Integrable (Moser)

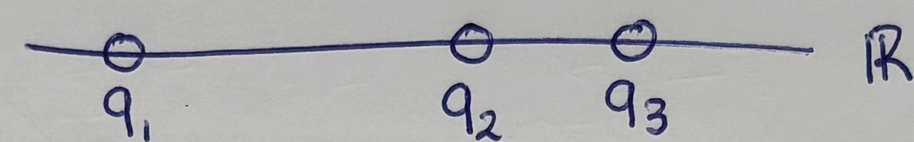
$$L_{jk} = p_j \delta_{jk} + \frac{i(1-\delta_{jk})}{q_j - q_k}$$

$$P_{jk} = -i\delta_{jk}\sum_{l\neq j}\frac{1}{(q_j-q_l)^2} + i\,\frac{1-\delta_{jk}}{(q_j-q_k)^2}$$

(q,p) solves CM $\iff \dot L = [P, L]$

$L = U L^0 U^{-1}$ w/ $\dot U = PU$, $U^0 = I$

Calogero-Moser Particles ($n<\infty$)



$$\left.\begin{aligned} &\dot q_j = p_j, \quad \dot p_j = \sum_{k\neq j}\frac{2}{(q_j-q_k)^3} \\ &H = \frac12\sum_j p_j^2 + \frac12\sum_{j\neq k}\frac{1}{(q_j-q_k)^2} \end{aligned}\right\} \text{CM}$$

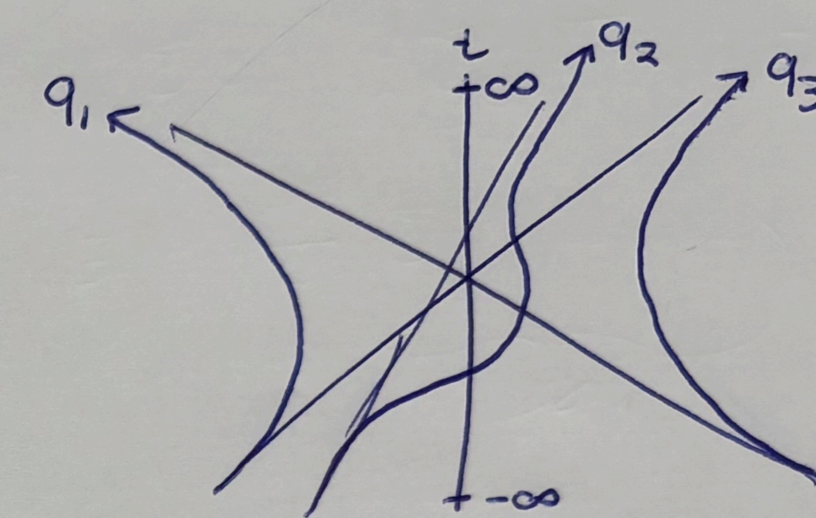
$$H = \frac12\langle L\vec1, L\vec1\rangle$$

1971 Calogero, quantum

1975 Moser, classical

Conservation Laws

eigenvals of L = $p_j^{\pm\infty}$ (distinct)



also, $\langle \vec1, L^k \vec1\rangle$ is conserved.

Explicit Formula (Olshanetsky – Perelomov)

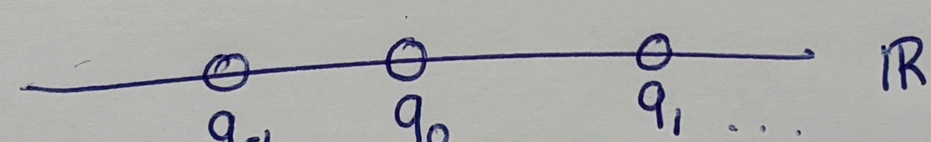
Define $Q_{jk} = \delta_{jk} q_j$.

Then

$$Q = U(Q^0 + tL^0)U^{-1}!$$

Particles are eigenvals of .

∞-many Particles (Killip-K-Vişan)



$$\dot q_j = p_j, \quad \dot p_j = \sum_{k\neq j}\frac{2}{(q_j-q_k)^3} \Big\} \text{CM}$$

~~Ingrimarson~~ Ingimarson – Pego, Wright 2023

CM approximates BO.

