Group Exercises

Exercise 1. Find $dy/dx$ in terms of $x$ and $y$ given the following equation

(1) \[ \frac{y}{x} + \frac{x}{y} = 2y \]

(2) \[ \tan(x^2y) = (x + y)^3 \]

Exercise 2. Find the extreme values of $f(x)$ on the given interval.

(1) \[ f(z) = z^5 - 80z, \quad [-3, 3] \]

(2) \[ f(y) = \sqrt{x + x^2} - 2\sqrt{x}, \quad [0, 4] \]

Exercise 3. Find all critical points the following functions and use the first derivative test to determine whether they are local maxima or local minima

(1) \[ y = \frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x + 4 \]

(2) \[ y = \frac{2x + 1}{x^2 + 1} \]
Implicit Differentiation

Thus far, we have developed formulas for when we have \( y \) explicitly written as a function of \( x \). However, what if \( y \) is related to \( x \) by an equation like the following?

\[
y^3 + \frac{1}{xy} = 16 - 9x^2y
\]

In this case, we can differentiate both sides of the equation and then gather all terms of \( \frac{dy}{dx} \) onto one side and solve for them.

In the case of the equation above,

\[
\frac{d}{dx} \left( y^3 + \frac{1}{xy} \right) = \frac{d}{dx} (16 - 9x^2y)
\]

\[
3y^2 \frac{dy}{dx} + \frac{-1}{xy^2} \frac{dy}{dx} + \frac{-1}{x^2y} = -18xy - 9x^3 \frac{dy}{dx}
\]

\[
3y^2 \frac{dy}{dx} + \frac{-1}{xy^2} \frac{dy}{dx} + 9x^3 \frac{dy}{dx} = \frac{1}{x^2y} - 18xy
\]

\[
\frac{dy}{dx} = \frac{\frac{1}{x^2y} - 18xy}{3y^2 + \frac{1}{xy^2} + 9x^3}
\]

\[
\frac{dy}{dx} = \frac{y - 18x^3y^3}{3y^4x^2 - x + 9x^5y^2}
\]

For implicit differentiation, it is important to go slowly and be careful about when you use various derivative rules. The chain rule will be especially helpful in these types of problems.

Extreme Values

**Extreme Values on an Interval**

Let \( f \) be a function on an interval \( I \) and let there exist \( a \in I \). We say that \( f(a) \) is

- **Absolute minimum** if \( f(a) \leq f(x) \) for all \( x \in I \)
- **Absolute maximum** if \( f(a) \geq f(x) \) for all \( x \in I \)

**Theorem. (Existence of Extrema on a Closed Interval)**: A continuous function \( f \) on a closed and bounded interval takes on both a minimum and a maximum value on \( I \).

Local Extrema and Critical Points

**Local Extrema**

We say that \( f(c) \) is a

- **Local minimum** occurring at \( x = c \) if \( f(c) \) is the minimum value of \( f \) on some open interval containing \( c \)
- **Local maximum** occurring at \( x = c \) if \( f(c) \) is the maximum value of \( f \) on some open interval containing \( c \)
Optimizing on a Closed Interval

It seems intuitively easy to find the absolute minimum and maximum of a function, but how do we find the local extrema? We can actually use the derivative to easily find the points that can be local extrema through the use of critical points.

**Critical Points**

A number \( c \) in the domain of \( f \) is called a **critical point** if either \( f'(c) = 0 \) or \( f'(c) \) does not exist.

Which benefit us through the following theorem:

**Theorem. (Fermat’s Theorem on Local Extrema)**: If \( f(c) \) is a local minimum or maximum, then \( c \) is a critical point of \( f \).

**Optimizing on a Closed Interval**

**Theorem. (Extreme Values on a Closed Interval)**: Let \( f \) be continuous on \([a, b]\) and let \( f(c) \) be the minimum or maximum value on \([a, b]\). Then \( c \) is either a critical point or one of the endpoints, \( a \) or \( b \).

**First Derivative Test**

Given a critical point, we can determine the nature of \( f \) at that point through the first derivative test.

**Theorem. (First Derivative Test)**: Let \( c \) be a critical point of \( f \). Then

- \( f'(x) \) changes from \(+\) to \(−\) implies that \( f(c) \) is a local maximum
- \( f'(x) \) changes from \(−\) to \(+\) implies that \( f(c) \) is a local minimum

**The Mean Value Theorem**

The mean value theorem is analogous to the intermediate value theorem. Based on the values of \( f \) on the endpoints of an interval, we can determine the existence of points in the interval with a specific derivative.

Before the general mean value theorem, we have a specific case:

**Theorem. (Rolle’s Theorem)**: Assume that \( f \) is continuous on \([a, b]\) and differentiable on \((a, b)\). If \( f(a) = f(b) \) then there exists number \( c \) between \( a \) and \( b \) such that \( f'(c) = 0 \).

This generalizes to:

**Theorem. (Mean Value Theorem)**: Assume that \( f \) is continuous on \([a, b]\) and differentiable on \((a, b)\). Then there exists some \( c \) in \((a, b)\) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}
\]