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**Week 1 - Limits and Continuity**

**Limits**

The limit of a function is what the function approaches as approaches some value .

More technically, we say that the limit of as approaches is equal to if can be made arbitrarily small by taking sufficiently close (but not equal) to . This is denoted by

\[
\lim_{x \to c} f(x) = L
\]

Without any other machinery, there are two main ways to investigate limits.

**Graphically:** Plot the function . As gets close to , what does get close to?

**Numerically:** Make a table of for values where is small and of values where is small. If both of these tables approach the same value, then the limit exists.

As defined, the limit is a **two-sided** idea. It matters not only what does for but also what does for . We can investigate these ideas separately using the left and right handed limits, which is the same as the limit but restricted to and respectively. The limit exists if and only if the left and right handed limits exist and are equal.


Basic Limit Laws

**Theorem.** If \( \lim_{x \to c} f(x) \) and \( \lim_{x \to c} g(x) \) exist, then

1. **Sum Law:** \( \lim_{x \to c} [f(x) + g(x)] \) exists and
   \[
   \lim_{x \to c} [f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)
   \]
2. **Constant Multiple Law:** For any \( k \in \mathbb{R} \), \( \lim_{x \to c} kf(x) \) exists and
   \[
   \lim_{x \to c} kf(x) = k \left( \lim_{x \to c} f(x) \right)
   \]
3. **Product Law:** \( \lim_{x \to c} f(x)g(x) \) exists and
   \[
   \lim_{x \to c} f(x)g(x) = \left( \lim_{x \to c} f(x) \right) \left( \lim_{x \to c} g(x) \right)
   \]
4. **Quotient Law:** If \( \lim_{x \to c} g(x) \neq 0 \) then \( \lim_{x \to c} \frac{f(x)}{g(x)} \) exists and
   \[
   \lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}
   \]
5. **Powers and Roots:** If \( p, q \) are integers with \( q \neq 0 \) then \( \lim_{x \to c} [f(x)]^{p/q} \) exists and
   \[
   \lim_{x \to c} [f(x)]^{p/q} = \left( \lim_{x \to c} f(x) \right)^{p/q}
   \]
   Provided that \( \lim_{x \to c} f(x) \geq 0 \) if \( q \) is even and that \( \lim_{x \to c} f(x) \neq 0 \) if \( p/q < 0 \).

   Note that this is easily applied to \( [f(x)]^n \) and \( [f(x)]^{1/n} \) for integer \( n \).

Continuity

**Continuous at a Point**

Assume that \( f(x) \) is defined on an open interval containing \( x = c \). Then \( f \) is **continuous** at \( x = c \) if

\[
\lim_{x \to c} f(x) = f(c)
\]

If the limit does not exist or is not equal to \( f(c) \), then we say that \( f(x) \) is **discontinuous** at \( x = c \).

So to show that a function \( f(x) \) is continuous at \( x = c \), we must show that

1. \( f(c) \) is defined
2. \( \lim_{x \to c} f(x) \) is defined
3. \( \lim_{x \to c} f(x) = f(c) \)
Indeterminate Forms  

If a function is continuous at all points in its domain, we simply say that it is **continuous**.

**Theorem. Laws of Continuity**: Sums, products, multiples, inverses and composites of continuous functions are continuous. The same holds for a quotient \( f(x)/g(x) \) at points where \( g(x) \neq 0 \).

Intuitively, we can think of continuous functions as having a graph that doesn't have any disconnections. That is, we can think of continuous functions as having graphs that could be drawn without your pencil leaving the paper.

### Indeterminate Forms

**Indeterminate Form**

If the formula for \( f(c) \) yields an undefined expression of the form

\[
\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \infty \cdot 0, \quad \text{or} \quad \infty - \infty
\]

Then we say that \( f(x) \) has an **indeterminate form** at \( x = c \) or that \( f(x) \) is **indeterminate** at \( x = c \).

Note that if \( f(x) \) is indeterminate at \( x = c \) then this does not imply that \( \lim_{x \to c} f(x) \) does not exist. Rather, it only implies that this limit is more difficult to find than by simply checking \( f(c) \).

If \( f(x) \) is indeterminate at \( x = c \), then we currently have two main methods to determine \( \lim_{x \to c} f(x) \)

1. (Best option for now.) Algebraically transform \( f(x) \) into a new expression that is defined and continuous at \( x = c \) and then evaluate by plugging \( c \) in. This is usually accomplished by factoring the top and bottom and then cancelling. More complicated, but equally useful, is multiplying the top and bottom by a conjugate of either the top or bottom.

2. (Worst option for now.) Examine the function graphically or numerically.

In the future, when we have derivatives and other machinery, we can evaluate these in more elegant ways.

### Practice Problems

**Exercise 1.** Use the basic limit laws to complete the following.

(a) Find

\[ \lim_{t \to 1} \frac{t^2 + 1}{(t^3 + 2)(t^4 + 1)} \]

(b) Can you apply the quotient rule to

\[ \frac{\sin(x)}{x} \]

Why or why not?
Exercise 2. Give an example where \( \lim_{x \to \infty} (f(x) + g(x)) \) exists but neither \( \lim_{x \to 0} f(x) \) nor \( \lim_{x \to 0} g(x) \) exist.

Exercise 3. Draw the graph of a function \( f : [0,5] \to \mathbb{R} \) such that \( f \) is right but not left continuous at \( x = 1 \), left but not right continuous at \( x = 2 \) and neither right nor left continuous at \( x = 3 \).

Exercise 4. Challenge: Give an example of functions \( f, g \) such that \( f(g(x)) \) is continuous but \( g \) has at least one discontinuity.

