Math 31A: Week 4 Discussion

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Practice Problems

Exercise 1. Calculate the following limits,

(1) \[ \lim_{x \to 1} \left( \frac{1}{1 - x} - \frac{2}{1 - x^2} \right) \]
(2) \[ \lim_{x \to 1} \left( \frac{1}{\sqrt{x} - 2} - \frac{4}{x - 4} \right) \]
(3) \[ \lim_{\theta \to \pi/2} (\sec \theta - \tan \theta) \]

Exercise 2. Identify points of discontinuity of the following functions, state why they are discontinuities, and give what type of discontinuity.

(1) \[ f(x) = \begin{cases} x + 1, & x < 1 \\ 1/x, & x \geq 1 \end{cases} \]
(2) \[ f(x) = \begin{cases} \frac{x^2 - 3x + 2}{|x-2|}, & x \neq 2 \\ 0, & x = 2 \end{cases} \]

Exercise 3. Use the IVT to show that the following have solutions

(1) \[ 2^x + 3^x = 4^x \]
(2) \[ \sqrt{x} + \sqrt{x + 2} = 3 \]
(3) \[ \cos(x) = \tan(2x) \quad \text{on } (0, 1) \]

Exercise 4. Use the limit definition of the derivative to calculate \( f'(x) \) when

(1) \[ f(x) = x^3 + 2x \]
(2) \[ f(x) = \sqrt{x + 4} \quad \text{on } x > -4 \]
(3) \[ f(x) = \frac{1}{1 - x} \quad \text{on } x \neq 1 \]
Exercise 5. *(Optional)* Using the intermediate value theorem, show that $\sqrt{2}$ exists.

Exercise 6. Using the intermediate value theorem, show that 

$$\cos(x) = \tan(x)$$

has a solution.

Exercise 7. Using the limit definition of the derivative, calculate the derivative of 

$$f(x) = x^3 + 2x$$

Exercise 8. Given 

$$f(x) = x - 2x^2$$

Use the limit definition to compute $f'(3)$ and find an equation of the tangent line.

Exercise 9. Sketch a graph of 

$$f(x) = x^{2/5}$$

and identify the points where $f'(c)$ does not exist.

Exercise 10. Calculate the derivative of 

$$g(x) = \frac{x^2 + 4x^{1/2}}{x^2}$$
Exercise 11. Evaluate the limit, if it exists. If not, determine whether the one-sided limits (finite or infinite.)

\[ \lim_{x \to 4} \frac{\sqrt{5 - x} - 1}{2 - \sqrt{x}} \]

Solution. From plugging in \( x = 4 \) into the expression, we get

\[ \frac{\sqrt{5 - 4} - 1}{2 - \sqrt{4}} = 0 \]

This is an indeterminate form, so we will attempt to simplify it before determining its identity. Multiplying by the conjugate of the numerator,

\[ \lim_{x \to 4} \frac{\sqrt{5 - x} - 1}{2 - \sqrt{x}} = \lim_{x \to 4} \frac{\sqrt{5 - x} - 1}{2 - \sqrt{x}} \left( \frac{\sqrt{5 - x} + 1}{\sqrt{5 - x} + 1} \right) \]

\[ = \lim_{x \to 4} \frac{5 - x - 1}{(2 - \sqrt{x})(\sqrt{5 - x} + 1)} \]

\[ = \lim_{x \to 4} \frac{4 - x}{(2 - \sqrt{x})(\sqrt{5 - x} + 1)} \]

Using the difference of squares formula, we can factor \( 4 - x = (2 - \sqrt{x})(2 + \sqrt{x}) \). Using this to simplify,

\[ \lim_{x \to 4} \frac{\sqrt{5 - x} - 1}{2 - \sqrt{x}} = \lim_{x \to 4} \frac{(2 - \sqrt{x})(2 + \sqrt{x})}{(2 - \sqrt{x})(\sqrt{5 - x} + 1)} \]

\[ = \lim_{x \to 4} \frac{2 + \sqrt{x}}{\sqrt{5 - x} + 1} \]

Because this expression is continuous at \( x = 4 \), we can plug in \( x = 4 \) to find the limit. Then,

\[ \lim_{x \to 4} \frac{\sqrt{5 - x} - 1}{2 - \sqrt{x}} = \frac{2 + \sqrt{4}}{\sqrt{5 - 4} + 1} = 2 \]