$$\lim_{x\to 1} \frac{x^4 - 1}{x^3 - 1}$$

Exercise 8. Using the squeeze theorem, evaluate the following limit

$$\lim_{x\to 0} \tan\left(x \cos\left(\sin\left(\frac{1}{x}\right)\right)\right)$$

Exercise 9. Do the following inequalities provide enough information to determine $\lim_{x\to 1} f(x)$ by the squeeze theorem?

(a) $4x - 5 \le f(x) \le x^2$

(b) $2x - 1 \le f(x) \le x^2$

(c) $4x - x^2 \le f(x) \le x^2 + 2$

Exercise 10. *(Challenging) :* Let $a_n, b_m \neq 0$. What are all possible values of the following limit and what conditions cause each limit?

$$\lim_{x\to\infty} \frac{a_n x^n + \cdots + a_0}{b_m x^m + \cdots + b_0}$$

Exercise 11. *(Challenging) :* Intuitively, explain why the limit at infinity of a rational function is what it is.

Week 3 - Intermediate Value Theorem and Derivatives

Intermediate Value Theorem

Theorem. (Intermediate Value Theorem) : If f is continuous on a closed interval $[a,b]$, then for every M, strictly between $f(a)$ and $f(b)$, there exists at least one value $c \in (a,b)$ such that $f(c) = M$.

This can be used to show that functions attain certain values somewhere. Perhaps most usefully, it can be shown that functions have zeros within different locations. Specifically,

Theorem. (Existence of Zeros) : If f is continuous on $[a,b]$ and one of $f(a)$ or $f(b)$ is negative and the other is positive (equivalently if $f(a)f(b)$ is negative) then f has a zero in $[a,b]$.

This is useful when faced with problems like "Show that $f(x) = g(x)$ for some x." because you can consider the function $f(x) - g(x)$ and then show that it has a zero somewhere.

The Derivative

The Derivative
The derivative of f at a point a, if it exists, is the limit

$$f'(a) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

When the derivative exists, we say that f is **differentiable** at a.

By definition, this implies that if f is differentiable at a, then $f'(a)$ is approximately $(f(x+h)-f(x))/h$ for small h. That is

Theorem. Let f be differentiable at a. For small h,

$$f'(a) \approx \frac{f(x+h)-f(x)}{h}$$

Using the derivative, we can find a tangent line to the graph of a function at any point. This is given by

Tangent Line
Let f be differentiable at a. Then the line tangent to the graph of $y = f(x)$ at $P = (a, f(a))$ is the line through P with slope $f'(a)$. The equation of this line is

$$y = f'(a)(x-a) + f(a)$$

Though the prime notation used above is popular, there is another notation that is just as popular, if not more so. This is called Leibniz notation

Leibniz Notation
Leibniz notation replaces the prime symbol with $\frac{d}{dx}$ instead. It is popular because it emphasizes how the derivative is found through a ratio of the change in f to the change in x. Additionally, it makes it clear which variable you are differentiating with respect to, in the case that there are multiple. It is specifically written as

$$f'(x) = \frac{df}{dx}$$

The Derivative as a Function

In the previous section, we defined what it means for f to be differentiable at a point a. We can now expand that into something much broader.

Differentiable
If the derivative of f exists for all $x \in (a, b)$, then we say that f is **differentiable on** (a, b). Similarly, if the derivative of f exists for all real numbers x (or if the derivative of f exists on the whole domain of f), then we say that f is **differentiable**.

Derivative Rules

There are many rules for derivatives that can make calculating them very quick. The most important ones for now are

Theorem. (Constant Rule) : The derivative of a constant is 0. Specifically,

$$\frac{d}{dx}c = 0$$

Theorem. (Power Rule) : For any exponent n,

$$\frac{d}{dx}x^n = nx^{n-1}$$

Theorem. (Linearity Rules) : The derivative of the sum of two functions is the sum of the derivatives. Specifically,

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$$

The derivative of a constant times a function is the constant times the derivative of the function. Specifically, for a function f and a constant c,

$$\frac{d}{dx}(cf) = c\frac{df}{dx}$$

Differentiability, Continuity, and Local Linearity

From the definition of the derivative, we find two nice properties

Theorem. (Differentiability Implies Continuity) : If f is differentiable at $x = c$ then f is continuous at $x = c$.

Theorem. (Local Linearity) : If f is differentiable at $x = c$ then in small neighborhoods of c, f is approximately equivalent to the tangent line at $x = c$.

Practice Problems

Exercise 12. *(Optional)* Using the intermediate value theorem, show that $\sqrt{2}$ exists.

Exercise 13. Using the intermediate value theorem, show that

$$\cos(x) = \tan(x)$$

has a solution.

Exercise 14. Using the limit definition of the derivative, calculate the derivative of

$$f(x) = x^3 + 2x$$

Exercise 15. Given

$$f(x) = x - 2x^2$$

Use the limit definition to compute $f'(3)$ and find an equation of the tangent line.

Exercise 16. Sketch a graph of

$$f(x) = x^{2/5}$$

and identify the points where $f'(c)$ does not exist.

Exercise 17. Calculate the derivative of

$$g(x) = \frac{x^2 + 4x^{1/2}}{x^2}$$

Math31A Week 4 - Midterm Review

Practice Problems

Exercise 18. Calculate the following limits,

(1)

$$\lim_{x\to 1}\left(\frac{1}{1-x}-\frac{2}{1-x^2}\right)$$

(2)

$$\lim_{x\to 1}\left(\frac{1}{\sqrt{x}-2}-\frac{4}{x-4}\right)$$

(3)

$$\lim_{\theta\to\pi/2}(\sec\theta-\tan\theta)$$

Exercise 19. Identify points of discontinuity of the following functions, state why they are discontinuities, and give what type of discontinuity.

(1)

$$f(x)=\begin{cases} x+1, & x<1\\ 1/x, & x\geq 1\end{cases}$$

(2)

$$f(x)=\begin{cases} \frac{x^2-3x+2}{|x-2|}, & x\neq 2\\ 0, & x=2\end{cases}$$

Exercise 20. Use the IVT to show that the following have solutions

(1)

$$2^x+3^x=4^x$$

(2)

$$\sqrt{x}+\sqrt{x+2}=3$$

(3)

$$\cos(x)=\tan(2x) \qquad \text{on } (0,1)$$

Exercise 21. Use the limit definition of the derivative to calculate $f'(x)$ when

(1)

$$f(x)=x^3+2x$$

(2)

$$f(x) = \sqrt{x+4} \qquad \text{on } x > -4$$

(3)

$$f(x) = \frac{1}{1-x} \qquad \text{on } x \neq 1$$

Week 5 - Post Midterm Review and Derivatives

Practice Problems

Exercise 22. Use the product rule to calculate the derivative of

(1)

$$f(x) = (3x-5)(2x^2-3)$$

(2)

$$f(x) = (t-8t^{-1})(t+t^2)$$

Find the tangent line at $t = 1$.

Exercise 23. Use the quotient rule to calculate the derivative of

(1)

$$f(x) = \frac{x+4}{x^2+x+1}$$

(2)

$$f(x) = \frac{z^2}{\sqrt{z}+z}$$

Exercise 24. Use the chain rule to calculate the derivative of

(1)

$$f(x) = \cos(x^3)$$

(2)

$$f(x) = \sqrt{9+x+\sin x}$$

Exercise 25. Calculate the following limits,

(1)

$$\lim_{\theta\to 0} \frac{\sin\theta - \sin\theta\cos\theta}{\theta}$$

(2)

$$\lim_{x\to 4} \frac{3x-12}{\sqrt{x}-2}$$

Exercise 26. Let $f(x) = 7xe^{x^2}$.

(1) Calculate the derivative $f'(x)$.

(2) Give the equation for the tangent line of f at $x = 0$.

Exercise 27. Calculate the following limit

$$\lim_{x\to\infty} \frac{12x^4 + 3x^2 + 4}{22x^4 + 15}$$

Exercise 28. Let f be defined by

$$f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

(1) Show that f is continuous at $x = 0$.

(2) Show that f is differentiable at 0 and $f'(0) = 1$ by using the limit definition of the derivative.
Hint:

$$\lim_{\theta\to 0} \frac{\sin\theta - \theta}{\theta^2} = 0$$

Week 6 - Implicit Differentiation and Extrema

Implicit Differentiation

Thus far, we have developed formulas for when we have y explicitly written as a function of x. However, what if y is related to x by an equation like the following?

$$y^3 + \frac{1}{xy} = 16 - 9x^2 y$$

In this case, we can differentiate both sides of the equation and then gather all terms of $\frac{dy}{dx}$ onto one side and solve for them.