Exercise 5. Challenge: Show that the following function is only continuous at \( x = 0 \),

\[
 f(x) = \begin{cases} 
 x, & \text{x is rational} \\
 -x, & \text{x is irrational} 
\end{cases}
\]
Squeeze Theorem

**Theorem. (Squeeze Theorem):** If there exists an open interval $I$ containing $c$ such that for all $x \in I \setminus \{c\}$ ($I$ excluding $c$)

$$\ell(x) \leq f(x) \leq u(x) \quad \text{and} \quad \lim_{x \to c} \ell(x) = \lim_{x \to c} u(x) = L$$

Then $\lim_{x \to c} f(x)$ exists and $\lim_{x \to c} f(x) = L$.

**Intuition:** Near, but not necessarily at, $c$, $f$ is bounded above by $u(x)$ and below by $\ell(x)$. So the limit of $f(x)$ must be between the limits of $\ell(x)$ and $u(x)$.

Significant Trigonometric Limits

Using the squeeze test and some cleverness, we can show that

**Theorem.**

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

**Theorem.**

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x} = 0$$

Limits at Infinity

Intuitively, limits at infinity behave exactly the same as limits at any other point. We say that $\lim_{x \to \infty} f(x) = L$ if we can make $|f(x) - L|$ arbitrarily small if we choose $x$ sufficiently large.

For exponents, we know that following rules,

**Theorem.** For all $n > 0$,

$$\lim_{x \to \infty} x^n = \infty \quad \text{and} \quad \lim_{x \to \infty} x^{-n} = 0$$

If $n$ is a positive whole number,

$$\lim_{x \to -\infty} x^n = \begin{cases} \infty, & \text{if } n \text{ is even} \\ -\infty, & \text{if } n \text{ is odd} \end{cases} \quad \text{and} \quad \lim_{x \to -\infty} x^{-n} = 0$$
Theorem. (Limits of a Rational Function): The asymptotic behavior of a rational function depends only on the leading terms of its numerator and denominator. If \( a_n, b_m \neq 0 \) then

\[
\lim_{x \to \pm \infty} \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0} = \frac{a_n}{b_m} \lim_{x \to \pm \infty} x^{n-m}
\]
Exercise 6. Draw the graph of a function \( f : [0,5] \to \mathbb{R} \) such that \( f \) is right but not left continuous at \( x = 1 \), left but not right continuous at \( x = 2 \) and neither right nor left continuous at \( x = 3 \).

Exercise 7. Evaluate the following limits using the identity

\[ a^3 - b^3 = (a - b)(a^2 + ab + b^2) \]

(a) \( \lim_{x \to 1} \frac{x^2 - 5x + 4}{x^3 - 1} \)

(b) \( \lim_{x \to 1} \frac{x^4 - 1}{x^3 - 1} \)

Exercise 8. Using the squeeze theorem, evaluate the following limit

\[ \lim_{x \to 0} \tan \left( x \cos \left( \sin \left( \frac{1}{x} \right) \right) \right) \]

Exercise 9. Do the following inequalities provide enough information to determine \( \lim_{x \to 1} f(x) \) by the squeeze theorem?

(a) \( 4x - 5 \leq f(x) \leq x^2 \)

(b) \( 2x - 1 \leq f(x) \leq x^2 \)

(c) \( 4x - x^2 \leq f(x) \leq x^2 + 2 \)

Exercise 10. (Challenging) : Let \( a_n, b_m \neq 0 \). What are all possible values of the following limit and what conditions cause each limit?

\[ \lim_{x \to \infty} \frac{a_n x^n + \cdots + a_0}{b_m x^m + \cdots + b_0} \]

Exercise 11. (Challenging) : Intuitively, explain why the limit at infinity of a rational function is what it is.
Week 3 - Intermediate Value Theorem and Derivatives

Intermediate Value Theorem

**Theorem. (Intermediate Value Theorem)**: If $f$ is continuous on a closed interval $[a, b]$, then for every $M$, strictly between $f(a)$ and $f(b)$, there exists at least one value $c \in (a, b)$ such that $f(c) = M$.

This can be used to show that functions attain certain values somewhere. Perhaps most usefully, it can be shown that functions have zeros within different locations. Specifically,

**Theorem. (Existence of Zeros)**: If $f$ is continuous on $[a, b]$ and one of $f(a)$ or $f(b)$ is negative and the other is positive (equivalently if $f(a)f(b)$ is negative) then $f$ has a zero in $[a, b]$.

This is useful when faced with problems like "Show that $f(x) = g(x)$ for some $x."$ because you can consider the function $f(x) - g(x)$ and then show that it has a zero somewhere.

The Derivative

**The Derivative**

The derivative of $f$ at a point $a$, if it exists, is the limit

$$f'(a) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

When the derivative exists, we say that $f$ is **differentiable** at $a$.

By definition, this implies that if $f$ is differentiable at $a$, then $f'(a)$ is approximately $(f(x + h) - f(x))/h$ for small $h$. That is

**Theorem.** Let $f$ be differentiable at $a$. For small $h$,

$$f'(a) \approx \frac{f(x + h) - f(x)}{h}$$

Using the derivative, we can find a tangent line to the graph of a function at any point. This is given by

**Tangent Line**

Let $f$ be differentiable at $a$. Then the line tangent to the graph of $y = f(x)$ at $P = (a, f(a))$ is the line through $P$ with slope $f'(a)$. The equation of this line is

$$y = f'(a)(x - a) + f(a)$$
The Derivative as a Function

Though the prime notation used above is popular, there is another notation that is just as popular, if not more so. This is called Leibniz notation.

**Leibniz Notation**

Leibniz notation replaces the prime symbol with \( \frac{d}{dx} \) instead. It is popular because it emphasizes how the derivative is found through a ratio of the change in \( f \) to the change in \( x \). Additionally, it makes it clear which variable you are differentiating with respect to, in the case that there are multiple. It is specifically written as

\[
f'(x) = \frac{df}{dx}
\]

**The Derivative as a Function**

In the previous section, we defined what it means for \( f \) to be differentiable at a point \( a \). We can now expand that into something much broader.

