DIMENSION IN THE REALM OF TRANSSERIES

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Abstract. Let $\mathbb{T}$ be the differential field of transseries. We establish some basic properties of the *dimension* of a definable subset of $\mathbb{T}^n$, also in relation to its *codimension* in the ambient space $\mathbb{T}^n$. The case of dimension 0 is of special interest, and can be characterized both in topological terms (discreteness) and in terms of the Herwig-Hrushovski-Macpherson notion of co-analyzability.

Introduction

The field of Laurent series with real coefficients comes with a natural derivation but is too small to be closed under integration and exponentiation. These defects are cured by passing to a certain canonical extension, the ordered differential field $\mathbb{T}$ of transseries. Transseries are formal series in an indeterminate $x > \mathbb{R}$, such as

$$\begin{aligned} &-3\,\mathrm{e}^{\mathrm{e}^x} + \mathrm{e}^{\frac{\mathrm{e}^x}{\log x} + \frac{\mathrm{e}^x}{\log^2 x} + \frac{\mathrm{e}^x}{\log^3 x} + \cdots} - x^{11} + 7 \\ &\quad + \frac{\pi}{x} + \frac{1}{x\log x} + \frac{1}{x\log^2 x} + \frac{1}{x\log^3 x} + \cdots \\ &\quad + \frac{2}{x^2} + \frac{6}{x^3} + \frac{24}{x^4} + \frac{120}{x^5} + \frac{720}{x^6} + \cdots \\ &\quad + \mathrm{e}^{-x} + 2\,\mathrm{e}^{-x^2} + 3\,\mathrm{e}^{-x^3} + 4\,\mathrm{e}^{-x^4} + \cdots, \end{aligned}$$

where $\log^2 x := (\log x)^2$, etc. Transseries, that is, elements of $\mathbb{T}$, are also the *logarithmic-exponential series* (LE-series, for short) from [5]; we refer to that paper, or to Appendix A of our book [1], for a detailed construction of $\mathbb{T}$.

What we need for now is that $\mathbb{T}$ is a real closed field extension of the field $\mathbb{R}$ of real numbers and that $\mathbb{T}$ comes equipped with a distinguished element $x > \mathbb{R}$, an exponential operation $\exp\colon \mathbb{T} \to \mathbb{T}$ and a distinguished derivation $\partial\colon \mathbb{T} \to \mathbb{T}$. The exponentiation here is an isomorphism of the ordered additive group of $\mathbb{T}$ onto the ordered multiplicative group $\mathbb{T}^{>}$ of positive elements of $\mathbb{T}$. The derivation ∂ comes from differentiating a transseries termwise with respect to x, and we set $f' := \partial(f)$, $f'' := \partial^2(f)$, and so on, for $f \in \mathbb{T}$; in particular, $x' = 1$, and ∂ is compatible with exponentiation: $\exp(f)' = f' \exp(f)$ for $f \in \mathbb{T}$. Moreover, the constant field of $\mathbb{T}$ is $\mathbb{R}$, that is, $\{f \in \mathbb{T} : f' = 0\} = \mathbb{R}$; see again [1] for details.

In Section 1 we define for any differential field K (of characteristic 0 in this paper) and any set $S \subseteq K^n$ its (differential-algebraic) dimension

$$\dim S \in \{-\infty, 0, 1, \dots, n\} \quad (\text{with } \dim S = -\infty \text{ iff } S = \emptyset).$$

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Some dimension properties hold in this generality, but for more substantial results we assume that $K = \mathbb{T}$ and S is *definable* in $\mathbb{T}$, in which case we have:

$$\dim S = n \iff S \text{ has nonempty interior in } \mathbb{T}^n.$$

Here $\mathbb{T}$ is equipped with its order topology, and each $\mathbb{T}^n$ with the corresponding product topology. This equivalence is shown in Section 3, where we also prove:

Theorem 0.1. *If $S \subseteq \mathbb{T}^m$ and $f\colon S \to \mathbb{T}^n$ are definable, then $\dim S \geqslant \dim f(S)$, for every $i \in \{0, \dots, m\}$ the set $B(i) := \big\{y \in \mathbb{T}^n : \dim f^{-1}(y) = i\big\}$ is definable, and $\dim f^{-1}\big(B(i)\big) = i + \dim B(i)$.*

In Section 4 we show that for definable nonempty $S \subseteq \mathbb{T}^n$,

$$\dim S = 0 \iff S \text{ is discrete.}$$

For $S \subseteq \mathbb{T}^n$ to be discrete means as usual that every point of S has a neighborhood in $\mathbb{T}^n$ that contains no other point of S. For example, $\mathbb{R}^n$ as a subset of $\mathbb{T}^n$ is discrete! Proving the backwards direction of the equivalence above involves an unusual cardinality argument. Both directions use key results from [1].

The rest of the paper is inspired by [1, Theorem 16.0.3], which suggests that for a definable set $S \subseteq \mathbb{T}^n$ to have dimension 0 amounts to S being controlled in some fashion by the constant field $\mathbb{R}$. In what fashion? Our first guess was that perhaps every definable subset of $\mathbb{T}^n$ of dimension 0 is the image of some definable map $\mathbb{R}^m \to \mathbb{T}^n$. (Every such image has indeed dimension 0.) It turns out, however, that the solution set of the algebraic differential equation $yy'' = (y')^2$ in $\mathbb{T}$, which has dimension 0, is *not* such an image: in Section 5 we show how this follows from a fact about automorphisms of $\mathbb{T}$ to be established in [2]. (In that section we call an image as above *parametrizable by constants*; we have since learned that it already has a name in the literature, namely, *internal to the constants*, a special case of a general model-theoretic notion; see [14, Section 7.3].)

The correct way to understand the model-theoretic meaning of dimension 0 is the concept of *co-analyzability* from [8]. This is the topic of Section 6, where we also answer positively a question that partly motivated our paper: given definable $S \subseteq \mathbb{T}^m$ and definable $f\colon S \to \mathbb{T}^n$, does there always exist an $e \in \mathbb{N}$ such that $|f^{-1}(y)| \leqslant e$ for all $y \in \mathbb{T}^n$ for which $f^{-1}(y)$ is finite? In other words, is the quantifier "there exist infinitely many" available for free?

We thank James Freitag for pointing us to the notion of co-analyzability.

1. Differential-algebraic Dimension

We summarize here parts of subsection 2.25 in [4], referring to that paper for proofs. Throughout this section K is a differential field (of characteristic zero with a single distinguished derivation, in this paper), with constant field $C \neq K$. Also, $Y = (Y_1, \dots, Y_n)$ is a tuple of distinct differential indeterminates, and $K\{Y\}$ the ring of differential polynomials in Y over K.

Generalities. Let a set $S \subseteq K^n$ be given. Then the differential polynomials $P_1, \dots, P_m \in K\{Y\}$ are said to be d**-algebraically dependent on** S if for some nonzero differential polynomial $F \in K\{X_1, \dots, X_m\}$,

$$F\big(P_1(y), \dots, P_m(y)\big) \ = \ 0 \ \text{ for all } y = (y_1, \dots, y_n) \in S;$$

if no such F exists, we say that $P_1, \dots, P_m$ are **d-algebraically independent on S,** and in that case we must have $m \leqslant n$; the prefix d stands for *differential.* For nonempty S we define the (differential-algebraic) **dimension** $\dim S$ of S to be the largest m for which there exist $P_1, \dots, P_m \in K\{Y\}$ that are d-algebraically independent on S, and if $S = \emptyset$, then we set $\dim S := -\infty$.

In particular, for nonempty S, $\dim S = 0$ means that for every $P \in K\{Y\}$ there exists a nonzero $F \in K\{X\}$, $X = X_1$, such that $F\big(P(y)\big) = 0$ for all $y \in S$. As an example, let $a \in K^n$ and consider $S = \{a\}$. For $P \in K\{Y\}$ we have $F(P(a)) = 0$ for $F(X) := X - P(a)$, so $\dim\{a\} = 0$. Also, $\dim C^n = 0$ by Lemma 1.1.

Of course, this notion of dimension is relative to K, and if we need to indicate the ambient K we write $\dim_K S$ instead of $\dim S$. But this will hardly be necessary, since $\dim_K S = \dim_L S$ for any differential field extension L of K.