In the case of the equation above,

$$\begin{aligned}
\frac{d}{dx}\left(y^3 + \frac{1}{xy}\right) &= \frac{d}{dx}\left(16 - 9x^2 y\right) \\
3y^2 \frac{dy}{dx} + \frac{-1}{xy^2}\frac{dy}{dx} + \frac{-1}{x^2 y} &= -18xy - 9x^3 \frac{dy}{dx} \\
3y^2 \frac{dy}{dx} + \frac{-1}{xy^2}\frac{dy}{dx} + 9x^3 \frac{dy}{dx} &= \frac{1}{x^2 y} - 18xy \\
\frac{dy}{dx} &= \frac{\frac{1}{x^2 y} - 18xy}{3y^2 + \frac{-1}{xy^2} + 9x^3} \\
\frac{dy}{dx} &= \frac{y - 18x^3 y^3}{3y^4 x^2 - x + 9x^5 y^2}
\end{aligned}$$

For implicit differentiation, it is important to go slowly and be careful about when you use various derivative rules. The chain rule will be especially helpful in these types of problems.

Extreme Values

Extreme Values on an Interval
Let f be a function on an interval I and let there exist $a \in I$. We say that $f(a)$ is

- **Absolute minimum** of f if $f(a) \leq f(x)$ for all $x \in I$
- **Absolute maximum** of f if $f(a) \geq f(x)$ for all $x \in I$

Theorem. (Existence of Extrema on a Closed Interval) : A continuous function f on a closed and bounded interval takes on both a minimum and a maximum value on I.

Local Extrema and Critical Points

Local Extrema
We say that $f(c)$ is a

- **Local minimum** occurring at $x = c$ if $f(c)$ is the minimum value of f on some open interval containing c
- **Local maximum** occurring at $x = c$ if $f(c)$ is the maximum value of f on some open interval containing c

It seems intuitively easy to find the absolute minimum and maximum of a function, but how do we find the local extrema? We can actually use the derivative to easily find the points that can be local extrema through the use of critical points

Critical Points
A number c in the domain of f is called a **critical point** if either $f'(c) = 0$ or $f'(c)$ does not exist.

Which benefit us through the following theorem

Theorem. (Fermat's Theorem on Local Extrema) : If $f(c)$ is a local minimum or maximum, then c is a critical point of f.

Optimizing on a Closed Interval

Theorem. (Extreme Values on a Closed Interval) : Let f be continuous on $[a, b]$ and let $f(c)$ be the minimum or maximum value on $[a, b]$. Then c is either a critical point or one of the endpoints, a or b.

First Derivative Test

Given a critical point, we can determine the nature of f at that point through the first derivative test.

Theorem. (First Derivative Test) : Let c be a critical point of f. Then

- $f'(x)$ changes from $+$ to $-$ implies that $f(c)$ is a local maximum
- $f'(x)$ changes from $-$ to $+$ implies that $f(c)$ is a local minimum

The Mean Value Theorem

The mean value theorem is analogous to the intermediate value theorem. Based on the values of f on the endpoints of an interval, we can determine the existence of points in the interval with a specific derivative.

Before the general mean value theorem, we have a specific case

Theorem. (Rolle's Theorem) : Assume that f is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$ then there exists number c between a and b such that $f'(c) = 0$.

This generalizes to

Theorem. (Mean Value Theorem) : Assume that f is continuous on $[a, b]$ and differentiable on (a, b). Then there exists some c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Practice Problems

Exercise 29. Find dy/dx in terms of x and y given the following equation

(1)

$$\frac{y}{x} + \frac{x}{y} = 2y$$

(2)

$$\tan(x^2 y) = (x + y)^3$$

Exercise 30. Find the extreme values of $f(x)$ on the given interval.

(1)

$$f(z) = z^5 - 80z, \qquad [-3, 3]$$

(2)

$$f(y) = \sqrt{x + x^2} - 2\sqrt{x}, \qquad [0, 4]$$

Exercise 31. Find all critical points the following functions and use the first derivative test to determine whether they are local maxima or local minima

(1)

$$y = \frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x + 4$$

(2)

$$y = \frac{2x+1}{x^2+1}$$

Week 7 - MVT, Second Derivatives, and L'Hopital's Rule

The Mean Value Theorem

The mean value theorem is analogous to the intermediate value theorem. Based on the values of f on the endpoints of an interval, we can determine the existence of points in the interval with a specific derivative.

Before the general mean value theorem, we have a specific case

Theorem. (Rolle's Theorem) : Assume that f is continuous on $[a,b]$ and differentiable on (a,b). If $f(a) = f(b)$ then there exists number c between a and b such that $f'(c) = 0$.

This generalizes to

Theorem. (Mean Value Theorem) : Assume that f is continuous on $[a,b]$ and differentiable on (a,b). Then there exists some c in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Monotonicity

The derivative helps us to classify whether a function is increasing or decreasing on any given interval. In particular,

Theorem.
$f'(x) > 0$ for $x \in (a,b)$ implies that f is increasing on (a,b)
$f'(x) < 0$ for $x \in (a,b)$ implies that f is decreasing on (a,b)

Between adjacent critical points, the derivative of a function cannot be 0. So we know that the function must be increasing or decreasing on these intervals.

Second Derivative

We have seen that the derivative can be used to see how the function behaves on given intervals, whether it is increasing or decreasing, maximum or minimum, etc. The second derivative can be used in much the same ways, except now it tells use how the first derivative behaves, which then tells us how the function behaves. So the process now has two steps.

In helping with this discussion, we define concave up and concave down as follows

Concavity
Let f be a differentiable function on an open interval (a, b). Then

- f is **concave up** on (a, b) if f' is increasing on (a, b)
- f is **concave down** on (a, b) if f' is decreasing on (a, b)

In the same manner that the first derivative was used to determine if f was increasing or decreasing, we can use the second derivative to determine if f' is increasing or decreasing. Therefore we arrive at the following test for concavity

Theorem. (Test for Concavity) : Assume that $f''(x)$ exists for all $x \in (a, b)$. Then

- If $f''(x) > 0$ for all $x \in (a, b)$ then f is concave up on (a, b)
- If $f''(x) < 0$ for all $x \in (a, b)$ then f is concave down on (a, b)

Just like how we cared about the points where the derivative was zero, we also care about the points where the second derivative is zero.

Point of Inflection
A point c is a **point of inflection** of f if the concavity changes from up to down at $x = c$.

We can test for points of inflection in the exact same way that we tested for critical points, by looking at the second derivative and seeing where it is 0 or undefined.

Theorem. (Test for Inflection Points) : If $f''(c) = 0$ or $f''(c)$ does not exists and $f''(x)$ changes sign at $x = c$, then f has a point of inflection at $x = c$.

Second Derivative Test

When we did the first derivative test to determine if a point was a local maximum, minimum, or neither, we checked whether the derivative went from positive to negative or vice versa. Now, we can look at this in terms of the second derivative. If the derivative goes from positive to negative at a point, then that means that the second derivative (if it exists) must be negative at that point because the derivative is decreasing. If the derivative goes from negative to positive at a point, then that means that the second derivative (if it exists) must be positive at that point because the derivative is increasing. This leads us to the following test to determine the nature of critical points.

Theorem. (Second Derivative Test for Critical Points) : Let c be a critical point of f. If $f''(c)$ exists, then

- $f''(c) > 0$ implies that $f(c)$ is a local maximum
- $f''(c) < 0$ implies that $f(c)$ is a local minimum
- $f''(c) = 0$ implies nothing. $f(c)$ may be a local max, local min, or neither

L'Hôpital's Rule

L'Hôpital's Rules is a valuable tool for computing certain limits, especially the limits of quotients when the quotient rule cannot be applied. It is stated as follows

Theorem. (L'Hôpital's Rule) : Suppose that f and g are differentiable on an open interval containing a and that $f(a) = g(a) = 0$. Also assume that $g'(x) \neq 0$ except possibly at a. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right exists or is infinite. This conclusion also holds if f and g are differentiable for x near (but not equal to) a and

$$\lim_{x\to a} f(x) = \pm\infty \qquad \lim_{x\to a} g(x) = \pm\infty$$

Furthermore, this rule is valid for one-sided limits.

Note that if you do L'Hôpital's rule once and still find an indeterminate form, then you may do it again until you don't.

All review materials and problems on this document are taken from Jon Rogawski's Single Variable Calculus, 4^{th} edition. Special thanks to him for these materials.

Practice Problems

Exercise 32. Suppose the $f(0) = 2$ and $f'(x) \leq 3$ for $x > 0$. Apply the MVT to the interval $[0,4]$ to show that $f(4) \leq 14$.
In addition, show that $f(x) \leq 2 + 3x$ for all $x > 0$.