**Differentiable**

If the derivative of \( f \) exists for all \( x \in (a, b) \), then we say that \( f \) is **differentiable on** \( (a, b) \). Similarly, if the derivative of \( f \) exists for all real numbers \( x \) (or if the derivative of \( f \) exists on the whole domain of \( f \) ), then we say that \( f \) is **differentiable**.

**Derivative Rules**

There are many rules for derivatives that can make calculating them very quick. The most important ones for now are

**Theorem. (Constant Rule)**: The derivative of a constant is 0. Specifically,

\[
\frac{d}{dx} c = 0
\]

**Theorem. (Power Rule)**: For any exponent \( n \),

\[
\frac{d}{dx} x^n = nx^{n-1}
\]

**Theorem. (Linearity Rules)**: The derivative of the sum of two functions is the sum of the derivatives. Specifically,

\[
\frac{d}{dx} (f + g) = \frac{df}{dx} + \frac{dg}{dx}
\]
The derivative of a constant times a function is the constant times the derivative of the function. Specifically, for a function $f$ and a constant $c$,

$$\frac{d}{dx}(cf) = c\frac{df}{dx}$$
Practice Problems

**WEEK 3 - INTERMEDIATE VALUE THEOREM AND DERIVATIVES**

Differentiability, Continuity, and Local Linearity

From the definition of the derivative, we find two nice properties

**Theorem. (Differentiability Implies Continuity):** If \( f \) is differentiable at \( x = c \) then \( f \) is continuous at \( x = c \).

**Theorem. (Local Linearity):** If \( f \) is differentiable at \( x = c \) then in small neighborhoods of \( c \), \( f \) is approximately equivalent to the tangent line at \( x = c \).

### Practice Problems

**Exercise 12.** *(Optional)* Using the intermediate value theorem, show that \( \sqrt{2} \) exists.

**Exercise 13.** Using the intermediate value theorem, show that

\[
\cos(x) = \tan(x)
\]

has a solution.

**Exercise 14.** Using the limit definition of the derivative, calculate the derivative of

\[
f(x) = x^3 + 2x
\]

**Exercise 15.** Given

\[
f(x) = x - 2x^2
\]

Use the limit definition to compute \( f'(3) \) and find an equation of the tangent line.

**Exercise 16.** Sketch a graph of

\[
f(x) = x^{2/5}
\]

and identify the points where \( f'(c) \) does not exist.

**Exercise 17.** Calculate the derivative of

\[
g(x) = \frac{x^2 + 4x^{1/2}}{x^2}
\]
Week 4 - Midterm Review

Practice Problems

Exercise 18. Calculate the following limits,

(1) \[ \lim_{x \to 1} \left( \frac{1}{1 - x} - \frac{2}{1 - x^2} \right) \]

(2) \[ \lim_{x \to 1} \left( \frac{1}{\sqrt{x} - 2} - \frac{4}{x - 4} \right) \]

(3) \[ \lim_{\theta \to \pi/2} (\sec \theta - \tan \theta) \]

Exercise 19. Identify points of discontinuity of the following functions, state why they are discontinuities, and give what type of discontinuity.

(1) \[ f(x) = \begin{cases} x + 1, & x < 1 \\ 1/x, & x \geq 1 \end{cases} \]

(2) \[ f(x) = \begin{cases} \frac{x^2 - 3x + 2}{|x - 2|}, & x \neq 2 \\ 0, & x = 2 \end{cases} \]

Exercise 20. Use the IVT to show that the following have solutions

(1) \[ 2^x + 3^x = 4^x \]

(2) \[ \sqrt{x} + \sqrt{x} + 2 = 3 \]

(3) \[ \cos(x) = \tan(2x) \quad \text{on} \ (0,1) \]

Exercise 21. Use the limit definition of the derivative to calculate \( f'(x) \) when

(1) \[ f(x) = x^3 + 2x \]
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<tr>
<td>(2)</td>
<td>$f(x) = \sqrt{x + 4}$ on $x &gt; -4$</td>
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<tr>
<td>(3)</td>
<td>$f(x) = \frac{1}{1 - x}$ on $x \neq 1$</td>
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Week 5 - Post Midterm Review and Derivatives

Practice Problems

**Exercise 22.** Use the product rule to calculate the derivative of

(1) \( f(x) = (3x - 5)(2x^2 - 3) \)

(2) \( f(x) = (t - 8t^{-1})(t + t^2) \)

Find the tangent line at \( t = 1 \).

**Exercise 23.** Use the quotient rule to calculate the derivative of

(1) \( f(x) = \frac{x + 4}{x^2 + x + 1} \)

(2) \( f(x) = \frac{z^2}{\sqrt{z} + z} \)

**Exercise 24.** Use the chain rule to calculate the derivative of

(1) \( f(x) = \cos(x^3) \)

(2) \( f(x) = \sqrt{9 + x + \sin x} \)

**Exercise 25.** Calculate the following limits,

(1) \( \lim_{\theta \to 0} \frac{\sin \theta - \sin \theta \cos \theta}{\theta} \)

(2) \( \lim_{x \to 4} \frac{3x - 12}{\sqrt{x} - 2} \)

**Exercise 26.** Let \( f(x) = 7xe^x^2 \).

(1) Calculate the derivative \( f'(x) \).
(2) Give the equation for the tangent line of \( f \) at \( x = 0 \).

**Exercise 27.** Calculate the following limit

\[
\lim_{x \to \infty} \frac{12x^4 + 3x^2 + 4}{22x^4 + 15}
\]

**Exercise 28.** Let \( f \) be defined by

\[
f(x) = \begin{cases} 
\sin \frac{x}{x}, & x \neq 0 \\
1, & x = 0
\end{cases}
\]

(1) Show that \( f \) is continuous at \( x = 0 \).