Below we also consider the structure (K, S): the differential field K equipped with the n-ary relation S. The following is a useful characterization of dimension in terms of *differential transcendence degree* (for which see [1, Section 4.1]):

Lemma 1.1. *Let (K^*, S^*) be a $|K|^+$-saturated elementary extension of (K, S) and assume S is not empty. Then*

$$\dim_K S \ = \ \max\big\{\textit{differential transcendence degree of } K\langle s\rangle \textit{ over } K:\ s \in S^*\big\}.$$

Here are some easy consequences of the definition of *dimension* and Lemma 1.1:

Lemma 1.2. *Let $S, S_1, S_2 \subseteq K^n$. Then:*

(i) *if S is finite and nonempty, then $\dim S = 0$; $\dim K^n = n$;*
(ii) $\dim S < n \iff S \subseteq \big\{y \in K^n:\ P(y) = 0\big\}$ *for some nonzero $P \in K\{Y\}$;*
(iii) $\dim(S_1 \cup S_2) = \max(\dim S_1, \dim S_2)$*;*
(iv) $\dim S^\sigma = \dim S$ *for each permutation σ of $\{1, \dots, n\}$, where*
$$S^\sigma \ := \ \big\{\big(y_{\sigma(1)}, \dots, y_{\sigma(n)}\big):\ (y_1, \dots, y_n) \in S\big\};$$
(v) *if $m \leqslant n$ and $\pi\colon K^n \to K^m$ is given by $\pi(y_1, \dots, y_n) = (y_1, \dots, y_m)$, then $\dim \pi(S) \leqslant \dim S$;*
(vi) *if $\dim S = m$, then $\dim \pi(S^\sigma) = m$ for some σ as in* (iv) *and π as in* (v).

The next two lemmas are not in [4], and are left as easy exercises:

Lemma 1.3. $\dim(S_1 \times S_2) = \dim S_1 + \dim S_2$ *for $S_1 \subseteq K^m$ and $S_2 \subseteq K^n$.*

Lemma 1.4. $\dim_K S = \dim_{K^*} S^*$ *in the situation of Lemma 1.1.*

Let now K^* be any elementary extension of K and suppose S is definable in K, say by the formula $\phi(y_1, \dots, y_n)$ in the language of differential fields with names for the elements of K. Let $S^* \subseteq (K^*)^n$ be defined in K^* by the same formula $\phi(y_1, \dots, y_n)$. Note that S^* does not depend on the choice of ϕ. We have the following easy consequence of Lemma 1.4:

Corollary 1.5. $\dim_K S = \dim_{K^*} S^*$.

Differential boundedness. For a set $S \subseteq K^{n+1}$ and $y \in K^n$ we define

$$S(y) \ := \ \big\{z \in K:\ (y, z) \in S\big\} \qquad \text{(the section of } S \text{ above } y).$$

We say that K is **d-bounded** if for every definable set $S \subseteq K^{n+1}$ there exist $P_1, \dots, P_m \in K\{Y, Z\}$ (with Z an extra indeterminate) such that if $y \in K^n$ and $\dim S(y) = 0$, then $S(y) \subseteq \{z \in K:\ P_i(y, z) = 0\}$ for some $i \in \{1, \dots, m\}$

with $P_i(y,Z) \neq 0$. (In view of Lemma 1.2(ii), this is equivalent to the differential field K being *differentially bounded* as defined on p. 203 of [4].) Here is the main consequence of d-boundedness, taken from [4]:

Proposition 1.6. *Assume K is* d*-bounded. Let $S \subseteq K^m$ and $f\colon S \to K^n$ be definable. Then $\dim S \geqslant \dim f(S)$. Moreover, for every $i \in \{0,\dots,m\}$ the set $B(i) := \big\{y \in K^n :\ \dim f^{-1}(y) = i\big\}$ is definable, and $\dim f^{-1}\big(B(i)\big) = i + \dim B(i)$.*

As $\mathbb{T}$ is d-bounded (see Section 3), this gives Theorem 0.1. Differentially closed fields are d-bounded, as pointed out in [4]. Guzy and Point [7] (see also [3]) show that existentially closed ordered differential fields, and Scanlon's d-henselian valued differential fields with many constants (see [1, Chapter 8]) are d-bounded.

2. Dimension and Codimension

This section will not be used in the rest of this paper, but is included for its own sake. The main result is Corollary 2.3. A byproduct of the treatment here is a simpler proof of [1, Theorem 5.9.1] that avoids the nontrivial facts about regular local rings used in [1], where we followed closely Johnson's proof in [10] of a more general result.

Let $y = (y_1,\dots,y_n)$ be a tuple of elements of a differential field extension of K, and let d be the differential transcendence degree of $F := K\langle y\rangle$ over K: there are $i_1 < \cdots < i_d$ in $\{1,\dots,n\}$ such that $y_{i_1},\dots,y_{i_d}$ are d-algebraically independent over K, but there are no $i_1 < \cdots < i_d < i_{d+1}$ in $\{1,\dots,n\}$ such that $y_{i_1},\dots,y_{i_d},y_{i_{d+1}}$ are d-algebraically independent over K. We wish to characterize d alternatively as follows: there should exist $n-d$ "independent" relations $P_1(y) = \cdots = P_{n-d}(y) = 0$, with all $P_i \in K\{Y\}$, but not more than $n-d$ such relations. The issue here is what "independent" should mean.

We say that a d-polynomial $P \in K\{Y\}$ has order at most $\vec{r} = (r_1,\dots,r_n) \in \mathbb{N}^n$ if $P \in K\big[Y_j^{(r)} :\ 1 \leqslant j \leqslant n,\ 0 \leqslant r \leqslant r_j\big]$. Given $P_1,\dots,P_m \in K\{Y\}$ of order at most $\vec{r} \in \mathbb{N}^n$, consider the $m \times n$-matrix over F with i,j-entry

$$\frac{\partial P_i}{\partial Y_j^{(r_j)}}(y) \qquad (i = 1,\dots,m,\ j = 1,\dots,n).$$

This matrix has rank $\leqslant \min(m,n)$. We say that $P_1,\dots,P_m$ are **strongly** d**-independent at** y if for some $\vec{r} \in \mathbb{N}^n$ with $P_1,\dots,P_m$ of order at most $\vec{r}$, this matrix has rank m; thus $m \leqslant n$ in that case.

Set $R := K\{Y\}$ and $\mathfrak{p} := \big\{P \in R : P(y) = 0\big\}$, a differential prime ideal of R. With these notations we have:

Lemma 2.1. *There are $P_1,\dots,P_{n-d} \in \mathfrak{p}$ that are strongly* d*-independent at y.*

Proof. Set $m := n-d$ and permute indices such that $y_{m+1},\dots,y_n$ is a differential transcendence base of $F = K\langle y\rangle$ over K. For $i = 1,\dots,m$, pick

$$P_i(Y_i, Y_{m+1},\dots,Y_n) \in K\{Y_i, Y_{m+1},\dots,Y_n\} \ \subseteq\ K\{Y\}$$

such that $P_i(Y_i, y_{m+1},\dots,y_n)$ is a minimal annihilator of y_i over $K\langle y_{m+1},\dots,y_n\rangle$. Let P_i have order r_i in Y_i. Then the minimality of P_i gives

$$\frac{\partial P_i}{\partial Y_i^{(r_i)}}(y_i, y_{m+1},\dots,y_n) \neq 0, \qquad (i = 1,\dots,m).$$

Next we take $r_{m+1}, \dots, r_n \in \mathbb{N}$ such that all P_i have order $\leqslant r_j$ in Y_j for $j = m+1, \dots, n$. Considering all P_i as elements of $K\{Y\}$ we see that $P_1, \dots, P_m$ have order $\leqslant (r_1, \dots, r_n)$, and that the $m \times m$ matrix

$$\left(\frac{\partial P_i}{\partial Y_j^{(r_j)}}(y) \right) \qquad (1 \leqslant i, j \leqslant m)$$

is diagonal, with nonzero determinant. □

We refer to [1, Section 5.4] for what it means for $P_1, \dots, P_m \in R$ to be d-*independent at* y. By [1, Lemma 5.4.7], if $P_1, \dots, P_m \in R$ are strongly d-independent at y, then they are d-independent at y (but the converse may fail). Below we show that if $P_1, \dots, P_m \in \mathfrak{p}$ are d-independent at y, then $m \leqslant n - d$.