Exercise 33. Determine the intervals on which the function is concave up or concave down and find the points of inflection.

(i) $y = (x-2)(1-x^3)$

(ii) $y = \theta - 2\sin\theta \quad$ on $[0, 2\pi]$

(iii) $y = \frac{x^4-1}{x}$

Exercise 34. Find all critical points and apply the second derivative test.

(i) $f(x) = x^5 - x^3$

(ii) $f(x) = 3x^4 - 8x^3 + 6x^2$

(iii) $f(x) = \frac{1}{x^2 - x + 2}$

(iv) $y = \sin^2 x + \cos x$

Exercise 35. Evaluate the following limits, using L'Hôpital's Rule where it applies

(i) $\lim_{x\to 4} \frac{x^3-64}{x^2+16}$

(ii) $\lim_{x\to 0} \frac{\sin 4x}{x^2+3x+1}$

(iii) $\lim_{x\to 0} \frac{\sin 2x}{\sin 7x}$

(iv) $\lim_{x\to 0}(\cos x)^{3/x^2}$

Hint: $a^x = e^{\log(a)x}$ and $\lim_{x\to a} e^{f(x)} = e^{\lim_{x\to a} f(x)}$ because e^x is continuous.

Exercise 36. *(Challenge)* Show that if $f(0) = g(0)$ and $f'(x) \le g'(x)$ for $x \ge 0$, then $f(x) \le g(x)$ for all $x \ge 0$. *Hint:* consider the function $h(x) = g(x) - f(x)$ and show that it is non-decreasing.

The following problem will most likely not be included on homeworks or exams, but it is cool, so I thought I would include it.

Exercise 37. *(Challenge)* Assume that f'' exists and $f''(x) = 0$ for all x. Show that $f(x) = mx + b$ where $m = f'(0)$ and $b = f(0)$.

Week 8 - Second Midterm Review and Newton's Method

Newton's Method

Newton's Method is a way of using the derivative of a function $f(x)$ to numerically find (or approximate) the roots $f(x) = 0$. It does this by starting with a guess point x_0, assuming that $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$ and then using this to determine a next best guess x_1. Iterating this process, we get better and better guesses $x_0, x_1, x_2, \dots$. More definitively, the process is defined as

Theorem. Newton's Method : To approximate a root of $f(x) = 0$,

(1) Choose an initial guess x_0 (close to the desired root if possible)

(2) Generate successive approximations $x_1, x_2, \dots$ where

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

As a good rule of thumb, if x_n and x_{n+1} agree to m decimal places, then you can usually safely assume that x_n agrees with a root to m decimal places.

Review Problems

Exercise 38. *(Mean Value Theorem):* Show that there exists some c in the given interval such that $f'(c)$ satisfies the relationship described.

(i) $f(x) = \sqrt{x}$ $[9, 25]$ $f'(c) = \frac{1}{8}$

(ii) $f(x) = x - \sin \pi x$ $[-1, 1]$ $f'(c) = 1$

(iii) $f(x) = (x-1)(x-3)$ $[1, 3]$ $f'(c) = 0$

For more practice, you can find a specific c satisfying the relationship.

Exercise 39. *(Implicit Differentiation) :* Give the equation for the tangent line of the given curve at the given point.

(i) $xy + x^2y^2 = 6$ at the point $(2, 1)$

(ii) $x^2 + \sin y = xy^2 + 1$ at the point $(1, 0)$

(iii) $2x^{1/2} + 4y^{-1/2} = xy$ at the point $(1, 4)$

(iv) $\sin(2x - y) = \frac{x^2}{y}$ at the point $(0, \pi)$

Exercise 40. *(Extrema) :* Find all local extrema of the following functions

(i) $f(x) = \frac{1}{\sin x + 4}$

(ii) $f(x) = 9x^{7/3} - 21x^{1/2}$

(iii) $f(x) = 3x^4 - 6x^3 + 6x^2$

(iv) $f(x) = \sin(x)\cos(x)$ *(potential challenge)*

Exercise 41. *(Concavity) :* Determine the intervals on which the function is concave up or down and find the points of inflection.

(i) $y = 10x^3 - x^5$

(ii) $y = (x-2)(1-x^3)$

(iii) $y = x^{7/2} - 35x^2$

(iv) $f(x) = \frac{x^3}{1+x}$

(v) $f(x) = \tan(x)$ *(potential challenge)*

Week 9 - Integrals

Definite Integral

Given a function f on an interval $[a,b]$, a surprisingly insightful question to ask is "What is the area between the graph of f and the x-axis?" We can determine this area through the definite integral. But that question alone leaves some aspects of the integral ambiguous. Such as "What about when $f(x)$ is negative?" and "How do you calculate that area when $f(x)$ isn't nice?" With these questions in mind, we can go and work up to define the definite integral.

Let $f : [a,b] \to \mathbb{C}$ be a continuous function. In order to find the area under the curve of f, we approximate the area by a series of rectangles and then let these rectangles become really small. In determining these rectangles, we need a few concepts

Partition P of Size N
A choice of points that divides $[a,b]$ into N subintervals (not necessarily of equal width). This is normally denoted as

$$P : a = x_0 < x_1 < \cdots < x_N = b$$

Sample Points
Given a partition P, we define the set of **sample points** $C = \{c_1, \ldots, c_N\}$ such that c_i is an element of the interval $[x_{i-1}, x_i]$ for all $i = 1, \ldots, N$. It doesn't matter which points are chosen, as long as they are within the specific interval.

Length of Subinterval
Given a partition P of $[a,b]$, the length of a subinterval $[x_{i-1}, x_i]$ is

$$\Delta x_i = x_i - x_{i-1}$$

Norm of Partition
Given a partition P of $[a,b]$, the norm of P, denoted $\|P\|$, is the maximum length of the subintervals. That is,

$$\|P\| = \max_i \Delta x_i$$

We will briefly connect these definitions back to the rectangls. A partition P gives breaks the interval $[a,b]$ down into pieces that will become the bases of our rectangles. The sample points C, give test points for the value of f on each of these intervals. We will take the height of each rectangle to be $f(c_i)$. Note here that $f(c_i)$ could be negative, at which point our rectangle has negative area. The lengths of the intervals gives us the length of the base of our rectangles. Finally, the norm of a partition gives one number to how "big" our partition is. By forcing the norm to become really small, we force our partition to become really fine and our rectangles to become really small.

With these, we define the Riemann sum of f over P as

Riemann Sum

The Riemann Sum of f over the partition P with sample points C is given by

$$R(f,P,C) = \sum_{i=1}^{N} f(c_i)\Delta x_i = f(c_1)\Delta x_1 + \cdots + f(c_N)\Delta x_N$$

Finally, to define the Riemann integral, we let the rectangles get really small. Equivalently, we let the norm of the Partition go to 0. Defining this explicitly,

Definite Integral

The definite integral of f over $[a,b]$ is the limit of Riemann sums as $\|P\| \to 0$. It is defined as

$$\int_a^b f(x)\,dx = \lim_{\substack{\|P\|\to 0 \\ \text{partitions } P \text{ with sample points } C}} R(f,P,C) = \lim_{\|P\|\to 0} \sum_{i=1}^{N} f(c_i)\Delta x_i$$

When this limit exists and is finite, we say that f is **integrable** over $[a,b]$.

This leads us to our first theorem

Theorem. (Almost Continuous Functions are Riemann Integrable) :
If f is continuous on $[a,b]$, or if f is continuous except at finitely many jump discontinuities, then f is integrable over $[a,b]$.

Properties of the Definite Integral

Given the definition earlier, we can show many properties of the definite integral. These will not be shown here, as the proofs are long and not particularly insightful. However, you should note that all of the following theorems make intuitive sense when the integral is thought of as the area under the curve.

Theorem. (Linearity of the Integral) : If f and g are integrable over $[a,b]$ and c is a real constant, then $f+g$ and cf are integrable and

$$\int_a^b \big(f(x)+g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$
$$\int_a^b c * f(x)\,dx = c\int_a^b f(x)\,dx$$

Reversing the Limits of Integration
Let f be a function. Then for $a < b$, we define

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Theorem. (Additivity for Adjacent Integrals) : Let $a \le b \le c$ and let f be integrable. Then

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

Theorem. (Comparison Theorem) : If f, g are integrable over $[a, b]$ and $f(x) \le g(x)$ for all x in $[a, b]$ then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

Indefinite Integral

The indefinite integral is difficult to motivate without defining it first and finding out what properties it has. Instead, we will arrive at it in a roundabout way by asking "What happens if we take a derivative but backwards?"

Antiderivate
A function F is an antiderivative of f on an open interval (a, b) if $F'(x) = f(x)$ for all x in (a, b).