(2) Show that \( f \) is differentiable at 0 and \( f'(0) = 1 \) by using the limit definition of the derivative.

*Hint:*

\[
\lim_{\theta \to 0} \frac{\sin \theta - \theta}{\theta^2} = 0
\]
Week 6 - Implicit Differentiation and Extrema

Implicit Differentiation

Thus far, we have developed formulas for when we have $y$ explicitly written as a function of $x$. However, what if $y$ is related to $x$ by an equation like the following?

$$y^3 + \frac{1}{xy} = 16 - 9x^2 y$$

In this case, we can differentiate both sides of the equation and then gather all terms of $\frac{dy}{dx}$ onto one side and solve for them.

In the case of the equation above,

$$\frac{d}{dx} \left( y^3 + \frac{1}{xy} \right) = \frac{d}{dx} (16 - 9x^2 y)$$

$$3y^2 \frac{dy}{dx} + \frac{-1}{xy^2} \frac{dy}{dx} + \frac{-1}{x^2 y} = -18xy - 9x^3 \frac{dy}{dx}$$

$$3y^2 \frac{dy}{dx} + \frac{-1}{xy^2} \frac{dy}{dx} + 9x^2 \frac{dy}{dx} = \frac{1}{x^2 y} - 18xy$$

$$\frac{dy}{dx} = \frac{\frac{1}{x^2 y} - 18xy}{3y^2 + \frac{1}{xy^2} + 9x^3}$$

$$\frac{dy}{dx} = \frac{y - 18x^3 y^3}{3y^4 x^2 - x + 9x^5 y^2}$$

For implicit differentiation, it is important to go slowly and be careful about when you use various derivative rules. The chain rule will be especially helpful in these types of problems.

Extreme Values

Extreme Values on an Interval

Let $f$ be a function on an interval $I$ and let there exist $a \in I$. We say that $f(a)$ is

- **Absolute minimum** of $f$ if $f(a) \leq f(x)$ for all $x \in I$
- **Absolute maximum** of $f$ if $f(a) \geq f(x)$ for all $x \in I$

**Theorem. (Existence of Extrema on a Closed Interval)**: A continuous function $f$ on a closed and bounded interval takes on both a minimum and a maximum value on $I$. 
Local Extrema and Critical Points

Local Extrema
We say that \( f(c) \) is a

- **Local minimum** occurring at \( x = c \) if \( f(c) \) is the minimum value of \( f \) on some open interval containing \( c \)

- **Local maximum** occurring at \( x = c \) if \( f(c) \) is the maximum value of \( f \) on some open interval containing \( c \)

It seems intuitively easy to find the absolute minimum and maximum of a function, but how do we find the local extrema? We can actually use the derivative to easily find the points that can be local extrema through the use of critical points

Critical Points
A number \( c \) in the domain of \( f \) is called a **critical point** if either \( f'(c) = 0 \) or \( f'(c) \) does not exist.

Which benefit us through the following theorem

**Theorem. (Fermat’s Theorem on Local Extrema)**: If \( f(c) \) is a local minimum or maximum, then \( c \) is a critical point of \( f \).

Optimizing on a Closed Interval

**Theorem. (Extreme Values on a Closed Interval)**: Let \( f \) be continuous on \([a, b]\) and let \( f(c) \) be the minimum or maximum value on \([a, b]\). Then \( c \) is either a critical point or one of the endpoints, \( a \) or \( b \).

First Derivative Test

Given a critical point, we can determine the nature of \( f \) at that point through the first derivative test.

**Theorem. (First Derivative Test)**: Let \( c \) be a critical point of \( f \). Then

- \( f'(x) \) changes from + to – implies that \( f(c) \) is a local maximum
- \( f'(x) \) changes from – to + implies that \( f(c) \) is a local minimum
The Mean Value Theorem

The mean value theorem is analogous to the intermediate value theorem. Based on the values of \( f \) on the endpoints of an interval, we can determine the existence of points in the interval with a specific derivative.

Before the general mean value theorem, we have a specific case

**Theorem. (Rolle's Theorem):** Assume that \( f \) is continuous on \([a, b]\) and differentiable on \((a, b)\). If \( f(a) = f(b) \) then there exists number \( c \) between \( a \) and \( b \) such that \( f'(c) = 0 \).

This generalizes to

**Theorem. (Mean Value Theorem):** Assume that \( f \) is continuous on \([a, b]\) and differentiable on \((a, b)\). Then there exists some \( c \) in \((a, b)\) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}
\]

Practice Problems

**Exercise 29.** Find \( \frac{dy}{dx} \) in terms of \( x \) and \( y \) given the following equation

(1) \[
\frac{y}{x} + \frac{x}{y} = 2y
\]

(2) \[
tan(x^2 y) = (x + y)^3
\]

**Exercise 30.** Find the extreme values of \( f(x) \) on the given interval.

(1) \[
f(z) = z^5 - 80z, \quad [-3, 3]
\]

(2) \[
f(y) = \sqrt{x + x^2} - 2\sqrt{x}, \quad [0, 4]
\]

**Exercise 31.** Find all critical points the following functions and use the first derivative test to determine whether they are local maxima or local minima

(1) \[
y = \frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x + 4
\]
(2) \[ y = \frac{2x + 1}{x^2 + 1} \]
Week 7 - MVT, Second Derivatives, and L'Hopital's Rule

The Mean Value Theorem

The mean value theorem is analogous to the intermediate value theorem. Based on the values of $f$ on the endpoints of an interval, we can determine the existence of points in the interval with a specific derivative.

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**Theorem. (Rolle's Theorem):** Assume that $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$. If $f(a) = f(b)$ then there exists number $c$ between $a$ and $b$ such that $f'(c) = 0$.