The notion of d-independence at y is more intrinsic and more flexible than that of strong d-independence at y. To discuss the former in more detail, we need some terminology from [1]. Let A be a commutative ring, $\mathfrak{p}$ a prime ideal of A, and M an A-module; then a family (f_i) of elements of M is said to be *independent at* $\mathfrak{p}$ if the family $(f_i + \mathfrak{p}M)$ of elements of the $A/\mathfrak{p}$-module $M/\mathfrak{p}M$ is linearly independent. Next, let A also be a differential ring extension of K. Then the K-algebra A yields the A-module $\Omega_{A|K}$ of Kähler differentials with the (universal) K-derivation

$$a \mapsto \mathrm{d}\, a \;=\; \mathrm{d}_{A|K}\, a \;:\; A \to \Omega_{A|K}.$$

Following Johnson [10] we make this A-module compatibly into an $A[\partial]$-module by $\partial(\mathrm{d}\, a) := \mathrm{d}\, \partial a$ for $a \in A$; a family of elements of $\Omega_{A|K}$ is said to be d-*independent* if this family is linearly independent in $\Omega_{A|K}$ viewed as an $A[\partial]$-module. This means for $a_1, \dots, a_m \in A$: the differentials $\mathrm{d}\, a_1, \dots, \mathrm{d}\, a_m \in \Omega_{A|K}$ are d-independent iff the family $\big(\mathrm{d}\, a_i^{(r)}\big)$ $(i = 1, \dots, m,\ r = 0, 1, 2, \dots)$ is linearly independent in the A-module $\Omega_{A|K}$; given also a prime ideal $\mathfrak{p}$ of A we say that $\mathrm{d}\, a_1, \dots, \mathrm{d}\, a_m$ are d-*independent at* $\mathfrak{p}$ if the family $(\mathrm{d}\, a_i^{(r)})$ is independent at $\mathfrak{p}$ in the A-module $\Omega_{A|K}$.

Returning to the differential ring extensions R and $F = K\langle y \rangle$ of K, the $R[\partial]$-module $\Omega_{R|K}$ is free on $\mathrm{d}\, Y_1, \dots, \mathrm{d}\, Y_n$, by [1, Lemma 1.8.11]. The $F[\partial]$-module $\Omega_{F|K}$ is generated by $\mathrm{d}\, y_1, \dots, \mathrm{d}\, y_n$, as shown in [1, Section 5.9]. In [1, Section 5.3] we assign to every finitely generated $F[\partial]$-module M a number $\operatorname{rank}(M) \in \mathbb{N}$, and we have $\operatorname{rank}(\Omega_{F|K}) = d$ by [1, Corollary 5.9.3].

The differential ring morphism $P \mapsto P(y)\colon R \to F$ is the identity on K, and makes $F \otimes_R \Omega_{R|K}$ into an $F[\partial]$-module as explained in [1, Section 5.9]. Note that the kernel of the above differential ring morphism $R \to F$ is the differential prime ideal $\mathfrak{p} = \{P \in R :\ P(y) = 0\}$ of R.

Lemma 2.2. *Suppose $P_1, \dots, P_m \in \mathfrak{p}$ are* d-*independent at y. Then $m \leqslant n - d$.*

Proof. We have a surjective $F[\partial]$-linear map $F \otimes_R \Omega_{R|K} \to \Omega_{F|K}$ sending $1 \otimes \mathrm{d}\, P$ to $\mathrm{d}\, P(y)$ for $P \in R$. Note that $1 \otimes \mathrm{d}\, P_1, \dots, 1 \otimes \mathrm{d}\, P_m$ are in the kernel of this map. By the equivalence (1) ⇔ (5) and Lemma 5.9.4 in [1], the d-independence of $P_1, \dots, P_m$ at y gives that $1 \otimes \mathrm{d}\, P_1, \dots, 1 \otimes \mathrm{d}\, P_m \in F \otimes_R \Omega_{R|K}$ are $F[\partial]$-independent (meaning: linearly independent in this $F[\partial]$-module). Since the $R[\partial]$-module $\Omega_{R|K}$ is free on $dY_1, \dots, dY_n$, the $F[\partial]$-module $F \otimes_R \Omega_{R|K}$ is free on $1 \otimes \mathrm{d}\, Y_1, \dots, 1 \otimes \mathrm{d}\, Y_n$, and so has rank n. To get $m + d \leqslant n$ it remains to use [1, Corollary 5.9.3] and the fact that $\operatorname{rank}(\Omega_{F|K}) = d$. □

Combining the previous two lemmas we conclude:

Corollary 2.3. *The codimension $n-d$ can be characterized as follows:*

$$\begin{aligned} n-d &= \max\{m : \textit{some } P_1,\dots,P_m \in \mathfrak{p} \textit{ are } \mathrm{d}\textit{-independent at } y\} \\ &= \max\{m : \textit{some } P_1,\dots,P_m \in \mathfrak{p} \textit{ are strongly } \mathrm{d}\textit{-independent at } y\}. \end{aligned}$$

This yields a strengthening of Theorem 5.9.1 and its Corollary 5.9.6 in [1]:

Corollary 2.4. *The following are equivalent:*

(i) *$y_1,\dots,y_n$ are* d*-algebraic over K;*
(ii) *there exist $P_1,\dots,P_n \in \mathfrak{p}$ that are* d*-independent at y;*
(iii) *there exist $P_1,\dots,P_n \in \mathfrak{p}$ that are are strongly* d*-independent at y.*

To formulate the above in terms of sets $S \subseteq K^n$ we recall that the *Kolchin topology on K^n* (called the *differential-Zariski topology on K^n* in [4]) is the topology on K^n whose closed sets are the sets

$$\{y \in K^n : \ P_1(y) = \cdots = P_m(y) = 0\} \quad (P_1,\dots,P_m \in K\{Y\}).$$

This is a noetherian topology, and so a Kolchin closed subset of K^n is the union of its finitely many irreducible components. For $S \subseteq K^n$ we let S^{Ko} be its Kolchin closure in K^n with respect to the Kolchin topology. Note that $\dim S = \dim S^{\mathrm{Ko}}$, since for all $P \in K\{Y\}$ we have: if $P = 0$ on S (that is, $P(y) = 0$ for all $y \in S$), then $P = 0$ on S^{Ko}.

Suppose S^{Ko} is irreducible. A **tuple of m independent relations on S** is defined to be a tuple $(P_1,\dots,P_m) \in K\{Y\}^m$ such that

(1) $P_1(y) = \cdots = P_m(y) = 0$ for all $y \in S$;
(2) $P_1,\dots,P_m$ are d-independent at some $y \in S$.

Similarly we define a **tuple of m strongly independent relations on S,** by replacing "d-independent" in (2) by "strongly d-independent". Every tuple of strongly independent relations on S is a tuple of independent relations on S. Since S^{Ko} is irreducible,

$$\mathfrak{p} := \{P \in K\{Y\} : \ P = 0 \text{ on } S\}$$

is a differential prime ideal of $K\{Y\}$. Letting $K\{y\} = K\{Y\}/\mathfrak{p}$ be the corresponding differential K-algebra (an integral domain) with $y = (y_1,\dots,y_n)$, $y_i = Y_i + \mathfrak{p}$, for $P \in K\{Y\}$ we have $P(y) = 0$ iff $P = 0$ on S. So the considerations above applied to y yield for $d := \dim S$ and irreducible S^{Ko}:

Corollary 2.5. *There is a tuple of m strongly independent relations on S for $m = n-d$, but there is no tuple of m independent relations on S for $m > n-d$.*

3. The Case of $\mathbb{T}$

The paper [4] contains an axiomatic framework for a reasonable notion of dimension for the definable sets in suitable model-theoretic structures with a topology. In this section we show that as a consequence of [1, Chapter 16] the relevant axioms are satisfied for $\mathbb{T}$ with its order topology.

To state the necessary facts about $\mathbb{T}$ from [1] we recall from that book that an *H-field* is an ordered differential field K with constant field C such that:

(H1) $\partial(a) > 0$ for all $a \in K$ with $a > C$;
(H2) $\mathcal{O} = C + \smallo$, where $\mathcal{O}$ is the convex hull of C in the ordered field K, and $\smallo$ is the maximal ideal of the valuation ring $\mathcal{O}$.