<u>always</u> $q_{j+1} > q_j$

LWP

Def. $\Omega^\mu = \{(q,p) : \|p\|_\infty < \mu, \ \inf_j q_{j+1}-q_j > 1/\mu\}$

<u>theorem</u>. Given $(q^0,p^0)\in\Omega^\mu$, $\exists!$ solution (q,p) to CM s.t.

$$q-q^0,\ p \in C_t^* \ell^\infty([-T,T]\times\mathbb{Z})$$

w/ $T \sim \frac{1}{1+\mu^4}$ and $(q,p)\in\Omega^{3\mu}$ $\forall t$.

<u>sketch</u>. contraction mapping

$q_{j+1}-q_j > 1/\mu \Rightarrow |q_j-q_k| > \frac{|j-k|}{\mu}$

so $|\dot p_j| \lesssim 1/\mu^3$.

~~Lax Op~~

Lax Operator

<u>theorem</u> $L : \ell^2 \to \ell^2$ bdd

$$\Updownarrow$$

$$(q,p) \in \Omega = \bigcup_{\mu > 0} \Omega^\mu.$$

Moreover $\|L\|_{op} = \mu \Rightarrow (q,p) \in \Omega^\mu$

and $(q,p) \in \Omega^\mu \Rightarrow \|L\|_{op} \leq 8\mu$.

<u>sketch</u> $\Downarrow$: immediate

$\Uparrow$: off-diagonals, $x_j = \frac{\lfloor q_j \mu \rfloor}{\mu} \in \mathbb{Z}/\mu$

$$\frac{i}{q_j - q_k} = \frac{i}{x_j - x_k} + \frac{i(x_j - x_k - q_j + q_k)}{(q_j - q_k)(x_j - x_k)} \quad \square$$

Integrability

<u>proposition</u> For (q,p) solving CM,

$L, P \in C_t B([-T,T])$,

$\dot{L} = [P, L]$,

$L = U L^\circ U^{-1}$ for $\dot{U} = PU$, $U^\circ = \mathrm{Id}$.

<u>sketch</u>:

$$\left.\begin{array}{l} L, P \in C_t B \\ \dot{L}_{jk} = [P,L]_{jk} \end{array}\right\} \Rightarrow \dot{L} = [P,L]$$

$\dot{U} = PU$, $U^\circ = \mathrm{Id}$ is LWP

$\partial_t U^{-1} L U = \ldots = 0$.

GWP

<u>theorem</u> Given $(q,p) \in \Omega$,

$\exists!$ solution (q,p) to CM

$q - q^\circ, p \in C_t \ell^\infty(\mathbb{R} \times \mathbb{Z})$

w/ $(q,p) \in \Omega^{\|L^\circ\|_{op}}$ $\forall t$.

Explicit Formula

Define $Q_{jk} = \delta_{jk} q_j$.

Then

$$\partial_t (U^{-1} Q U) = U^{-1} Q P U + U^{-1} \dot{Q} U - U^{-1} P Q U$$

$$= U^{-1}(\dot{Q} + [Q,P]) U$$

$$= U^{-1} L U$$

$$= L^\circ.$$

Therefore

$$Q = U(Q^\circ + t L^\circ) U^{-1}.$$

Details

<u>lemma</u> Let

$$D(Q) = \{ x \in \ell^2 : q_j x_j \in \ell^2_j \}.$$

Then $D(Q) = D(Q^\circ)$. and

$U, U^{-1}, P : D(Q^\circ) \to D(Q^\circ)$.

<u>sketch</u>

$$q_j x_j - q_j^\circ x_j = \underbrace{(q_j - q_j^\circ)}_{\ell^\infty} x_j \in \ell^2.$$

Conservation Laws

Motivation: for CM w/ $n<\infty$,

$\langle \vec{1}, L\vec{1} \rangle =$ total momentum

$\langle \vec{1}, L^2\vec{1} \rangle = 2 \cdot$ Hamiltonian.

How do we make sense of $L\vec{1}$?