This generalizes to

**Theorem. (Mean Value Theorem):** Assume that $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Then there exists some $c$ in $(a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Monotonicity

The derivative helps us to classify whether a function is increasing or decreasing on any given interval. In particular,

**Theorem.**

- $f'(x) > 0$ for $x \in (a, b)$ implies that $f$ is increasing on $(a, b)$
- $f'(x) < 0$ for $x \in (a, b)$ implies that $f$ is decreasing on $(a, b)$

Between adjacent critical points, the derivative of a function cannot be 0. So we know that the function must be increasing or decreasing on these intervals.

Second Derivative

We have seen that the derivative can be used to see how the function behaves on given intervals, whether it is increasing or decreasing, maximum or minimum, etc. The second derivative can be used in much the same ways, except now it tells use how the first derivative behaves, which then tells us how the function behaves. So the process now has two steps.

In helping with this discussion, we define concave up and concave down as follows
Concavity
Let \( f \) be a differentiable function on an open interval \((a, b)\). Then

- \( f \) is **concave up** on \((a, b)\) if \( f' \) is increasing on \((a, b)\)
- \( f \) is **concave down** on \((a, b)\) if \( f' \) is decreasing on \((a, b)\)

In the same manner that the first derivative was used to determine if \( f \) was increasing or decreasing, we can use the second derivative to determine if \( f' \) is increasing or decreasing. Therefore we arrive at the following test for concavity

**Theorem. (Test for Concavity):** Assume that \( f''(x) \) exists for all \( x \in (a, b) \). Then

- If \( f''(x) > 0 \) for all \( x \in (a, b) \) then \( f \) is concave up on \((a, b)\)
- If \( f''(x) < 0 \) for all \( x \in (a, b) \) then \( f \) is concave down on \((a, b)\)

Just like how we cared about the points where the derivative was zero, we also care about the points where the second derivative is zero.

**Point of Inflection**
A point \( c \) is a **point of inflection** of \( f \) if the concavity changes from up to down at \( x = c \).

We can test for points of inflection in the exact same way that we tested for critical points, by looking at the second derivative and seeing where it is 0 or undefined.

**Theorem. (Test for Inflection Points):** If \( f''(c) = 0 \) or \( f''(c) \) does not exists and \( f''(x) \) changes sign at \( x = c \), then \( f \) has a point of inflection at \( x = c \).

**Second Derivative Test**

When we did the first derivative test to determine if a point was a local maximum, minimum, or neither, we checked whether the derivative went from positive to negative or vice versa. Now, we can look at this in terms of the second derivative. If the derivative goes from positive to negative at a point, then that means that the second derivative (if it exists) must be negative at that point because the derivative is decreasing. If the derivative goes from negative to positive at a point, then that means that the second derivative (if it exists) must be positive at that point because the derivative is increasing. This leads us to the following test to determine the nature of critical points.

**Theorem. (Second Derivative Test for Critical Points):** Let \( c \) be a critical point of \( f \). If \( f''(c) \) exists, then
• \( f''(c) > 0 \) implies that \( f(c) \) is a local maximum
• \( f''(c) < 0 \) implies that \( f(c) \) is a local minimum
• \( f''(c) = 0 \) implies nothing. \( f(c) \) may be a local max, local min, or neither

**L'Hôpital's Rule**

L'Hôpital's Rules is a valuable tool for computing certain limits, especially the limits of quotients when the quotient rule cannot be applied. It is stated as follows

**Theorem. (L'Hôpital's Rule) :** Suppose that \( f \) and \( g \) are differentiable on an open interval containing \( a \) and that \( f(a) = g(a) = 0 \). Also assume that \( g'(x) \neq 0 \) except possibly at \( a \). Then

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}
\]

if the limit on the right exists or is infinite. This conclusion also holds if \( f \) and \( g \) are differentiable for \( x \) near (but not equal to) \( a \) and

\[
\lim_{x \to a} f(x) = \pm \infty \quad \lim_{x \to a} g(x) = \pm \infty
\]

Furthermore, this rule is valid for one-sided limits.

Note that if you do L'Hôpital's rule once and still find an indeterminate form, then you may do it again until you don't.

"All review materials and problems on this document are taken from Jon Rogawski's Single Variable Calculus, 4th edition. Special thanks to him for these materials."

**Practice Problems**

**Exercise 32.** Suppose the \( f(0) = 2 \) and \( f'(x) \leq 3 \) for \( x > 0 \). Apply the MVT to the interval \([0,4]\) to show that \( f(4) \leq 14 \).
In addition, show that \( f(x) \leq 2 + 3x \) for all \( x > 0 \).

**Exercise 33.** Determine the intervals on which the function is concave up or concave down and find the points of inflection.

(i) \( y = (x - 2)(1 - x^3) \)
(ii) \( y = \theta - 2\sin \theta \) on \([0,2\pi]\)
(iii) \( y = \frac{x^4 - 1}{x} \)
**Exercise 34.** Find all critical points and apply the second derivative test.

(i) \( f(x) = x^5 - x^3 \)

(ii) \( f(x) = 3x^4 - 8x^2 + 6x^2 \)

(iii) \( f(x) = \frac{1}{x^3 - x + 2} \)

(iv) \( y = \sin^2 x + \cos x \)

**Exercise 35.** Evaluate the following limits, using L'Hôpital's Rule where it applies

(i) \( \lim_{x \to 4} \frac{x^3 - 64}{x^2 + 16} \)

(ii) \( \lim_{x \to 0} \frac{\sin 4x}{x^2 + 3x + 1} \)

(iii) \( \lim_{x \to 0} \frac{\sin 2x}{\sin 7x} \)

(iv) \( \lim_{x \to 0} (\cos x)^{3/x^2} \)

*Hint:* \( a^x = e^{\log(a) x} \) and \( \lim_{x \to a} e^{f(x)} = e^{\lim_{x \to a} f(x)} \) because \( e^x \) is continuous.