Let K be an H-field, and let $\mathcal{O}$ and $\mathcal{o}$ be as in (H2). Thus K is a valued field with valuation ring $\mathcal{O}$. The valuation topology on K equals its order topology if $C \neq K$. We consider K as an $\mathcal{L}$-structure, where

$$\mathcal{L} \ := \ \{0, 1, +, -, \times, \partial, P, \preccurlyeq\}$$

is the language of ordered valued differential fields. The symbols 0, 1, $+$, $-$, $\times$, ∂ are interpreted as usual in K, and P and $\preccurlyeq$ encode the ordering and the valuation:

$$P(a) \iff a > 0, \qquad a \preccurlyeq b \iff a \in \mathcal{O}b \qquad (a, b \in K).$$

Given $a \in K$ we also write a' instead of $\partial(a)$, and we set $a^\dagger := a'/a$ for $a \neq 0$.

The real closed (and thus ordered) differential field $\mathbb{T}$ is an H-field, and in [1] we showed that it is a model of a model-complete $\mathcal{L}$-theory T^{nl}. The models of the latter are exactly the H-fields K satisfying the following (first-order) conditions:

(1) K is Liouville closed;
(2) K is ω-free;
(3) K is newtonian.

(An H-field K is said to be *Liouville closed* if it is real closed and for all $a \in K$ there exists $b \in K$ with $a = b'$ and also a $b \in K^\times$ such that $a = b^\dagger$; for the definition of "ω-free" and "newtonian" we refer to the Introduction of [1].) Since "Liouville closed" includes "real closed", the ordering (and thus the valuation ring) of any model of T^{nl} is definable in the underlying differential field of the model. We shall prove the dimension results in this paper for all models of T^{nl}: working in this generality plays a role even when our main interest is in $\mathbb{T}$. So in the rest of this section we fix an arbitrary model K of T^{nl}, that is, K *is a Liouville closed ω-free newtonian H-field*. Lemma 1.2(ii) and [1, Corollary 16.6.4] yield:

Corollary 3.1. *For definable $S \subseteq K^n$,*

$$\dim S = n \iff S \text{ has nonempty interior in } K^n.$$

To avoid confusion with the Kolchin topology, we consider K here and below as equipped with its order topology, and K^n with the corresponding product topology. Combining the previous corollary with (iv)–(vi) in Lemma 1.2 yields a topological characterization of dimension:

Corollary 3.2. *For nonempty definable $S \subseteq K^n$, $\dim S$ is the largest $m \leqslant n$ such that for some permutation σ of $\{1, \dots, n\}$, the subset $\pi_m(S^\sigma)$ of K^m has nonempty interior; here $\pi_m(x_1, \dots, x_n) := (x_1, \dots, x_m)$ for $(x_1, \dots, x_n) \in K^n$.*

In particular, if $S \subseteq K^n$ is semialgebraic in the sense of the real closed field K, then $\dim S$ agrees with the usual semialgebraic dimension of S over K.

To get that K is d-bounded, we introduce two key subsets of K, namely $\Lambda(K)$ and $\Omega(K)$. They are defined by the following equivalences, for $a \in K$:

$$a \in \Lambda(K) \iff a = -y^{\dagger\dagger} \text{ for some } y \succ 1 \text{ in } K,$$
$$a \in \Omega(K) \iff 4y'' + ay = 0 \text{ for some } y \in K^\times.$$

To describe these sets more concretely for $K = \mathbb{T}$, set $\ell_0 := x$ and $\ell_{n+1} := \log \ell_n$, so ℓ_n is the nth iterated logarithm of x in $\mathbb{T}$. Then for $f \in \mathbb{T}$,

$$f \in \Lambda(\mathbb{T}) \iff f \ < \ \frac{1}{\ell_0} + \frac{1}{\ell_0\ell_1} + \cdots + \frac{1}{\ell_0\ell_1\cdots\ell_n} \quad \text{for some } n,$$
$$f \in \Omega(\mathbb{T}) \iff f \ < \ \frac{1}{\ell_0^2} + \frac{1}{\ell_0^2\ell_1^2} + \cdots + \frac{1}{\ell_0^2\ell_1^2\cdots\ell_n^2} \quad \text{for some } n,$$

by [1, Example after 11.8.19; Proposition 11.8.20 and Corollary 11.8.21]. The set $\Lambda(K)$ is closed downward in K: if $a \in K$ and $a < b \in \Lambda(K)$, then $a \in \Lambda(K)$; and $\Lambda(K)$ has an upper bound in K but no least upper bound; these properties also hold for $\Omega(K)$ instead of $\Lambda(K)$. From Chapter 16 of [1] we need that T^{nl} has a certain extension by definitions $T^{\mathrm{nl}}_{\Lambda\Omega}$ that has QE: the language of $T^{\mathrm{nl}}_{\Lambda\Omega}$ is $\mathcal{L}$ augmented by two extra binary relation symbols R_Λ and R_Ω, to be interpreted in K according to

$$aR_\Lambda b \iff a \in \Lambda(K)b, \qquad aR_\Omega b \iff a \in \Omega(K)b.$$

(The language of $T^{\mathrm{nl}}_{\Lambda\Omega}$ in [1, Chapter 16] is slightly different, but yields the same notion of what is quantifier-free definable. The version here is more convenient for our purpose.) Using that $\Lambda(K)$ and $\Omega(K)$ are open-and-closed in K, it is routine (but tedious) to check that K satisfies the differential analogue of [4, 2.15] that is discussed on p. 203 of that paper in a general setting. Thus:

Corollary 3.3. *K is* d*-bounded; in particular, $\mathbb{T}$ is* d*-bounded.*

Moreover, [4, p. 203] points out the following consequence (extending Corollary 3.1):

Corollary 3.4. *Every nonempty definable set $S \subseteq K^n$ has nonempty interior in the Kolchin closure S^{Ko} of S in K^n.*

(By our earlier convention, the *interior* here refers to the topology on S^{Ko} induced by the product topology on K^n that comes from the order topology on K.) For nonempty definable $S \subseteq K^n$ with closure $\mathrm{cl}(S)$ in K^n we have

$$\dim\big(\mathrm{cl}(S) \setminus S\big) \ < \ \dim S.$$

This is analogous to [4, 2.23], but the proof there doesn't go through. We intend to show this dimension decrease in a follow-up paper.

4. Dimension 0 = Discrete

Let K be a Liouville closed ω-free newtonian H-field, with the order topology on K and the corresponding product topology on each K^n. Corollary 16.6.11 in [1] and its proof yields the following equivalences for definable $S \subseteq K$:

$$\dim S = 0 \iff S \text{ has empty interior} \iff S \text{ is discrete.}$$

We now extend part of this to definable subsets of K^n. The proof of one of the directions is rather curious and makes full use of the resources of [1].

Proposition 4.1. *For definable nonempty $S \subseteq K^n$:*

$$\dim S = 0 \iff S \textit{ is discrete.}$$

Proof. For $i = 1, \dots, n$ we let $\pi_i \colon K^n \to K$ be given by $\pi_i(a_1, \dots, a_n) = a_i$. If $\dim S = 0$, then $\dim \pi_i(S) = 0$ for all i, so $\pi_i(S)$ is discrete for all i, hence the cartesian product $\pi_1(S) \times \cdots \times \pi_n(S) \subseteq K^n$ is discrete, and so is its subset S.

Now for the converse. Assume $S \subseteq K^n$ is discrete. We first replace K by a suitable countable elementary substructure over which S is defined and S by its corresponding trace. Now that K is countable we next pass to its completion K^{c} as defined in [1, Section 4.4], which by [1, 14.1.6] is an elementary extension of K. Replacing K by K^{c} and S by the corresponding extension, the overall effect is that we have arranged K to be *uncountable,* but with a *countable* base for its topology. Then the discrete set S is countable, so $\pi_i(S) \subseteq K$ is countable for each i, hence with empty interior, so $\dim \pi_i(S) = 0$ for all i, and thus $\dim S = 0$. $\square$

Corollary 4.2. *If $S \subseteq K^n$ is definable and discrete, then there is a neighborhood U of $0 \in K^n$ such that $(s_1 + U) \cap (s_2 + U) = \emptyset$ for all distinct $s_1, s_2 \in S$.*

Proof. Let $S \subseteq K^n$ $(n \geqslant 1)$ be nonempty, definable, and discrete. For $y \in K^n$ we set $|y| := \max_i |y_i|$. The set $D := \big\{|a - b| : \ a, b \in S\big\}$ is the image of a definable map $S^2 \to K$, so D is definable with $\dim D = 0$ and $0 \in D$. Thus D is discrete, so $(-\varepsilon, \varepsilon) \cap D = \{0\}$ for some $\varepsilon \in K^{>}$, which gives the desired conclusion. $\square$

In particular, any definable discrete subset of K^n is closed in K^n.