Now, unlike the derivative, the antiderivative is not necessarily unique. This comes from the fact that the derivative of a constant is 0, and so adding a constant to a function doesn't change its derivative. This leads to the following theorem

Theorem. (The General Antiderivative) : Let $y = F(x)$ be an antiderivative of $y = f(x)$ on (a, b). Then every antiderivative of f on (a, b) is of the form $y = F(x) + c$ for some constant c.

From this, we define the indefinite integral. The connection between this and the definite integral will be seen soon, but for now you will have to trust that they connect.

Indefinite Integral
The notation

$$\int f(x)\,dx = F(x) + c \quad \text{means that} \quad F'(x) = f(x)$$

We say that $F(x) + c$ is the general antiderivative or **indefinite integral** of $y = f(x)$.

Properties of the Indefinite Integral

The indefinite integral has many of the same properties as the definite integral. Namely,

Theorem. (Linearity of the Indefinite Integral) :

$$\int \big(f(x)+g(x)\big)\,dx = \int f(x)dx + \int g(x)dx$$
$$\int c * f(x)dx = c\int f(x)dx$$

Important Indefinite Integrals

Using the power rule for derivatives and the standard trig function derivatives, we can derive a couple of important indefinite integrals. In particular,

Theorem. (Power Rule for Indefinite Integral) :

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

Theorem. (Basic Trigonometric Integrals) :

$$\int \sin x dx = -\cos x + c \qquad \int \cos x dx = \sin x + c$$
$$\int \sec^2 x dx = \tan x + c \qquad \int \csc^2 x dx = -\cot x + c$$
$$\int \sec x \tan x dx = \sec x + c \qquad \int \csc x \cot x dx = -\csc x + c$$

The Fundamental Theorem of Calculus

Now that we've defined the indefinite integral, it's time to show why it is such an important concept. That importance, and the connection between the indefinite and definite integrals, comes from the fundamental theorem of calculus. There are two parts to the fundamental theorem of calculus, both of which we give here

Theorem. (The First Fundamental Theorem of Calculus) : Let $a < b$ and let f be continuous on (a,b). If F is an antiderivative of f on $[a,b]$ then

$$\int_a^b f(x)dx = F(b) - F(a) = F(x)\Big|_{x=a}^{b}$$

Theorem. (The Second Fundamental Theorem of Calculus) : Let f be continuous on an open interval I and let a be in I. Then the area function

$$A(x) = \int_a^x f(t)dt$$

is an antiderivative of f on I. So

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

The first fundamental theorem of calculus tells us that taking the indefinite integral of a function (finding the antiderivative) gives us a way to compute any integral of the function quickly and easily. The second fundamental theorem of calculus tells us that any continuous function has an antiderivative and that the integral is, intuitively, the inverse of the derivative, and vice versa.

With these tools, we can calculate lots of integrals quickly and easily!

Practice Problems

Exercise 42. Calculate the integral of the following functions on the following intervals

(1)

$$f(x) = \begin{cases} 10, & x \text{ in } [0, 0.5) \\ 25, & x \text{ in } [0.5, 1.5) \\ 15, & x \text{ in } [1.5, 2) \\ 20, & x \text{ in } [2, 3] \end{cases} \qquad \text{on } [0, 3]$$

(2)

$$f(x) = \begin{cases} x, & x \text{ in } [0, 1) \\ 2 - x, & x \text{ in } [1, 2) \\ -3, & x \text{ in } [2, 3] \end{cases} \qquad \text{on } [0, 3]$$

Exercise 43. Express the following in one integral

(1)

$$\int_0^3 f(x)dx + \int_3^7 f(x)dx$$

(2)

$$\int_2^9 f(x)dx - \int_4^9 f(x)dx$$

(3)

$$\int_7^3 f(x)dx + \int_3^9 f(x)dx$$

Exercise 44. *Challenge :* Let f be an odd function. Show that

$$\int_{-a}^{a} f(x)\,dx = 0$$

Exercise 45. Find the indefinite integral of f and check your answer by differentiating

(1)
$$f(x) = 18x^2$$

(2)
$$f(x) = x^{-3/5}$$

(3)
$$f(x) = 2\cos x - 9\sin x$$

(4)
$$f(x) = 4x^7 - 3\cos x$$

Week 10 - FTC and Change of Variables

The Fundamental Theorem of Calculus

Now that we've defined the indefinite integral, it's time to show why it is such an important concept. That importance, and the connection between the indefinite and definite integrals, comes from the fundamental theorem of calculus. There are two parts to the fundamental theorem of calculus, both of which we give here

Theorem. (The First Fundamental Theorem of Calculus) : Let $a < b$ and let f be continuous on (a, b). If F is an antiderivative of f on $[a, b]$ then

$$\int_a^b f(x)dx = F(b) - F(a) = F(x)\Big|_{x=a}^{b}$$

Theorem. (The Second Fundamental Theorem of Calculus) : Let f be continuous on an open interval I and let a be in I. Then the area function

$$A(x) = \int_a^x f(t)dt$$

is an antiderivative of f on I. So

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

The first fundamental theorem of calculus tells us that taking the indefinite integral of a function (finding the antiderivative) gives us a way to compute any integral of the function quickly and easily. The second fundamental theorem of calculus tells us that any continuous function has an antiderivative and that the integral is, intuitively, the inverse of the derivative, and vice versa.

With these tools, we can calculate lots of integrals quickly and easily!

The Substitution Method

The substitution method is to integrals what the chain rule is to derivatives. It is a way of composing and uncomposing functions to make integrals easier. Intuitively, it can be viewed as doing the chain rule in reverse, which is reflected in how it looks.

Theorem. (The Substituion Method) : If $F'(x) = f(x)$ and u is a differentiable function whose range includes the domain of f, then

$$\int f(u(x))u'(x)dx = F(u(x)) + C$$

Unfortunately, as it stands, it is a difficult theorem to apply. It is very useful, but it is constraining in its requirements and form. Thankfully, we can generalize it to the change of variable formula as follows

Theorem. (Change of Variables Formula) :

$$\int \underbrace{f(u(x))}_{f(u)} \underbrace{u'(x)dx}_{du} = \int f(u)du$$

Finally, we can take this change of variables and apply it to the definite integral case, which is done as follows

Theorem. (Change of Variables for Definite Integral) : If u' is continuous on $[a, b]$ and f is continuous on the range of u, then

$$\int_a^b f(u(x))u'(x)dx = \int_{u(a)}^{u(b)} f(u)du$$

Practice Problems

Exercise 46. Evaluate the following integrals

(a)
$$\int_0^2 (12x^5 + 3x^2 - 4x)dx$$

(b)
$$\int_{1/16}^1 t^{1/4} dt$$

(c)
$$\int_0^5 |x^2 - 4x + 3| dx$$

(d)
$$\int_0^\pi |\cos x| dx$$

(e)
$$\int_{-2}^3 f(x)dx \qquad \text{where} f(x) = \begin{cases} 12 - x^2, & \text{for } x \le 2 \\ x^3, & \text{for } x > 2 \end{cases}$$

Exercise 47. Find explicit formulas for the functions represented by the following integrals

(a)
$$f(x) = \int_0^x \sin u\, du$$

(b)

$$k(x) = \int_1^{x^2} t\,dt$$

(c)

$$h(x) = \int_{x/2}^{x/4} \sec^2 u\,du$$

(d)

$$g(x) = \int_2^{\sqrt{x}} \frac{dt}{t^2}$$

Exercise 48. Write out the following integrals in terms of u and du. Then evaluate

(a)
$$\int t\sqrt{t^2+1}\,dt, \qquad u = t^2+1$$

(b)
$$\int \frac{t^3}{(4-2t^4)^{11}}\,dt, \qquad u = 4-2t^4$$

(c)
$$\int \sqrt{4x-1}\,dx, \qquad u = 4x-1$$

Exercise 49. Evaluate the indefinite integral

(a)
$$\int \frac{x}{\sqrt{x^2+9}}\,dx$$

(b)
$$\int x^2\sqrt{x^3+1}\,dx$$

(c)
$$\int x(3x+8)^{11}\,dx$$

(d)
$$\int x^3(x^2-1)^{3/2}\,dx$$

(e)
$$\int \frac{\sin x\cos x}{\sqrt{\sin x+1}}\,dx$$

Chapter 2

Math 31B

Week 1 - Exponentials and Inverse Functions

Derivative of an Exponential Functions

Exponential Function
For some $b > 0, b \neq 1$, an exponential function with base b, is $f(x) = b^x$. The most famous example of this is $b = e$.