**Exercise 36.** *(Challenge)* Show that if \( f(0) = g(0) \) and \( f'(x) \leq g'(x) \) for \( x \geq 0 \), then \( f(x) \leq g(x) \) for all \( x \geq 0 \). *Hint:* consider the function \( h(x) = g(x) - f(x) \) and show that it is non-decreasing.

The following problem will most likely not be included on homeworks or exams, but it is cool, so I thought I would include it.

**Exercise 37.** *(Challenge)* Assume that \( f'' \) exists and \( f''(x) = 0 \) for all \( x \). Show that \( f(x) = mx + b \) where \( m = f'(0) \) and \( b = f(0) \).
Week 8 - Second Midterm Review and Newton’s Method

Newton’s Method

Newton’s Method is a way of using the derivative of a function $f(x)$ to numerically find (or approximate) the roots $f(x) = 0$. It does this by starting with a guess point $x_0$, assuming that $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$ and then using this to determine a next best guess $x_1$. Iterating this process, we get better and better guesses $x_0, x_1, x_2, \ldots$. More definitively, the process is defined as

**Theorem. Newton’s Method:** To approximate a root of $f(x) = 0$,

1. Choose an initial guess $x_0$ (close to the desired root if possible)
2. Generate successive approximations $x_1, x_2, \ldots$ where

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

As a good rule of thumb, if $x_n$ and $x_{n+1}$ agree to $m$ decimal places, then you can usually safely assume that $x_n$ agrees with a root to $m$ decimal places.

Review Problems

**Exercise 38.** (Mean Value Theorem): Show that there exists some $c$ in the given interval such that $f'(c)$ satisfies the relationship described.

(i) $f(x) = \sqrt{x}$ \quad [9, 25] \quad f'(c) = \frac{1}{8}$
(ii) $f(x) = x - \sin \pi x$ \quad [-1, 1] \quad f'(c) = 1
(iii) $f(x) = (x - 1)(x - 3)$ \quad [1, 3] \quad f'(c) = 0

For more practice, you can find a specific $c$ satisfying the relationship.

**Exercise 39.** (Implicit Differentiation): Give the equation for the tangent line of the given curve at the given point.

(i) $xy + x^2 y^2 = 6$ at the point $(2, 1)$
(ii) $x^2 + \sin y = xy^2 + 1$ at the point $(1, 0)$
(iii) $2^{x^{1/2}} + 4^{y^{-1/2}} = xy$ at the point $(1, 4)$
(iv) $\sin(2x - y) = \frac{x^2}{y}$ at the point $(0, \pi)$
Exercise 40. (Extrema) : Find all local extrema of the following functions

(i) \( f(x) = \frac{1}{\sin x + 4} \)
(ii) \( f(x) = 9x^{7/3} - 21x^{1/2} \)
(iii) \( f(x) = 3x^4 - 6x^3 + 6x^2 \)
(iv) \( f(x) = \sin(x) \cos(x) \) (potential challenge)

Exercise 41. (Concavity) : Determine the intervals on which the function is concave up or down and find the points of inflection.

(i) \( y = 10x^3 - x^5 \)
(ii) \( y = (x - 2)(1 - x^3) \)
(iii) \( y = x^{7/2} - 35x^2 \)
(iv) \( f(x) = \frac{x^3}{1 + x} \)
(v) \( f(x) = \tan(x) \) (potential challenge)
Week 9 - Integrals

Definite Integral

Given a function $f$ on an interval $[a, b]$, a surprisingly insightful question to ask is "What is the area between the graph of $f$ and the x-axis?" We can determine this area through the definite integral. But that question alone leaves some aspects of the integral ambiguous. Such as "What about when $f(x)$ is negative?" and "How do you calculate that area when $f(x)$ isn't nice?" With these questions in mind, we can go and work up to define the definite integral.

Let $f : [a, b] \rightarrow \mathbb{C}$ be a continuous function. In order to find the area under the curve of $f$, we approximate the area by a series of rectangles and then let these rectangles become really small. In determining these rectangles, we need a few concepts.

### Partition $P$ of Size $N$

A choice of points that divides $[a, b]$ into $N$ subintervals (not necessarily of equal width). This is normally denoted as

$$P : a = x_0 < x_1 < \cdots < x_N = b$$

### Sample Points

Given a partition $P$, we define the set of sample points $C = \{c_1, \ldots, c_N\}$ such that $c_i$ is an element of the interval $[x_{i-1}, x_i]$ for all $i = 1, \ldots, N$. It doesn't matter which points are chosen, as long as they are within the specific interval.

### Length of Subinterval

Given a partition $P$ of $[a, b]$, the length of a subinterval $[x_{i-1}, x_i]$ is

$$\Delta x_i = x_i - x_{i-1}$$

### Norm of Partition

Given a partition $P$ of $[a, b]$, the norm of $P$, denoted $\|P\|$, is the maximum length of the subintervals. That is,

$$\|P\| = \max_i \Delta x_i$$

We will briefly connect these definitions back to the rectangles. A partition $P$ gives breaks the interval $[a, b]$ down into pieces that will become the bases of our rectangles. The sample points $C$, give test points for the value of $f$ on each of these intervals. We will take the height of each rectangle to be $f(c_i)$. Note here that $f(c_i)$ could be negative, at which point our rectangle has negative area. The lengths of the intervals gives us the length of the base of our rectangles. Finally, the norm of a partition gives one number to how "big" our partition is. By forcing the norm to become really small, we force our partition to become really fine and our rectangles to become really small.
With these, we define the Riemann sum of $f$ over $P$ as

**Riemann Sum**

The Riemann Sum of $f$ over the partition $P$ with sample points $C$ is given by

$$R(f, P, C) = \sum_{i=1}^{N} f(c_i)\Delta x_i = f(c_1)\Delta x_1 + \cdots + f(c_N)\Delta x_N$$

Finally, to define the Riemann integral, we let the rectangles get really small. Equivalently, we let the norm of the Partition go to 0. Defining this explicitly,

**Definite Integral**

The definite integral of $f$ over $[a, b]$ is the limit of Riemann sums as $\|P\| \to 0$. It is defined as

$$\int_{a}^{b} f(x) \, dx = \lim_{\|P\| \to 0} \sum_{i=1}^{N} f(c_i)\Delta x_i$$

When this limit exists and is finite, we say that $f$ is integrable over $[a, b]$.