5. Parametrizability by Constants

Let K be a Liouville closed $\boldsymbol{\omega}$-free newtonian H-field. Then K induces on its constant field C just C's structure as a real closed field, by [1, 16.0.2(ii)], that is, a set $X \subseteq C^m$ is definable in K iff X is semialgebraic in the sense of C.

Let $S \subseteq K^n$ be definable. We say that S is **parametrizable by constants** if $S \subseteq f(C^m)$ for some m and some definable map $f \colon C^m \to K^n$; equivalently, $S = f(X)$ for some injective definable map $f \colon X \to K^n$ with semialgebraic $X \subseteq C^m$ for some m. (The reduction to injective f uses the fact mentioned above about the induced structure on C.) For example, if $P \in K\{Y\}$ is a differential polynomial of degree 1 in a single indeterminate Y, then the set $\big\{y \in K : \ P(y) = 0\big\}$ is either empty or a translate of a finite-dimensional C-linear subspace of K, and so this set is parametrizable by constants. The definable sets in K^n for $n = 0, 1, 2, \dots$ that are parametrizable by constants make up a very robust class: it is closed under taking definable subsets, and under some basic logical operations: taking finite unions (in the same K^n), cartesian products, and images under definable maps. Moreover:

Lemma 5.1. *Let $S \subseteq K^n$ and $f \colon S \to C^m$ be definable, and let $e \in \mathbb{N}$ be such that $|f^{-1}(c)| \leqslant e$ for all $c \in C^m$. Then S is parametrizable by constants.*

Proof. By partitioning S appropriately we reduce to the case that for all $c \in f(S)$ we have $|f^{-1}(c)| = e$. Using the lexicographic ordering on K^n this yields definable injective $g_1, \dots, g_e \colon f(S) \to K^n$ such that $f^{-1}(c) = \big\{g_1(c), \dots, g_e(c)\big\}$ for all $c \in f(S)$. Thus $S = g_1\big(f(S)\big) \cup \cdots \cup g_e\big(f(S)\big)$ is parametrizable by constants. $\square$

Suppose $S \subseteq K^n$ be definable. Note that if S is parametrizable by constants, then $\dim S \leqslant 0$. The question arises if the converse holds: does it follow from $\dim S = 0$ that S is parametrizable by constants? We show that the answer is negative for $K = \mathbb{T}$ and the set

$$\big\{y \in \mathbb{T} : \ yy'' = (y')^2\big\} \ = \ \{a\,\mathrm{e}^{bx} : \ a, b \in \mathbb{R}\}.$$

This set has dimension 0 and we claim that it is not parametrizable by constants. (The map $(a,b) \mapsto a\,\mathrm{e}^{bx} : \mathbb{R}^2 \to \mathbb{T}$ would be a parametrization of this set by constants if exp were definable in $\mathbb{T}$; we return to this issue at the end of this section.) To justify this claim we appeal to a special case of results from [2]:

For any finite set $A \subseteq \mathbb{T}$ there exists an automorphism of the differential field $\mathbb{T}$ over A that is not the identity on $\{\mathrm{e}^{bx} : b \in \mathbb{R}\}$.

The claimed nonparametrizability by constants follows when we combine this fact with the observation that if $f\colon \mathbb{R}^m \to \mathbb{T}$ is definable in $\mathbb{T}$, say over the finite set $A \subseteq \mathbb{T}$, then any automorphism of the differential field $\mathbb{T}$ over A fixes each real number, and so it fixes each value of the function f.

Below Y is a single indeterminate, and for $P \in K\{Y\}$ we let

$$\mathrm{Z}(P) \ := \ \big\{y \in K : \ P(y) = 0\big\}.$$

Thus $\mathrm{Z}\big(YY'' - (Y')^2\big) = \{a\,\mathrm{e}^{bx} : a, b \in \mathbb{R}\}$ for $K = \mathbb{T}$ and $YY'' - (Y')^2$ has order 2. What about the parametrizability of $\mathrm{Z}(P)$ for P of order 1? In the next two lemmas we consider the special case $P(Y) = F(Y)Y' - G(Y)$ where $F, G \in C[Y]^{\neq}$ have no common factor of positive degree.

Lemma 5.2. *If $\frac{F}{G} = c\frac{\partial R}{\partial Y}/R$ for some $c \in C^{\times}$, $R \in C(Y)^{\times}$, or $\frac{F}{G} = \frac{\partial R}{\partial Y}$ for some $R \in C(Y)^{\times}$, then $\mathrm{Z}(P)$ is parametrizable by constants.*

Proof. Suppose $\frac{F}{G} = c\frac{\partial R}{\partial Y}/R$ where $c \in C^{\times}$, $R \in C(Y)^{\times}$. Since K is Liouville closed we can take $b \in K^{\times}$ with $b^{\dagger} = 1/c$. Set $S := \big\{y \in \mathrm{Z}(P) : G(y) \neq 0,\ R(y) \neq 0, \infty\big\}$. Then for $y \in S$ we have

$$0 \ = \ G(y)\left(\tfrac{F(y)}{G(y)}\, y' - 1\right) \ = \ G(y)\left(c\big(\tfrac{\partial R}{\partial Y}/R\big)(y)\, y' - 1\right) \ = \ G(y)\big(cR(y)^{\dagger} - 1\big)$$

and so $R(y) \in C^{\times} b$. It is clear that we can take $e \in \mathbb{N}$ such that the definable map $f\colon S \to C$ given by $f(y) := R(y)/b$ for $y \in S$ satisfies $|f^{-1}(c)| \leqslant e$ for all $c \in C$. Hence S, and thus $\mathrm{Z}(P)$, is parametrizable by constants by Lemma 5.1. Next, suppose that $\frac{F}{G} = \frac{\partial R}{\partial Y}$ where $R \in C(Y)$. Take $x \in K$ with $x' = 1$ and set $S := \big\{y \in \mathrm{Z}(P) : G(y) \neq 0,\ R(y) \neq \infty\big\}$. As before we obtain for $y \in S$ that $R(y) \in x + C$, and so $\mathrm{Z}(P)$ is parametrizable by constants. □

Let $Q \in K\{Y\}$ be irreducible and let a be an element of a differential field extension of K with minimal annihilator Q over K. We say that Q **creates a constant** if $C_{K\langle a\rangle} \neq C$. (This is related to the concept of "nonorthogonality to the constants" in the model theory of differential fields; see [12, Proposition 2.6].) Note that our $P = F(Y)Y' - G(Y)$ is irreducible in $K\{Y\}$.

Lemma 5.3. *P creates a constant iff $\frac{F}{G} = c\frac{\partial R}{\partial Y}/R$ for some $c \in C^{\times}$, $R \in C(Y)^{\times}$, or $\frac{F}{G} = \frac{\partial R}{\partial Y}$ for some $R \in C(Y)^{\times}$.*

Proof. The forward direction holds by Rosenlicht [15, Proposition 2]. For the backward direction, take an element a of a differential field extension of K with minimal annihilator P over K. Consider first the case $\frac{F}{G} = c\frac{\partial R}{\partial Y}/R$ where $c \in C^{\times}$ and $R \in C(Y)^{\times}$. Take $b \in K^{\times}$ with $b^{\dagger} = 1/c$. As in the proof of Lemma 5.2 we obtain $0 = P(a) = G(a)\big(cR(a)^{\dagger} - 1\big)$ with $G(a) \neq 0$, and thus $R(a)/b \in C_{K\langle a\rangle}$ and $R(a)/b \notin K$. The case $\frac{F}{G} = \frac{\partial R}{\partial Y}$ with $R \in C(Y)^{\times}$ is handled likewise. □

The following proposition therefore generalizes Lemma 5.2:

Proposition 5.4. *If $P \in K\{Y\}$ is irreducible of order 1 and creates a constant, then $\mathrm{Z}(P)$ is parametrizable by constants.*

Before we give the proof of this proposition, we prove two lemmas, in both of which we let $P \in K\{Y\}$ be irreducible of order 1 such that $\mathrm{Z}(P)$ is infinite.