By definition, we then compute that for all $b > 0, b \neq 1$,

$$\frac{d}{dx} b^x = \lim_{h \to 0} \frac{b^{x+h} - b^x}{h} = b^x \lim_{h \to 0} \frac{b^h - 1}{h}$$

For now, we let $m(b) = \lim_{h \to 0}(b^h - 1)/h$ and note that $m(e) = 1$.

Inverse Functions

Inverse Function
Let there be a function f with domain D and range R. Suppose there exists a function g with domain R such that for any $x \in D$ and for any $y \in R$,

$$g(f(x)) = x \text{ and } f(g(y)) = y$$

If this holds then we denote $g = f^{-1}$.

NOTE : We use f^{-1} to denote the inverse of f. This should not be confused with the reciprocal of f, which is denoted $1/f$.

Week 2 - Logarithms

Fix some $b > 0, b \neq 1$ and consider the exponential function $f(x) = b^x$. If we graph this function, then we see that it passes the horizontal line test and so it has an inverse $f^{-1}(x)$. We denote this inverse by $\log_b(x)$ and note that

$$b^{\log_b x} = x \text{ and } \log_b\left(b^x\right) = x.$$

For $b = e$, we let $\ln(x) = \log_e(x)$.

Analogous to exponentials, logarithms have nice properties when it comes to multiplication, division, and exponentiation.

Theorem. *(Properties of Logarithms) :*
(Products) : $\log_b(xy) = \log_b(x) + \log_b(y)$
(Quotients) : $\log_b(x/y) = \log_b(x) - \log_b(y)$
(Powers) : $\log_b(x^y) = y\log_b x$

We can also use these properties to change the base of a logarithm as follows

Theorem. *(Change of Bases) :* For all bases a, b,

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$$

In particular, for all bases b,

$$\log_b(x) = \frac{\ln(x)}{\ln(b)}$$

Derivative of Logarithms

We start with the simplest case of the natural logarithm. Using the inverse derivative formula,

$$\frac{d}{dx}\ln(x) = \frac{1}{e^{\ln(x)}} = \frac{1}{x}$$

For an arbitrary base $b > 0, b \neq 1$, we can find the derivative by first changing basis. Doing so

$$\frac{d}{dx}\log_b(x) = \frac{d}{dx}\frac{\ln x}{\ln b} = \frac{1}{x\ln b}$$

For general functions $f(x)$, the chain rule then gives us the logarithmic derivative,

$$\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)}$$

In particular, this implies that for any differentiable function f,

$$f'(x) = f(x)\frac{d}{dx}\ln(f(x))$$

This allows us to use the nice properties of the natural logarithm to simplify some calculations.

Exercise 50. Calculate the derivative of $f(x)$ by first differentiating $\ln(f(x))$.

(a) $f(x) = \frac{(x+1)(x+2)}{x+3}$

(b) $f(x) = \frac{e^x(x^2+1)}{\sin(x)\cos(x)}$

(c) $f(x) = (\sin(x))^x$

(d) $f(x) = x^{x^2+2}$

Antiderivative of $1/x$

As shown earlier, we know that $(\ln x)' = 1/x$, so the the antiderivative of $1/x$ is $\ln x$. However, we need to be careful about where this is defined when using this. The function $1/x$ is only defined for $x \neq 0$ and the function $\ln(x)$ is only defined for $x > 0$. Therefore, the anti-derivative of $1/x$ for $x > 0$ is $\ln(x)$, but what about $x < 0$?

For $x < 0$, we know that $|x| = -x$. Then $d/dx(|x|) = -1$ and so

$$\frac{d}{dx}\ln|x| = \frac{-1}{|x|} = \frac{1}{x}$$

for all $x < 0$. Therefore for all $x \neq 0$, $d/dx(\ln|x|) = 1/x$.

A word of caution though, this anti-derivative only exists when $x \neq 0$. If you try to use this to integrate $1/x$ across 0, this will fail.

Exercise 51. Explain why the fundamental theorem of calculus cannot be applied in the following situation

$$\int_{-1}^{1} \frac{1}{x} dx$$

Inverse Trig Functions

When considering whether different functions have inverses, all we need to figure out is if the function is one-to-one (otherwise called injective). We call a function **one-to-one** if $f(x) = f(y)$ implies that $x = y$. Intuitively, this is saying that any output of f can only be created by a single input. If we have this property, then we can define an inverse of f and if we don't have this property then we can't.

But what about the function $\sin(x)$? We know what the graph of $\sin(x)$ looks like and it definitely isn't one-to-one. In fact, every output has infinitely many inputs that could have created it! In this case, we can create an inverse function if we restrict the domain of $\sin(x)$.

By default, we consider $\sin(x)$ as a function that takes in any real number and spits out a real number. But if we restrict $\sin(x)$ so that it only takes in values $-\pi/2 \leq x \leq \pi/2$, then $\sin(x)$ becomes one-to-one. On this region, we define $\sin^{-1}(x)$ as the inverse of $\sin(x)$.

Similarly, we can restrict $\cos(x)$ to the region $[0, \pi]$ to define $\cos^{-1}(x)$ and $\tan(x)$ to the region $(-\pi/2, \pi/2)$ to define $\tan^{-1}(x)$.

Using the inverse derivative formula and some clever trigonometry that I won't repeat here, we find the derivatives of inverse trig functions

Theorem. *(Derivatives of Inverse Trig Functions) :*

1. $\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$
2. $\frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}$
3. $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$

Week 3 - L'Hopital's Rule

L'Hôpital's Rules is a valuable tool for computing certain limits, especially the limits of quotients when the quotient rule cannot be applied. It is stated as follows

Theorem. (L'Hôpital's Rule) : Suppose that f and g are differentiable on an open interval containing a and that $f(a) = g(a) = 0$. Also assume that $g'(x) \neq 0$ except possibly at a. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right exists or is infinite. This conclusion also holds if f and g are differentiable for x near (but not equal to) a and

$$\lim_{x \to a} f(x) = \pm\infty \qquad \lim_{x \to a} g(x) = \pm\infty$$

Furthermore, this rule is valid for one-sided limits.

Note that if you do L'Hôpital's rule once and still find an indeterminate form, then you may do it again until you don't.

Exercise 52. Evaluate the following limits, using L'Hôpital's Rule where it applies

(i) $\lim_{x \to 4} \frac{x^3-64}{x^2+16}$

(ii) $\lim_{x \to 0} \frac{\sin 4x}{x^2+3x+1}$

(iii) $\lim_{x \to 0} \frac{\sin 2x}{\sin 7x}$

(iv) $\lim_{x \to 0} (\cos x)^{3/x^2}$

Hint : $a^x = e^{\log(a)x}$ and $\lim_{x \to a} e^{f(x)} = e^{\lim_{x \to a} f(x)}$ because e^x is continuous.

(v) $\lim_{x \to \infty} \left(1 + \frac{r}{x}\right)^x = e^r$

(vi) $\lim_{x \to 0^+} x^{\sin x}$

We can also use L'Hopital's rule at infinity via

Theorem. Assume that f, g are differentiable on (b, ∞) and that $g'(x) \neq 0$ on (b, ∞). If $\lim_{x \to \infty} f(x)$ and $\lim_{x \to \infty} g(x)$ both are 0 or both are $\pm\infty$ then

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}$$

provided that the limit on the right exists. This also holds for $x \to -\infty$.

Growth of Functions

By considering the limits of the quotient of functions, we can also determine how fast functions grow relative to one another as $x \to \infty$. If a function f grows faster than g, then we would expect that eventually, f is so big compared to f that g/f is really really small. With this intuition, we define

> We say that $f(x)$ grows faster than $g(x)$ if
>
> $$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0$$
>
> we denote this by $g \ll f$. Equivalently, $g \ll f$ if $\lim_{x\to\infty} f(x)/g(x) = \infty$.

Week 4 - Integration by Parts

When we first worked with derivatives, we had a slew of nice properties. We could take the derivatives of sums, differences, products, quotients, scalar multiplication, and compositions. When we then defined integration, we had some similar properties. The integral of a sum is the sum of the integrals, similar to the sum rule for derivation. We could take the integrals of scalar multiples or the difference of two functions. We could even do something similar to the chain rule with u-substitution. However, we did not develop any tools for taking the integrals of products or quotients. We fix that hole in our knowledge with integration by parts.

The product rule for differentiation tells us that

$$(fg)' = f'g + fg'$$

If we then take the indefinite integral of both sides with respect to x, we find that

$$fg = \int f'g\,dx + \int fg'\,dx$$

Rearranging then yields

$$\int fg'\,dx = fg - \int fg'\,dx$$

Which is the integration by parts formula. Intuitively, integration by parts allows us to take a derivative from one term in our integral and move that derivative to another term, provided we spit out an additional term of fg.