Equilibrium

$$L_0 = i\pi H = \frac{i(1-\delta_{jk})}{j-k}$$

Then

$$(L-L_0)\vec{1} = p_j + i\sum_{k\neq j}\left(\frac{1}{q_j-q_k} - \frac{1}{j-k}\right)$$

$$= p_j + i\underbrace{\sum_{k\neq j}\frac{q_k-q_j-k+j}{(q_j-q_k)(j-k)}}_{*}$$

proposition

$(L-L_0)\vec{1} \in \ell^2$ if $\underbrace{p\in \ell^2 \text{ and } q_{j+1}-q_j-1 \in \ell^2_j}_{\Gamma}$.

sketch:

$$* = \sum_{m\neq 0}\frac{q_{j+m}-q_j-m}{(q_{j+m}-q_j)m}$$

$$= \sum_{m\neq 0}\frac{q_{j+m}-q_j-m}{m^2} + \text{error}$$

$$= \sum_{m>0}\frac{\sum_{\ell=1}^{m} q_{j+\ell}-q_{j+\ell-1}-1}{m^2} - \sum_{m>0}\frac{\sum_{\ell=-m}^{-1} q_{j+\ell}-q_{j+\ell-1}-1}{m^2} + \text{error}$$

$$= \sum_{\ell\neq 0}(q_{j+\ell}-q_{j+\ell-1}-1)\cdot \operatorname{sign}(\ell)\cdot \underbrace{\sum_{m\geq|\ell|}\frac{1}{m^2}}_{=\frac{1}{\ell}+\text{error}}$$

$$= H(q_j-q_{j-1}-1) + \text{error}.$$

$\square$

can skip

theorem Let (q,p) solve CM.

① $(q^0,p^0)\in\Gamma \Rightarrow (q,p)\in\Gamma$

② Define $f=(L-L_0)\vec{1}$. Then $f\in C_t^1\ell^2$ w/

- $\dot{f}=Pf$ and $f=Uf^0$.

③ $\langle f, L^k f\rangle$ is conserved $k\geq 0$.

④ Define $K = \frac{1}{2}\|f\|^2_{\ell^2}$

$$= \frac{1}{2}\sum_j p_j^2 + \frac{1}{2}\sum_{j}\left[\sum_{k\neq j}\left(\frac{1}{q_j-q_k} - \frac{1}{j-k}\right)\right]^2$$

Then K is a Hamiltonian:

$$\frac{\partial K}{\partial q_j} = -\sum_{k\neq j}\frac{2}{(q_j-q_k)^2}$$

$$\frac{\partial K}{\partial p_j} = p_j.$$

<u>sketch</u>

$$\frac{\partial K}{\partial q_j} = \left\langle \sum_{n\neq k}\left(\frac{1}{q_n-q_k} - \frac{1}{n-k}\right),\ -\delta_{jk}\sum_{\ell\neq j}\frac{1}{(q_j-q_\ell)^2} + (1-\delta_{jk})\frac{1}{(q_k-q_j)^2}\right\rangle_k$$

$$= -\sum_{k\neq j}\sum_{n\neq j}\frac{1}{(q_j-q_k)^2}\left(\frac{1}{q_n-q_j} - \cancel{\frac{1}{n-j}}\right) + \sum_{k\neq j}\sum_{n\neq k}\frac{1}{(q_j-q_k)^2}\left(\frac{1}{q_n-q_k} - \cancel{\frac{1}{n-k}}\right)$$

$$= -\sum_{k\neq j}\frac{2}{(q_j-q_k)^3} + \cancel{\sum_{\substack{n,k\neq j\\ n\neq k}}\frac{1}{(q_j-q_k)(q_n-q_k)(q_n-q_j)}}$$

□

Intuition

More naturally

$$E = \frac{1}{2}\sum_j p_j^2 + \frac{1}{2}\sum_{j\neq k}\left[\frac{1}{(q_j-q_k)^2} - \frac{1}{(j-k)^2}\right].$$

Then

$$\underset{\substack{\uparrow\\ \ell^2}}{K} = \underset{\substack{\uparrow\\ \ell^1}}{E} + \frac{\pi^2}{3}\sum_j \underset{\substack{\uparrow\\ \ell^1}}{(q_{j+1} - q_j - 1)}$$

<u>sketch</u>: <u>hard</u>