Lemma 5.5. *Let $Q \in K[Y,Y'] \subseteq K\{Y\}$. Then $\mathrm{Z}(P) \subseteq \mathrm{Z}(Q)$ iff $Q \in PK[Y,Y']$.*

Proof. Suppose $\mathrm{Z}(P) \subseteq \mathrm{Z}(Q)$ but $Q \notin PK[Y,Y']$. Put $F := K(Y)$. By Gauss' Lemma, P viewed as element of $F[Y']$ is irreducible and $Q \notin PF[Y']$. Thus there are $A, B \in K[Y,Y']$, $D \in K[Y]^{\neq}$ with $D = AP + BQ$. Then $\mathrm{Z}(P) \subseteq \mathrm{Z}(D)$ is finite, a contradiction. □

Lemma 5.6. *There is an element a in an elementary extension of K with minimal annihilator P over K.*

Proof. Given $Q_1, \dots, Q_n \in K[Y,Y']^{\neq}$ with $\deg_{Y'} Q_i < \deg_{Y'} P$ for $i = 1, \dots, n$, the previous lemma applied to $Q := Q_1 \cdots Q_n$ yields some $y \in K$ with $P(y) = 0$ and $Q_i(y) \neq 0$ for all $i = 1, \dots, n$. Now use compactness. □

Proof of Proposition 5.4. We can assume that $S := \mathrm{Z}(P)$ is infinite. The preceding lemma yields an element a in an elementary extension of K with $P(a) = 0$ and $Q(a) \neq 0$ for all $Q \in K[Y,Y']^{\neq}$ with $\deg_{Y'} Q < d := \deg_{Y'} P$. In particular, a is transcendental over K. Since P creates a constant, $K\langle a \rangle = K(a,a')$ has a constant $c \notin C$. We have $c = A(a)/B(a)$ with $A \in K[Y,Y']$, $\deg_{Y'} A < d$, $B \in K[Y]^{\neq}$. From $c' = 0$ we get $A'(a)B(a) - A(a)B'(a) = 0$, so

$$A'(Y)B(Y) - A(Y)B'(Y) \;=\; D(Y)P(Y) \;\text{ in } K\{Y\} \;\text{ with }\; D \in K[Y].$$

Hence for $y \in S$ with $B(y) \neq 0$ we have $\big(A(y)/B(y)\big)' = 0$, that is, $A(y)/B(y) \in C$. Thus for $S_B := \big\{y \in S : \; B(y) \neq 0\big\}$ we have a definable map

$$f\colon S_B \to C, \qquad f(y) \;:=\; A(y)/B(y).$$

Since c is transcendental over K, a is algebraic over $K(c)$, say

$$F_0(c)a^e + F_1(c)a^{e-1} + \cdots + F_e(c) \;=\; 0,$$

where $F_0, F_1, \dots, F_e \in K[Z]$ have no common divisor of positive degree in $K[Z]$. Let $G := \partial P/\partial Y'$ be the separant of P. Then $G(a) \neq 0$, $K\big[a, a', 1/B(a), 1/G(a)\big]$ is a differential subring of $K(a,a')$, and every $y \in S_B$ with $G(y) \neq 0$ yields a differential ring morphism

$$\phi_y \;:\; K\big[a, a', 1/B(a), 1/G(a)\big] \to K$$

that is the identity on K with $\phi_y(a) = y$; see the subsection on minimal annihilators in [1, Section 4.1]. Moreover, $c = A(a)/B(a) \in K\big[a, a', 1/B(a), 1/G(a)\big]$, and so for $y \in S_B$ with $G(y) \neq 0$ we have $\phi_y(c) = A(y)/B(y) = f(y)$, so

$$F_0\big(f(y)\big)y^e + F_1\big(f(y)\big)y^{e-1} + \cdots + F_e\big(f(y)\big) \;=\; 0.$$

Set $S_{B,G} := \big\{y \in S_B : \; G(y) \neq 0\big\}$. Then $S \setminus S_{B,G}$ is finite, and the above shows that for all $z \in f(S_{B,G})$ we have $|f^{-1}(z) \cap S_{B,G}| \leqslant e$. Now use Lemma 5.1. □

Freitag [6] proves a generalization of Lemma 5.3. Nishioka ([13], see also [11, p. 90]) gives sufficient conditions on irreducible differential polynomials of order 1 to create a constant, involving the concept of "having no movable singularities"; this can be used to give further examples of $P \in K\{Y\}$ of order 1 whose zero set is parametrizable by constants. But we do not know whether $\mathrm{Z}(P)$ is parametrizable by constants for every $P \in K\{Y\}$ of order 1.

Open problems. The definable set

$$\big\{y \in \mathbb{T} :\ yy'' = (y')^2\big\} \ =\ \{a\,\mathrm{e}^{bx} :\ a,b \in \mathbb{R}\} \subseteq \mathbb{T}^2$$

is the image of the map $(a,b) \mapsto a\,\mathrm{e}^{bx}\colon\ \mathbb{R}^2 \to \mathbb{T}^2$, and so by the above negative result this map is not definable in the differential field $\mathbb{T}$. But it is definable in the *exponential* differential field $(\mathbb{T}, \exp)$, where exponentiation on $\mathbb{T}$ is taken as an extra primitive. This raises the question whether parametrizability by constants holds in an extended sense where the parametrizing maps are allowed to be definable in $(\mathbb{T}, \exp)$. More precisely, *if $S \subseteq \mathbb{T}^n$ is definable in $\mathbb{T}$ with $\dim S = 0$, does there always exist an m and a map $f\colon \mathbb{R}^m \to \mathbb{T}^n$, definable in $(\mathbb{T}, \exp)$, with $S \subseteq f(\mathbb{R}^n)$?* (It is enough to have this for $n = 1$ and $S = \big\{y \in \mathbb{T} :\ P(y) = 0\big\}$, $P \in \mathbb{T}\{Y\}^{\neq}$.)

This is of course related to the issue whether the results in [1, Chapter 16] about $\mathbb{T}$ generalize to its expansion $(\mathbb{T}, \exp)$. In particular, *is the structure induced on $\mathbb{R}$ by $(\mathbb{T}, \exp)$ just the exponential field structure of $\mathbb{R}$?*

It would be good to know more about the order types of discrete definable subsets of Liouville closed $\boldsymbol{\omega}$-free newtonian H-fields K. For example, can any such set have order type ω, or more generally, have an initial segment of order type ω?

6. Dimension 0 = Co-Analyzable Relative to the Constant Field

Parametrizability by constants was our first guess of the model-theoretic significance of [1, Theorem 16.0.3] which says that a Liouville closed $\boldsymbol{\omega}$-free newtonian H-field has no proper differentially-algebraic H-field extension with the same constants. As we saw, this guess failed on the set of zeros of $YY'' - (Y')^2$. We subsequently realized that the notion of *co-analyzability* from [8] fits exactly our situation. Below we expose what we need from that paper, and next we apply it to $\mathbb{T}$.

Co-analyzability. We adopt here the model-theory notations of [1, Appendix B]. Let $\mathcal{L}$ be a first-order language with a distinguished unary relation symbol C. For convenience we assume $\mathcal{L}$ is *one-sorted.* Let $\boldsymbol{M} = (M; \dots)$ be an $\mathcal{L}$-structure and let $C^{\boldsymbol{M}} \subseteq M$ (or just C if $\boldsymbol{M}$ is clear from the context) be the interpretation of the symbol C in $\boldsymbol{M}$; we assume $C \neq \emptyset$.

Assume $\boldsymbol{M}$ is ω-saturated. Let $S \subseteq M^n$ be definable. By recursion on $r \in \mathbb{N}$ we define what makes S **co-analyzable in r steps** (tacitly: relative to $\boldsymbol{M}$ and C):

- (C_0) S is co-analyzable in 0 steps iff S is finite;
- (C_{r+1}) S is co-analyzable in $r+1$ steps iff for some definable set $R \subseteq C \times M^n$,
  - (a) the natural projection $C \times M^n \to M^n$ maps R onto S;
  - (b) for each $c \in C$, the section $R(c) := \big\{s \in M^n : (c,s) \in R\big\}$ above c is co-analyzable in r steps.