If we took a definite integral of our product rule first, then the fundamental theorem of calculus and a similar calculation would tell us that

$$\int_a^b fg'\,dx = fg|_a^b - \int_a^b f'g\,dx$$

Finally, we remember that $dg = g'dx$ and $df = f'dx$ so we can rewrite this nicely as

$$\int f\,dg = fg - \int g\,df$$

and similarly for the definite integral case. Most commonly, this is stated with u, v instead of f, g.

Applying integration by parts is often a big challenge. It's simple to understand what it does, but it is not simple to understand how to use that. To understand this, let's look at a simple example. Consider the integral

$$\int xe^x\,dx$$

Integration by parts allows us to move a derivative from either x or e^x to the other term. If we briefly ignore the fact that we have to take a derivative from x or e^x, let's consider what happens once we move the derivative over.

If we move the derivative to e^x, then e^x turns into e^x. Which doesn't really help us since nothing changes. Instead, if we move the derivative over to the x, then x becomes 1, which is great! That would simplify our integral down to only one term.

To move the derivative to the x, we need to steal a derivative from the e^x. Equivalently, we can take an anti-derivative of e^x. The anti-derivative of e^x is e^x, so that's easy.

Putting all these steps together, we can evaluate the integral. Often we will use the notation $\int u\,dv = uv - \int v\,du$ for integration by parts, so let's use that here. We want to put the derivative on x, so we have $u = x$. We want to take the derivative from e^x, so we have $dv = e^x dx$. Then $du = x' dx = dx$ and $v = e^x$ by taking a derivative and anti-derivative respectively. This implies

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + c$$

as desired.

Exercise 53. Use integration by parts to evaluate the following

(a) $\int \arcsin(x)\,dx$

(b) $\int x^3 \cos(x^2)\,dx$

(c) $\int x^3 \ln x\,dx$

(d) $\int x5^x\,dx$

(e) $\int x\cos(5x)\,dx$

(f) $\int re^{r/2}\,dx$

(g) $\int x^2 \sin(\pi x)\,dx$

(h) $\int \ln(2x+1)$

(i) $\int t\sec^2(2t)\,dt$

(j) $\int (\ln x)^2\,dx$

(k) $\int e^{-\theta}\cos(2\theta)\,d\theta$

(l) $\int_1^2 \frac{\ln x}{x^2}\,dx$

(m) $\int_0^1 (x^2+1)e^{-x}\,dx$

(n) $\int_1^{\sqrt{3}} \arctan(1/x)\,dx$

(o) $\int \sin(\ln x)\,dx$

(p) $\int \cos(x)\ln(\sin(x))\,dx$

Week 5 - Partial Fractions and Numerical Integration

Partial Fraction Decomposition

We turn our attention now to integrals of the form

$$\int \frac{P(x)}{Q(x)}dx$$

Where P, Q are polynomials in x. If P has a larger degree than Q, then we can use long division to simplify P/Q. If P has a smaller or equal degree to Q, then we use partial fractions to simplify.

We know that when we have two fractions and want to add them, that we find a common denominator and then combine. For example,

$$\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$$

Partial fractions is this exact same process, except in reverse, such as

$$\frac{2}{x^2-1}=\frac{2}{(x-1)(x+1)}=\frac{(x+1)-(x-1)}{(x-1)(x+1)}=\frac{1}{x-1}-\frac{1}{x+1}$$

Using this for integration, we can find that

$$\int \frac{2}{x^2-1}dx=\int \frac{1}{x-1}dx-\int \frac{1}{x+1}dx=\ln(x-1)-\ln(x+1)+c$$

In general, we have the following theorem

Theorem. *(Partial Fraction Decomposition) :* Suppose that P, Q are polynomials with $Q(x) = q_1^{k_1}(x)q_2^{k_2}(x)\dots q_n^{k_n}(x)$ where q_j are irreducible polynomials of degree d_j and $\deg(P) \le n$. Then there exists polynomials $p_{1,1},\dots,p_{1,k_1},p_{2,1},\dots,p_{2,k_2},\dots,p_{n,k_n}$ such that $p_{i,j}$ is of degree at most d_i and

$$\frac{P(x)}{Q(x)}=\frac{p_{1,1}(x)}{q_1(x)}+\frac{p_{1,2}(x)}{q_1^2(x)}+\dots+\frac{p_{1,k_1}(x)}{q_1^{k_1}(x)}+\dots+\frac{p_{n,k_n}(x)}{q_n^{k_n}(x)}=\sum_{i=1}^{n}\sum_{j=1}^{k_i}\frac{p_{i,j}(x)}{q_i^j(x)}$$

Intuitively, we need to have a term for each power of each factor on the denominator, and each numerator is the degree of the polynomial in the denominator, ignoring the outside power. Understanding this theorem in full generality is difficult, so it's best to do a good number of practice problems to master it.

Exercise 54. Find the partial fraction decomposition of the following rational functions :

(a) $\int \frac{dx}{(x-2)(x-4)}$

(b) $\int \frac{dx}{(x-1)^2(x-2)^2}$

(c) $\int \frac{4x^2-20}{(2x+5)^3}$

(d) $\int \frac{6x^2+2}{x^2+2x-3}dx$

(e) $\int \frac{10dx}{(x-1)^2(x^2+9)}$

Numerical Integration

When we first defined integration, we defined it by using skinny rectangles to approximate the area under the curve. So over an interval $[a,b]$, we took a partition $a = x_0 < x_1 < \cdots < x_n = b$ of $[a,b]$ and chose a point c_i from $[x_{i-1}, x_i]$ for $i = 1,\ldots,n$. Then we approximated $\int_a^b f(x)dx$ as

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$$

In practice, we have full control over both the partition $x_0,\ldots,x_n$ and the points chosen $c_1,\ldots,c_n$. For the purposes of this class, we will only work with evenly spaced partitions, so $x_i - x_{i-1} = \Delta x$ regardless of i. We have a few different systematic ways to choose the point c_i, but we focus now on the midpoint rule.

The midpoint rule

For each interval $[x_{i-1}, x_i]$, we choose c_i to be the midpoint, $c_i = \frac{1}{2}(x_i + x_{i-1})$. Since we want evenly spaced intervals, for any n we let $x_0 = a, x_1 = a + (b-a)/n, x_2 = a + 2(b-a)/n, \ldots, x_n = b$. With this method, we have the error bound

$$\left| \int_a^b f(x)dx - \underbrace{\frac{b-a}{n}\big(f(c_1) + \cdots + f(c_n)\big)}_{M_n} \right| \leq \frac{(b-a)^3}{24n^2} \max_{x\in[a,b]} |f''(x)|$$

Trapezoidal Rule

Instead of using rectangles to approximate our area, we can be more accurate by using trapezoids. For each interval $[x_{i-1}, x_i]$, we can make a trapezoid with vertices $x_{i-1}, x_i, f(x_i), f(x_{i-1})$. This trapezoid will then have area

$$A_i = (x_i - x_{i-1})\frac{f(x_i) + f(x_{i-1})}{2} = \frac{(b-a)(f(x_i) + f(x_{i-1}))}{2n}$$

We can then approximate $\int_a^b f(x)dx$ with these n trapezoids, with which we get the error bounds

$$\left| \int_a^b f(x)dx - \underbrace{\frac{b-a}{2n}\big(f(a) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(b)\big)}_{T_n} \right| \leq \frac{(b-a)^2}{12n^2} \max_{x\in[a,b]} |f''(x)|$$

As presented here, this trapezoid rule does not fit into the c_i method that we gave above. However, if f is continuous then the intermediate value theorem tells us that there exists some c_i in $[x_{i-1}, x_i]$ such that $f(c_i) = \frac{1}{2}\big(f(x_i) + f(x_{i-1})\big)$, so this isn't too far of a departure.

Exercise 55. Calculate T_n, M_n for the value of n indicated

1. $\int_0^{\pi/2} \sqrt{\sin(x)}dx$, $n = 6$
2. $\int_1^2 \ln(x)dx$, $n = 5$
3. $\int_0^1 e^{-x^2}dx$, $n = 5$

Week 6 - Arc Length, Surface Area, and Improper Integrals

This is best explained via picture, but a valiant effort is made here.