This leads us to our first theorem

**Theorem. (Almost Continuous Functions are Riemann Integrable)**: If $f$ is continuous on $[a, b]$, or if $f$ is continuous except at finitely many jump discontinuities, then $f$ is integrable over $[a, b]$.

**Properties of the Definite Integral**

Given the definition earlier, we can show many properties of the definite integral. These will not be shown here, as the proofs are long and not particularly insightful. However, you should note that all of the following theorems make intuitive sense when the integral is thought of as the area under the curve.

**Theorem. (Linearity of the Integral)**: If $f$ and $g$ are integrable over $[a, b]$ and $c$ is a real constant, then $f + g$ and $cf$ are integrable and

$$\int_{a}^{b} (f(x) + g(x)) \, dx = \int_{a}^{b} f(x) \, dx + \int_{a}^{b} g(x) \, dx$$

$$\int_{a}^{b} cf(x) \, dx = c \int_{a}^{b} f(x) \, dx$$
Reversing the Limits of Integration

Let $f$ be a function. Then for $a < b$, we define

$$
\int_a^b f(x) \, dx = -\int_b^a f(x) \, dx
$$

Theorem. (Additivity for Adjacent Integrals): Let $a \leq b \leq c$ and let $f$ be integrable. Then

$$
\int_a^c f(x) \, dx = \int_a^b f(x) \, dx + \int_b^c f(x) \, dx
$$

Theorem. (Comparison Theorem): If $f, g$ are integrable over $[a, b]$ and $f(x) \leq g(x)$ for all $x$ in $[a, b]$ then

$$
\int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx
$$

Indefinite Integral

The indefinite integral is difficult to motivate without defining it first and finding out what properties it has. Instead, we will arrive at it in a roundabout way by asking "What happens if we take a derivative but backwards?"

Antiderivative

A function $F$ is an antiderivative of $f$ on an open interval $(a, b)$ if $F'(x) = f(x)$ for all $x$ in $(a, b)$.

Now, unlike the derivative, the antiderivative is not necessarily unique. This comes from the fact that the derivative of a constant is 0, and so adding a constant to a function doesn’t change its derivative. This leads to the following theorem

Theorem. (The General Antiderivative): Let $y = F(x)$ be an antiderivative of $y = f(x)$ on $(a, b)$. Then every antiderivative of $f$ on $(a, b)$ is of the form $y = F(x) + c$ for some constant $c$.

From this, we define the indefinite integral. The connection between this and the definite integral will be seen soon, but for now you will have to trust that they connect.

Indefinite Integral

The notation

$$
\int f(x) \, dx = F(x) + c
$$

means that $F'(x) = f(x)$

We say that $F(x) + c$ is the general antiderivative or indefinite integral of $y = f(x)$. 

The indefinite integral has many of the same properties as the definite integral. Namely,

**Theorem. (Linearity of the Indefinite Integral):**
\[
\int (f(x) + g(x)) \, dx = \int f(x) \, dx + \int g(x) \, dx \\
\int c \cdot f(x) \, dx = c \int f(x) \, dx
\]

**Important Indefinite Integrals**

Using the power rule for derivatives and the standard trig function derivatives, we can derive a couple of important indefinite integrals. In particular,

**Theorem. (Power Rule for Indefinite Integral):**
\[
\int x^n \, dx = \frac{x^{n+1}}{n+1} + c
\]

**Theorem. (Basic Trigonometric Integrals):**
\[
\int \sin x \, dx = -\cos x + c \\
\int \cos x \, dx = \sin x + c \\
\int \sec^2 x \, dx = \tan x + c \\
\int \csc^2 x \, dx = -\cot x + c \\
\int \sec x \tan x \, dx = \sec x + c \\
\int \csc x \cot x \, dx = -\csc x + c
\]

**The Fundamental Theorem of Calculus**

Now that we’ve defined the indefinite integral, it’s time to show why it is such an important concept. That importance, and the connection between the indefinite and definite integrals, comes from the fundamental theorem of calculus. There are two parts to the fundamental theorem of calculus, both of which we give here.

**Theorem. (The First Fundamental Theorem of Calculus):** Let \( a < b \) and let \( f \) be continuous on \((a, b)\). If \( F \) is an antiderivative of \( f \) on \([a, b]\) then
\[
\int_a^b f(x) \, dx = F(b) - F(a) = F(x)\bigg|_{x=a}^b
\]
Theorem. (The Second Fundamental Theorem of Calculus): Let \( f \) be continuous on an open interval \( I \) and let \( a \) be in \( I \). Then the area function

\[
A(x) = \int_a^x f(t) \, dt
\]

is an antiderivative of \( f \) on \( I \). So

\[
\frac{d}{dx} \int_a^x f(t) \, dt = f(x)
\]

The first fundamental theorem of calculus tells us that taking the indefinite integral of a function (finding the antiderivative) gives us a way to compute any integral of the function quickly and easily. The second fundamental theorem of calculus tells us that any continuous function has an antiderivative and that the integral is, intuitively, the inverse of the derivative, and vice versa.

With these tools, we can calculate lots of integrals quickly and easily!