We call S **co-analyzable** if S is co-analyzable in r steps for some r.

Thus in (C_{r+1}) the set R gives rise to a covering $S = \bigcup_{c \in C} R(c)$ of S by definable sets $R(c)$ that are co-analyzable in r steps. Of course, the definable set $C^r \subseteq M^r$ is the archetype of a definable set that is co-analyzable in r steps. Note that if S is co-analyzable in 1 step, then the ω-saturation of $\boldsymbol{M}$ yields for R as in (C_1) a uniform bound $e \in \mathbb{N}$ such that $|R(c)| \leqslant e$ for all $c \in C$. This ω-saturation gives likewise an automatic uniformity in (C_{r+1}) that enables us to extend the notion of co-analyzability appropriately to arbitrary $\boldsymbol{M}$ (not necessarily ω-saturated). Before doing this, we mention some easy consequences of the definition above where we do assume $\boldsymbol{M}$ is ω-saturated. First, if the definable set $S \subseteq M^n$ is co-analyzable in r steps, then S is co-analyzable in $r+1$ steps: use induction on r. Second, if the definable set $S \subseteq M^n$ is co-analyzable in r steps, then so is any definable subset of S, and the image $f(S)$ under any definable map $f\colon S \to M^m$. Third, if the definable sets $S_1, S_2 \subseteq M^n$ are co-analyzable in r_1 and r_2 steps, respectively, then $S_1 \cup S_2$ is co-analyzable in $\max(r_1, r_2)$ steps. Finally, if the definable sets $S_1 \subseteq M^{n_1}$ and $S_2 \subseteq M^{n_2}$ are co-analyzable in r_1 steps and r_2 steps, respectively, then $S_1 \times S_2 \subseteq M^{n_1+n_2}$ is co-analyzable in $r_1 + r_2$ steps. In any case, the class of co-analyzable definable sets is clearly very robust.

Next we extend the notion above to arbitrary $\boldsymbol{M}$, not necessarily ω-saturated. Let $S \subseteq M^n$ be definable. Define an **r-step co-analysis of S** by recursion on $r \in \mathbb{N}$ as follows: for $r = 0$ it is an $e \in \mathbb{N}$ with $|S| \leqslant e$. For $r = 1$ it is a tuple (e, R) with $e \in \mathbb{N}$ and definable $R \subseteq C \times M^n$ such that the natural projection $C \times M^n \to M^n$ maps R onto S, and $|R(c)| \leqslant e$ for all $c \in C$. Given $r \geqslant 1$, an $(r+1)$-step co-analysis of S is a tuple $(e, R_1, \dots, R_{r+1})$ with $e \in \mathbb{N}$ and definable sets

$$R_i \subseteq C \times M^n \times M^{d_i} \times \cdots \times M^{d_r} \quad (i = 1, \dots, r+1,\ d_1, \dots, d_r \in \mathbb{N}),$$

(so $R_{r+1} \subseteq C \times M^n$), such that the natural projection $C \times M^n \to M^n$ maps R_{r+1} onto S, and for each $c \in C$ there exists $b \in M^{d_r}$ for which the tuple $\big(e, R_1^b, \dots, R_r^b\big)$ is an r-step co-analysis of $R_{r+1}(c) \subseteq S$. (Here we use the following notation for a relation $R \subseteq P \times Q$: for $q \in Q$ we set $R^q := \{p \in P :\ (p, q) \in R\}$.)

For model-theoretic use the reader should note the following uniformity with respect to parameters from M^m: let $e, R_1, \dots, R_{r+1}, S$ be given with $e \in \mathbb{N}$, 0-definable $R_i \subseteq M^m \times C \times M^{d_i} \times \cdots \times M^{d_r}$ for $i = 1, \dots, r+1$, and 0-definable $S \subseteq M^m \times M^n$. Then the set of $a \in M^m$ such that $\big(e, R_1(a), \dots, R_{r+1}(a)\big)$ is an $(r+1)$-step co-analysis of $S(a)$ is 0-definable. Moreover, one can take a defining $\mathcal{L}$-formula for this subset of M^m that depends only on e and given defining $\mathcal{L}$-formulas for $R_1, \dots, R_{r+1}, S$, not on $\boldsymbol{M}$.

If $\boldsymbol{M}$ is ω-saturated, then a definable set $S \subseteq M^n$ can be shown to be co-analyzable in r steps iff there exists an r-step co-analysis of S. (To go from co-analyzable in r steps to an r-step co-analysis requires the uniformity noted above.) Thus for arbitrary $\boldsymbol{M}$ and definable $S \subseteq M^n$ we can define without ambiguity S to be co-analyzable in r steps if there exists an r-step co-analysis of S; likewise, S is defined to be co-analyzable if S is co-analyzable in r steps for some r. After the proof of Lemma 6.3 we give an example of a definable $S \subseteq \mathbb{T}$ that is co-analyzable in 2 steps but not in 1 step (relative to $\mathbb{T}$ and $\mathbb{R}$).

Let $S \subseteq M^n$ be definable and $\boldsymbol{M}^*$ an elementary extension of $\boldsymbol{M}$. We denote by $S^* \subseteq (M^*)^n$ the extension of S to $\boldsymbol{M}^*$: choose an $\mathcal{L}_M$-formula $\varphi(x)$, where $x = (x_1, \dots, x_n)$, with $S = \varphi^{\boldsymbol{M}}$, and set $S^* := \varphi^{\boldsymbol{M}^*}$. Then for a tuple $(e, R_1, \dots, R_{r+1})$ with $e, r \in \mathbb{N}$ and definable $R_i \subseteq C \times M^n \times M^{d_i} \times \cdots \times M^{d_r}$ for $i = 1, \dots, r+1$ we have: $(e, R_1, \dots, R_{r+1})$ is an $(r+1)$-step co-analysis of S iff $(e, R_1^*, \dots, R_{r+1}^*)$ is an $(r+1)$-step co-analysis of S^*. Here is [8, Proposition 2.4]:

Proposition 6.1. *Let the language $\mathcal{L}$ be countable and let T be a complete $\mathcal{L}$-theory such that $T \vdash \exists x C(x)$. Then the following conditions on an $\mathcal{L}$-formula $\varphi(x)$ with $x = (x_1, \dots, x_n)$ are equivalent:*

(i) *for some model $\boldsymbol{M}$ of T, $\varphi^{\boldsymbol{M}}$ is co-analyzable;*
(ii) *for every model $\boldsymbol{M}$ of T, $\varphi^{\boldsymbol{M}}$ is co-analyzable;*
(iii) *for every model $\boldsymbol{M}$ of T, if $C^{\boldsymbol{M}}$ is countable, then so is $\varphi^{\boldsymbol{M}}$;*
(iv) *for all models $\boldsymbol{M} \preccurlyeq \boldsymbol{M}^*$ of T, if $C^{\boldsymbol{M}} = C^{\boldsymbol{M}^*}$, then $\varphi^{\boldsymbol{M}} = \varphi^{\boldsymbol{M}^*}$.*

The equivalence (i) $\Leftrightarrow$ (ii) and the implication (ii) $\Rightarrow$ (iii) are clear from the above, and (iii) $\Rightarrow$ (iv) holds by Vaught's two-cardinal theorem [9, Theorem 12.1.1]. The contrapositive of (iv) $\Rightarrow$ (i) is obtained in [8] by an omitting types argument.

Application to $\mathbb{T}$. Let $\mathcal{L}$ be the language of ordered valued differential fields from Section 3, except that we consider it as having in addition a distinguished unary relation symbol C; an H-field is construed as an $\mathcal{L}$-structure as before, with C in addition interpreted as its constant field.