One application of these new integration techniques is computing arc length. Suppose we have a curve parameterized by $(x, f(x))$ for $a \leq x \leq b$. To compute the length of this curve, we break it down into small steps, Δx, which induce small jumps in our function $\Delta f(x)$. Adding up all of these small segments, we find that our arclength is approximately given by

$$L \approx \sum \sqrt{(\Delta x)^2 + \left(\Delta f(x)\right)^2}$$

where the sum is taken over all the segments of the curve. To get equality here, we take our segments to be infinitely small and then we take infinitely many segments. This causes $\Delta x \to dx$ and $\Delta f \to df$ in an intuitive, but informal sense. Additionally, this transforms our sum into an integral, which gives us our equation for arc length,

$$L = \int_a^b \sqrt{dx^2 + df^2} = \int_a^b \sqrt{1 + \left(\frac{df}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

An extension of this concept is to compute the surface area of a curve rotated around the x axis. Following the same reasoning as above, we can approximate the surface area of this rotated curve by

$$L \approx \sum 2\pi f(x) \sqrt{(\Delta x)^2 + \left(\Delta f(x)\right)^2}$$

with the $2\pi f(x)$ following since the circumference of a circle is $2\pi r$. Then by the same limiting argument,

$$SA = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2}\, dx$$

Exercise 56. Find the arc length of $y = \frac{1}{12}x^3 + x^{-1}$ for $1 \leq x \leq 2$.

Exercise 57. Show that a spherical cap of height h and radius R has a surface area of $2\pi Rh$. *(height is taken from the top of the sphere down)*

Improper Integrals

When we defined Riemann integration, there were a few flaws inherent in the definition.

(i) **Infinite region :** If the region of integration is infinite, such as $\int_0^\infty f(x)dx$ or $\int_{-\infty}^5 f(x)dx$.

(ii) **Infinite function :** If the function f has an infinite discontinuity, such as the discontinuity of $1/x$ at 0 or $\ln(x)$ at 0.

In these cases, Riemann integration fails to converge. However, that doesn't mean that the notion of "area under the curve", which motivated the definition of Riemann integration, is ill-defined. Often, we can still define integration in these situations, we just need to be more clever in how we use Riemann integration. In these cases, we call these improper Riemann integrals.

To evaluate an improper Riemann integral, we use a limit to cut out the problem area. If we have region that extends to infinity, then we instead extend it to some large number R and then take a limit. For example,

$$\int_1^\infty x^{-2}dx = \lim_{R\to\infty}\int_1^R x^{-2}dx = \lim_{R\to\infty} 1-\frac{1}{R} = 1$$

Similarly, if we have an infinity discontinuity, then we use limits to cut out the discontinuity before continuing. For example,

$$\int_0^2 x^{-1/2}dx = \lim_{\varepsilon\to 0}\int_\varepsilon^2 x^{-1/2}dx = \lim_{\varepsilon\to 0} 2\sqrt{2}-2\sqrt{\varepsilon} = 2\sqrt{2}$$

When multiple issues arise, we will need to use multiple limits to cut out all of the issues. With these methods, we find two useful theorems.

Theorem. *(Comparison Test) :* Suppose that f, g satisfy

$$0 \le f(x) \le g(x)$$

on a region (a, b) where a, b might be $\pm\infty$. Then

$$\int_a^b f(x)dx \text{ converges} \implies \int_a^b g(x)dx \text{ converges}$$
$$\int_a^b g(x)dx \text{ diverges} \implies \int_a^b f(x)dx \text{ diverges}$$

Theorem. *(p-test):* For all real numbers a,

$$\int_a^\infty x^{-p}dx$$

converges if and only if $p > 1$.
Similarly,

$$\int_0^a x^{-p}dx$$

converges if and only if $p < 1$.

Exercise 58. Compute the following integrals or show that they do not exist.

(a) $\int_2^\infty x^{-3}dx$

(b) $\int_{-\infty}^0 x^2e^x dx$

(c) For which values of a does $\int_0^\infty e^{ax}dx$ converge?

(d) $\int_{-\infty}^{\infty} e^{-|x|^2} dx$

Exercise 59. State whether the following integrals converge or diverge

1. $\int_{-\infty}^{\infty} e^{-|x|^2} dx$
2. $\int_1^{\infty} (x^4+1)^{1/2} dx$
3. $\int_1^{\infty} (x^3+4)^{-1} dx$
4. $\int_0^1 \frac{dx}{x(2x+5)}$

Week 7 - Sequences and Series

A sequence is an infinity list of numbers $a_1, a_2, a_3, a_4, \ldots$, often denoted (a_n). Some classic examples are

$$\begin{array}{lll} & 1,2,3,4,5,6,7,\ldots & \text{rule}: a_n = n \\ \text{(Fibonacci)} & 0,1,1,2,3,5,8,\ldots & \text{rule}: a_n = a_{n-1} + a_{n-2} \\ & -1,1,-1,1,-1,1,-1,\ldots & \text{rule}: a_n = (-1)^n \end{array}$$

We define the limit of a sequence in the same way that we defined the limit of a function. Namely,

Limit of a Sequence
A sequence $a_1, a_2, a_3, \ldots$ converges to some limit L if for all $\varepsilon > 0$ there exists an $N \geq 1$ such that if $n \geq N$ then

$$|a_n - L| < \varepsilon$$

As before, we don't concern ourselves much with the precise definition and will not use it often.

Very often, it will be easier to work with a function rather than a sequence. With functions, we have the ability to use L'Hôpital's rule and other continuity tools that do not exist with sequences. For this, we have the following theorem

Theorem. If $\lim_{x\to\infty} f(x)$ exists then the sequence $a_n = f(n)$ converges and

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$$

Similarly, we may use the continuity of functions to evaluate limits.

Theorem. If f is continuous at $\lim_{n\to\infty} a_n = L$, then

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L)$$

We have a few specific examples of sequences that are easier to work with than others, namely

Special Sequences
A sequence is **bounded from above** if there exists M such that $a_n \leq M$ for all n.
A sequence is **bounded from below** if there exists m such that $a_n \geq m$ for all n.
A sequence is **increasing** if $a_{n+1} > a_n$ for all n.
A sequence is **decreasing** if $a_{n+1} < a_n$ for all n.
A sequence is **non-increasing** if $a_{n+1} \leq a_n$ for all n.
A sequence is **non-decreasing** if $a_{n+1} \geq a_n$ for all n.
A sequence is **monotonic** if it is increasing or decreasing.

With these, we have the following limit laws

Theorem. If (a_n) is non-decreasing (includes increasing) and bounded above by M, then $\lim_{n\to\infty} a_n \leq M$ exists.
If (a_n) is non-increasing (includes decreasing) and bounded from below by m, then $\lim_{n\to\infty} a_n \geq m$ exists.

Exercise 60. Determine the limits of the following sequences, if they exist

(i)
$$a_n = \sqrt{4 + \frac{1}{n}}$$

(ii)
$$a_n = e^{4n/(3n+9)}$$

(iii)
$$a_n = \tan^{-1}(e^{-n})$$

Series

Now that we have these long sequences of numbers, the next natural question is what can we do with them? Namely, can we add up all the numbers? For this question, we look at infinite series. Given a sequence (a_n), we look at the infinite sum
$$\sum_{n=1}^{\infty} a_n$$
and determine whether this sum converges or diverges, and in the case of convergence, what it converges to.

To view this series formally, we need to introduce a limit. To do this, we write
$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n = \lim_{N\to\infty} S_N$$
where S_N is the N^{th} partial sum.

With this notation, we say that an infinite series $\sum_{n=1}^{\infty} a_n$ converges if and only if the partial sums (S_n) converge as a sequence. With this definition, just as with sequences, we have all the usual properties of limits. An immediate consequence of this definition is the following theorem

Theorem. (Divergence Test) : If $\sum_{n=1}^{\infty} a_n$ exists then $\lim_{n\to\infty} a_n = 0$.

Some particularly special examples of these series are as follows

Telescoping Series

A series is telescoping if it is of the form

$$\sum_{n=1}^{\infty}(a_{n+1}-a_n) = \lim_{N\to\infty} a_{N+1}-a_N+a_N-a_{N-1}+\cdots-a_1 = \lim_{n\to\infty} a_n - a_1$$

Geometric Series

A series is geometric if it is of the form

$$\sum_{n=1}^{\infty} cr^n$$

It converges if and only if $|r|<1$.

Geometric series will become very important when we discuss Taylor series.

Exercise 61. Evaluate the following series or show that they do not exist

(a)
$$\sum_{n=1}^{\infty}\frac{1}{n(n+3)}$$

(b)
$$\sum_{n=1}^{\infty}(-1)^n\frac{n-1}{n}$$

(c)
$$\sum_{n=1}^{\infty}\frac{8+2^n}{5^n}$$

Week 8 - Positive Series

We now consider specifically positive series. That is, series of the form $\sum_{n=1}^{\infty} a_n$ where $a_n > 0$. In this case, the question of whether a series converges is simpler because the partial sums S_N are increasing. This means that either the partial sums S_N converge to some limit S or they diverge to infinity.

We recall that an increasing sequence will converge if and only if it is bounded above. Therefore, a positive series $\sum_{n=1}^{\infty} a_n$ will converge if and only if its partial sums S_N are bounded from above.