Practice Problems

Exercise 42. Calculate the integral of the following functions on the following intervals

(1)

\[
f(x) = \begin{cases} 
10, & x \text{ in } [0, 0.5) \\
25, & x \text{ in } [0.5, 1.5) \\
15, & x \text{ in } [1.5, 2) \\
20, & x \text{ in } [2, 3]
\end{cases}
\]

on \([0, 3]\)

(2)

\[
f(x) = \begin{cases} 
x, & x \text{ in } [0, 1) \\
2 - x, & x \text{ in } [1, 2) \\
-3, & x \text{ in } [2, 3]
\end{cases}
\]

on \([0, 3]\)

Exercise 43. Express the following in one integral

(1)

\[
\int_0^3 f(x) \, dx + \int_3^7 f(x) \, dx
\]

(2)

\[
\int_2^9 f(x) \, dx - \int_4^9 f(x) \, dx
\]

(3)

\[
\int_7^3 f(x) \, dx + \int_3^9 f(x) \, dx
\]
Exercise 44. Challenge: Let $f$ be an odd function. Show that

$$\int_{-a}^{a} f(x) \, dx = 0$$

Exercise 45. Find the indefinite integral of $f$ and check your answer by differentiating

1. $f(x) = 18x^2$
2. $f(x) = x^{-3/5}$
3. $f(x) = 2\cos x - 9\sin x$
4. $f(x) = 4x^7 - 3\cos x$
WEEK 10 - FTC AND CHANGE OF VARIABLES

Week 10 - FTC and Change of Variables

The Fundamental Theorem of Calculus

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**Theorem. (The First Fundamental Theorem of Calculus):** Let \( a < b \) and let \( f \) be continuous on \((a, b)\). If \( F \) is an antiderivative of \( f \) on \([a, b]\) then

\[
\int_a^b f(x) \, dx = F(b) - F(a) = F(x) \bigg|_{x=a}^b
\]

**Theorem. (The Second Fundamental Theorem of Calculus):** Let \( f \) be continuous on an open interval \( I \) and let \( a \) be in \( I \). Then the area function

\[
A(x) = \int_a^x f(t) \, dt
\]

is an antiderivative of \( f \) on \( I \). So

\[
\frac{d}{dx} \int_a^x f(t) \, dt = f(x)
\]

The first fundamental theorem of calculus tells us that taking the indefinite integral of a function (finding the antiderivative) gives us a way to compute any integral of the function quickly and easily. The second fundamental theorem of calculus tells us that any continuous function has an antiderivative and that the integral is, intuitively, the inverse of the derivative, and vice versa.

With these tools, we can calculate lots of integrals quickly and easily!

The Substitution Method

The substitution method is to integrals what the chain rule is to derivatives. It is a way of composing and uncomposing functions to make integrals easier. Intuitively, it can be viewed as doing the chain rule in reverse, which is reflected in how it looks.

**Theorem. (The Substitution Method):** If \( F'(x) = f(x) \) and \( u \) is a differentiable function whose range includes the domain of \( f \), then

\[
\int f(u(x))u'(x) \, dx = F(u(x)) + C
\]

Unfortunately, as it stands, it is a difficult theorem to apply. It is very useful, but it is constraining in its
requirements and form. Thankfully, we can generalize it to the change of variable formula as follows

**Theorem. (Change of Variables Formula):**

\[ \int_{f(u)}^{f(u)} f(u(x)) u'(x) \, dx = \int_{u(a)}^{u(b)} f(u) \, du \]

Finally, we can take this change of variables and apply it to the definite integral case, which is done as follows

**Theorem. (Change of Variables for Definite Integral):** If \( u' \) is continuous on \([a, b]\) and \( f \) is continuous on the range of \( u \), then

\[ \int_{a}^{b} f(u(x)) u'(x) \, dx = \int_{u(a)}^{u(b)} f(u) \, du \]

**Practice Problems**

**Exercise 46.** Evaluate the following integrals

(a) \[ \int_{0}^{2} (12x^5 + 3x^2 - 4x) \, dx \]

(b) \[ \int_{1/16}^{1} t^{1/4} \, dt \]

(c) \[ \int_{0}^{5} |x^2 - 4x + 3| \, dx \]

(d) \[ \int_{0}^{\pi} |\cos x| \, dx \]

(e) \[ \int_{-2}^{3} f(x) \, dx \] where \( f(x) = \begin{cases} 12 - x^2, & \text{for } x \leq 2 \\ x^3, & \text{for } x > 2 \end{cases} \]

**Exercise 47.** Find explicit formulas for the functions represented by the following integrals

(a) \[ f(x) = \int_{0}^{x} \sin u \, du \]
(b) \[ k(x) = \int_{1}^{x^2} t \, dt \]

(c) \[ h(x) = \int_{x/2}^{x/4} \sec^2 u \, du \]

(d) \[ g(x) = \int_{2}^{\sqrt{x}} \frac{dt}{t^2} \]
Exercise 48. Write out the following integrals in terms of $u$ and $du$. Then evaluate

(a) \[ \int t \sqrt{t^2 + 1} \, dt, \quad u = t^2 + 1 \]

(b) \[ \int \frac{t^3}{(4 - 2t^4)^{11}} \, dt, \quad u = 4 - 2t^4 \]

(c) \[ \int \sqrt{4x - 1} \, dx, \quad u = 4x - 1 \]

Exercise 49. Evaluate the indefinite integral

(a) \[ \int \frac{x}{\sqrt{x^2 + 9}} \, dx \]

(b) \[ \int x^2 \sqrt{x^3 + 1} \, dx \]

(c) \[ \int x(3x + 8)^{11} \, dx \]

(d) \[ \int x^3 (x^2 - 1)^{3/2} \, dx \]

(e) \[ \int \frac{\sin x \cos x}{\sqrt{\sin x + 1}} \, dx \]