Let K be a Liouville closed $\boldsymbol{\omega}$-free newtonian H-field and $P \in K\{Y\}^{\neq}$. If $K \preccurlyeq K^*$ and K and K^* have the same constants, then P has the same zeros in K and K^*, by [1, Theorem 16.0.3]. Thus the zero set $\mathrm{Z}(P) \subseteq K$ is co-analyzable by Proposition 6.1 applied to the $\mathcal{L}_A$-theory $T := \mathrm{Th}(K_A)$ where A is the finite set of nonzero coefficients of P. In fact:

Proposition 6.2. *Let $S \subseteq K^n$ be definable, $S \neq \emptyset$. Then*
$$S \text{ is co-analyzable} \iff \dim S = 0.$$

Proof. Suppose $\dim S = 0$. Then for $i = 1, \dots, n$ and the ith coordinate projection $\pi_i \colon K^n \to K$ we have $\dim \pi_i(S) = 0$, and thus $\pi_i(S) \subseteq \mathrm{Z}(P_i)$ with $P_i \in K\{Y\}^{\neq}$. Since each $\mathrm{Z}(P_i)$ is co-analyzable and $S \subseteq \mathrm{Z}(P_1) \times \cdots \times \mathrm{Z}(P_n)$, we conclude that S is co-analyzable. Conversely, assume that S is co-analyzable, say in r steps. To get $\dim S = 0$ we can arrange that K is ω-saturated. Using $\dim C = 0$ and induction on r it follows easily from the behavior of dimension in definable families (Theorem 0.1) that $\dim S = 0$. $\square$

Let $\dim_C S$ be the least $r \in \mathbb{N}$ such that S is co-analyzable in r steps, for nonempty definable $S \subseteq K^n$ with $\dim S = 0$ (and $\dim_C \emptyset := -\infty$). It is easy to show that $\dim_C S$ coincides with the usual semialgebraic dimension of S (with respect to the real closed field C) when $S \subseteq C^n$ is semialgebraic. In general, $\dim_C S$ behaves much like a dimension function, and it would be good to confirm this by showing for example that for definable $S_i \subseteq K^{n_i}$ with $\dim S_i = 0$ for $i = 1, 2$ we have
$$\dim_C S_1 \times S_2 \ = \ \dim_C S_1 + \dim_C S_2.$$
(We do know that the quantity on the left is at most that on the right.) Another question is whether $\dim_C \mathrm{Z}(P) \leqslant \mathrm{order}(P)$ for $P \in K\{Y\}^{\neq}$.

Towards the uniform finiteness property mentioned at the end of the introduction, we introduce a condition that is equivalent to co-analyzability.

Let K be ω-saturated and $S \subseteq K^n$ be definable. By recursion on $r \in \mathbb{N}$ we define what makes S **fiberable by C in r steps**: for $r = 0$ it means that S is finite; S is fiberable by C in $(r+1)$ steps iff there is a definable map $f \colon S \to C$ such that $f^{-1}(c)$ is fiberable by C in r steps for every $c \in C$.

Lemma 6.3. *S is co-analyzable in r steps iff S is fiberable by C in r steps.*

Proof. By induction on r. The case $r = 0$ is trivial. Assume S is co-analyzable in $(r+1)$ steps, so we have a definable $R \subseteq C \times K^n$ that is mapped onto S under the natural projection $C \times K^n \to K^n$ and such that $R(c)$ is co-analyzable in r steps for all r. For $s \in S$ the definable nonempty set $R^s \subseteq C$ is a finite union of intervals and points, and so we can pick a point $f(s) \in R^s$ such that the resulting function $f\colon S \to C$ is definable. Then $f^{-1}(c) \subseteq R(c)$ is co-analyzable in r steps for all $c \in C$, and so fiberable by C in r steps by the inductive assumption. Thus f witnesses that S is fiberable by C in $(r+1)$ steps. The other direction is clear. $\square$

As an example, consider $S = \mathrm{Z}\big(YY''-(Y')^2\big)$. Then we have a definable (surjective) function $f\colon S \to C$ given by $f(y) = y^\dagger$ for nonzero $y \in S$, and $f(0) = 0$. For $c \in C^\times$ we take any $y \in S$ with $f(y) = c$, and then $f^{-1}(c) = C^\times y$; also $f^{-1}(0) = C$. Thus f witnesses that S is fiberable by C in two steps. Moreover, S is not fiberable by C in one step: if it were, Lemma 5.1 would make S parametrizable by constants, which we know is not the case.

An advantage of fiberability by C over co-analyzability is that for $f\colon S \to C$ and $R \subseteq C \times S$ witnessing these notions the fibers $f^{-1}(c)$ in $S = \bigcup_c f^{-1}(c)$ are pairwise disjoint, which is not necessarily the case for the sections $R(c)$ in $S = \bigcup_c R(c)$. Below we use the equivalence

$$S \text{ is finite} \iff f(S) \text{ is finite and every fiber } f^{-1}(c) \text{ is finite.}$$

to obtain the uniform finiteness property mentioned at the end of the introduction. We state this property here again in a slightly different form, with K any Liouville closed $\boldsymbol{\omega}$-free newtonian H-field:

Proposition 6.4. *Let $D \subseteq K^m$ and $S \subseteq D \times K^n$ be definable. Then there exists an $e \in \mathbb{N}$ such that $|S(a)| \leqslant e$ whenever $a \in D$ and $S(a)$ is finite.*

Proof. We first consider the special case that $n = 1$ and $S(a) \subseteq C$ for all $a \in D$. By [1, 16.0.2(ii)] a subset of C is definable in K iff it is semialgebraic in the sense of C. Thus $S(a)$ is finite iff it doesn't contain any interval (b, c) in C with $b < c$ in C; the uniform bound then follows by a routine compactness argument. Next we reduce the general case to this special case.

First, using Proposition 1.6 we shrink D to arrange that $\dim S(a) = 0$ for all $a \in D$. Next, we arrange that K is ω-saturated, so $S(a)$ is fiberable by C for every $a \in D$. Saturation allows us to reduce further to the case that for a fixed $r \in \mathbb{N}$ every section $S(a)$ is fiberable by C in $(r+1)$ steps. We now proceed by induction on r. Model-theoretic compactness yields a definable function $f\colon S \to C$ such that for every $a \in D$ the function $f_a\colon S(a) \to C$ given by $f_a(s) = f(a, s)$ witnesses that $S(a)$ is fiberable by C in $(r+1)$ steps, that is, $f_a^{-1}(c)$ is fiberable by C in r steps for all $c \in C$.

Inductively we have $e \in \mathbb{N}$ such that $|f_a^{-1}(c)| \leqslant e$ whenever $a \in D, c \in C$, and $f_a^{-1}(c)$ is finite. The special case we did in the beginning of the proof gives $d \in \mathbb{N}$ such that $|f_a(S(a))| \leqslant d$ whenever $a \in D$ and $f_a(S(a))$ is finite. For $a \in D$ we have $S(a) = \bigcup_c f_a^{-1}(c)$, so if $S(a)$ is finite, then $|S(a)| \leqslant de$. $\square$

To fully justify the use of saturation/model-theoretic compactness in the proof above requires an explicit notion of "r-step fibration by C" (analogous to that of "r-step co-analysis") that makes sense for any K, not necessarily ω-saturated. We leave this to the reader, and just note a nice consequence: if $S \subseteq K^n$ is definable, infinite, and $\dim S = 0$, then S has the same cardinality as C. (This reduces to the fact that any infinite semialgebraic subset of C has the same cardinality as C.) In particular, there is no countably infinite definable set $S \subseteq \mathbb{T}$.

As an application of the material above we show that the differential field K does not eliminate imaginaries. More precisely:

Corollary 6.5. *No definable map $f\colon K^{\times} \to K^n$ is such that for all $a, b \in K^{\times}$,*

$$a \asymp b \iff f(a) = f(b).$$

Proof. By [1, Lemmas 16.6.10, 14.5.10] there exists an elementary extension of K with the same constant field C as K and whose value group has greater cardinality than C. Suppose $f\colon K^{\times} \to K^n$ is definable such that for all $a, b \in K^{\times}$ we have: $a \asymp b \Leftrightarrow f(a) = f(b)$. We can arrange that the value group of K has greater cardinality than C, and so $f(K^{\times}) \subseteq K^n$ has dimension > 0. Every fiber $f^{-1}(p)$ with $p \in f(K^{\times})$ is a nonempty open subset of $K^{\times}$, so has dimension 1, and thus $\dim K^{\times} > 1$ by d-boundedness of K, a contradiction. $\square$

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