However, showing that these partial sums are bounded from above is not always easy. So to help with this, we develop a few tools.

Theorem. (Integral Test): Let f be a positive, continuous, decreasing function on $[1,\infty)$ and let $a_n = f(n)$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges.

An immediate consequence of this is

Theorem. (p-series) : $\sum_{n=1}^{\infty} n^{-p}$ converges if and only if $p > 1$.

Similar to improper integrals, we also have a comparison test for series, which states

Theorem. (Comparison Test) : Suppose there exists two sequences a_n, b_n such that $0 \le a_n \le b_n$ for all $n \ge M$. If $\sum b_n$ converges then $\sum a_n$ converges and if $\sum a_n$ diverges then $\sum b_n$ diverges. (The purpose of M here is to specify that we only need the bound $a_n \le b_n$ for large n, since we can always deal with small n anyways.)

Exercise 62. Use the integral test on the following series

(a)
$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{3/2}}$$

(b)
$$\sum_{n=1}^{\infty} ne^{-n^2}$$

(c)
$$\sum_{n=1}^{\infty} \frac{1}{n(n+5)}$$

Exercise 63. Use the comparison test on the following series

(a)
$$\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+1}}$$

(b)

$$\sum_{n=3}^{\infty} \frac{3n+5}{n(n-1)(n-2)}$$

(c)

$$\sum_{n=1}^{\infty} \frac{e^n+n}{e^{2n}-n^2}$$

Week 9 - Series Convergence

We develop a number of theorems to help with determining whether series converge.

Theorem. *(Limit Comparison Test) :* Suppose that $a_n, b_n > 0$ and that $L = \lim_{n\to\infty} \frac{a_n}{b_n}$. Then

(i) If $L > 0$ then $\sum a_n$ converges if and only if $\sum b_n$ converges.

(ii) If $L = \infty$ then $\sum a_n$ converging implies that $\sum b_n$ converges.

(iii) If $L = 0$ then $\sum a_n$ divering implies that $\sum b_n$ diverges.

Absolute Convergence
The series $\sum a_n$ converges absolutely if $\sum |a_n|$ converges.

Theorem. If $\sum |a_n|$ converges then $\sum a_n$ converges.

Theorem. *(Alternating Series Test) :* Suppose that $b_n > 0$ decreases to 0. Then $\sum (-1)^n b_n$ converges if $\lim_{n\to\infty} b_n = 0$.

Conditional Convergence
A series $\sum a_n$ converges conditionally if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Theorem. *(Ratio Test):* Suppose that

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

(i) If $\rho < 1$ then $\sum a_n$ converges absolutely.

(ii) If $\rho > 1$ then $\sum a_n$ diverges.

(iii) $\rho = 1$ then the test is inconclusive.

Theorem. *(Root Test):* Suppose that

$$L = \lim_{n\to\infty} |a_n|^{1/n}$$

(i) If $L < 1$ then $\sum a_n$ converges absolutely.

(ii) If $L > 1$ then $\sum a_n$ diverges.

(iii) If $L = 1$ then the test is inconclusive.

Exercise 64. Determine whether the following series converge absolutely, conditionally, or not at all.

(a)
$$\sum_{n=1}^{\infty} \frac{\sin(\pi n/4)}{n^2}$$

(b)
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{1+1/n}$$

(c)
$$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}$$

(d)
$$\sum_{n=1}^{\infty} (-1)^n n^2 e^{-n^3/3}$$

Exercise 65. Use the ratio test on the following series.

(a)
$$\sum_{n=1}^{\infty} \frac{1}{5^n}$$

(b)
$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

(c)
$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}\ln n}$$

(d)
$$\sum_{n=1}^{\infty} \frac{1}{\ln n}$$

Exercise 66. Use the root test on the following series.

(a)
$$\sum_{n=0}^{\infty} \frac{1}{10^n}$$

(b)
$$\sum_{n=1}^{\infty} \frac{1}{n^n}$$

(c)
$$\sum_{n=0}^{\infty} \left(\frac{n}{3n+1}\right)^n$$

(d)

$$\sum_{n=1}^{\infty} \left(2 + \frac{1}{n}\right)^n$$

Week 10 - Taylor Series and Power Series

Back in Math 31A, we found that we could approximate differentiable functions by the tangent line as

$$f(x) \approx f(a) + f'(a)(x-a)$$

for any real number a. If we allow ourselves to take this approximation further, using all of the derivatives instead of just the first, then we can transform our approximation into an equality. Doing so, we end up with a power series (infinite polynomial), called the Taylor series, given by

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

Now the existence of a Taylor series relies on our function being infinitely differentiable and sufficiently nice. Regardless, if we are only finitely differentiable, then this gives us an approximation to our function if we truncate this infinite series to only the first $n+1$ terms. Doing so, we get the n^{th} Taylor polynomial, which we denote by

$$T_n(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

As usual with an approximation of this form, we have an associated error bound.

Theorem. Supposed that $f^{(n+1)}$ is $(n+1)$-times continuously differentiable and fix some real numbers a, x. Let K be a number such that

$$\max_{\substack{t \in [a,x] \\ \text{or } t \in [x,a]}} |f^{(n+1)}(t)| \leq K$$

so K is an upper bound for the $(n+1)^{st}$ derivative of f. Then

$$|f(x) - T_n(x)| \leq K \frac{|x-a|^{n+1}}{(n+1)!}$$

Note that if we center the Taylor series at $a = 0$ then we call it a **Maclaurin series**.

Some common example of Taylor series that would be good to know are

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \ldots$$

$$\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \ldots$$

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \ldots$$

$$\ln(x) = (x-1) - \frac{1}{2}(x-1)^2 + \cdots + \frac{(-1)^{n-1}}{n}(x-1)^n + \ldots$$
$$\frac{1}{1-x} = 1 + x + x^2 + \ldots$$

Exercise 67. Calculate T_3 of the given function at the given center

(a) $f(x) = \tan x$, $a = 0$

(b) $f(x) = e^{-x} + e^{-2x}$, $a = 0$

(c) $f(x) = x^2 e^{-x}$, $a = 1$

(d) $f(x) = \ln(x)/x$, $a = 1$

Exercise 68. Calculate the Taylor series of the given function centered at the given point

(a) $f(x) = \frac{1}{1+x}$, $a = 0$

(b) $f(x) = e^x$, $a = 1$

(c) $f(x) = \sin 3\theta$, $a = 0$

(d) $f(x) = x^{-2}$, $a = 1$

Exercise 69. Let T_n be the Taylor polynomial for $f(x) = \ln x$ at $a = 1$ and let $c > 1$. Show that

$$|\ln c - T_n(c)| \leq \frac{|c-1|^{n+1}}{n+1}$$

Power Series

In the previous section, we started with a function and found a power series that represented it. In this section, we start with a power series and determine whether it defines a function.

Power Series
A power series with center a is an infinite series

$$F(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

Whether or not a power series converges depends on the coefficients c_n and the distance $|x-a|$. Given this, we have three possibilities, based on the radius of convergence

Radius of Convergence
Given a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, the radius of convergence is a value $0 \le R \le \infty$ such that

(i) If $R = 0$ then $\sum_{n=0}^{\infty} c_n(x-a)^n$ only converges at $x = a$.

(ii) If $R = \infty$ then $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges absolutely for all x.

(iii) If $0 < R < \infty$ then $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges absolutely for $|x-a| < R$, diverges for $|x-a| > R$ and might converge or diverge at $|x-a| = R$.

Given this, we define a similar notion

Interval of Convergence
The interval of convergence of $\sum_{n=0}^{\infty} c_n(x-a)^n$ is the interval containing all values of x such that the series converges at x.

To calculate the radius of convergence, we use the ratio test or the root test, as presented earlier.

Once we understand where a power series converges, the next natural question is to figure out what we can say about it as a function. Since this is a calculus course, we always care about limits and integrals. To this end, we find the following theorem

Theorem. *(Differentiation and Integration Term-by-Term):* Suppose that $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R > 0$. Then $F(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ is differentiable on $(a-R, a+R)$ and for any $x \in (a-R, a+R)$,

$$F'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

$$\int F(x)dx = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1} + c$$

Exercise 70. Find the interval of convergence

(a)
$$\sum_{n=8}^{\infty} n^7 x^n$$

(b)
$$\sum_{n=0}^{\infty} \frac{8^n}{n!} x^n$$

(c)
$$\sum_{n=1}^{\infty} (-1)^n n^5 (x-7)^n$$

(d)

$$\sum_{n=12}^{\infty} e^n (x-2)^n$$

(e)

$$\sum_{n=1}^{\infty} \frac{x^{3n+2}}{\ln n}$$