Solving linear differential equations over H-fields

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Overview.

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1. Differential rings and fields; linear differential operators.

A **differential ring** is a ring K (here: always commutative and containing $\mathbb{Q}$) equipped with a **derivation** D, i.e., a map $D\colon K \to K$ satisfying

$$D(f+g) = D(f) + D(g), \quad D(fg) = fD(g) + gD(f) \qquad (f, g \in K).$$

Usually write, for $f \in K$:

$$f' = D(f), f'' = D^2(f),\ \dots,\ f^{(n)} = D^n(f), n > 0.$$

A **differential field** is a differential ring K whose underlying ring is a field. In this case $C = C_K := \{f \in K : f' = 0\}$ forms a subfield of K, called the **constant field** of K.

For a nonzero element f of a differential field put

$$f^\dagger := f'/f \qquad \text{(the **logarithmic derivative of** } f\text{)}.$$

Let K be a differential field. We put

$$K[D] = \text{the ring of linear differential operators over } K.$$

Formally, $K[D]$ is a ring with 1 containing K as a subring (with the same 1), with a distinguished element D, such that $K[D]$, as a left-module over K, is free with basis

$$D^0, D^1, D^2, \ldots, \qquad \text{with } D^0 = 1,\ D^1 = D,\ D^m \neq D^n \text{ for } m \neq n,$$

and such that $Da = aD + a'$ for all $a \in K$.

Every $A \in K[D]$ can be written as

$$A = a_0 + a_1 D + \cdots + a_n D^n \quad (a_0, \ldots, a_n \in K).$$

If $a_n \neq 0$, then we say that A has **order** n. Put $\mathrm{order}(0) := -\infty$. Then

$$\mathrm{order}(AB) = \mathrm{order}(A) + \mathrm{order}(B) \quad \text{for all } A, B \in K[D].$$

Let R be a differential ring extension of K. With A as above we obtain a C-linear operator

$$y \mapsto A(y) := a_0 y + a_1 y' + \cdots + a_n y^{(n)} \colon R \to R.$$

Multiplication in $K[D] \longleftrightarrow$ composition of C-linear operators:

$$(AB)(y) = A\big(B(y)\big) \quad \text{for } A, B \in K[D] \text{ and } y \in R.$$

One calls A of positive order **irreducible** if there are no $A_1, A_2 \in K[D]$ of positive order with $A = A_1 A_2$.

The kernel of $A \in K[D]$ acting as C-linear operator on K,

$$\ker A := \big\{y \in K : A(y) = 0\big\},$$

is a C-linear subspace of K of dimension $\leqslant n$ if $0 \leqslant \operatorname{order} A \leqslant n$.

Division with remainder. *For* $A, B \in K[D]$, $B \neq 0$, *there exist unique* $Q, R \in K[D]$ *with* $A = QB + R$ *and* $\operatorname{order} R < \operatorname{order} B$.

As a consequence we obtain: *Let* $A \in K[D]$ *be of order* $n > 0$, *and* $u^\dagger = a \in K$ *with* $u \neq 0$ *from some differential field extension of* K. *Then*

$$A = B \cdot (D - a) \text{ for some } B \in K[D] \quad \Longleftrightarrow \quad A(u) = 0.$$

Proof. Write $A = B \cdot (D - a) + b$, $B \in K[D]$, and note $A(u) = bu$. $\square$

Here is another useful fact about zeros of differential operators:

Suppose that K *is real closed, and* u *is a nonzero element in a differential field extension of* $K(i)$, $i^2 = -1$, *such that* $u^\dagger \in K(i)$. *Then*

$$B(u) = 0 \qquad \text{for some } B \in K[D] \text{ of order } 2.$$

2. Differential valuations.

Let K be a differential field, and let

$$f \mapsto v(f) = vf \colon K^\times = K \setminus \{0\} \to \Gamma$$

be a (Krull) valuation of K, extended to K by $v(0) := \infty > \Gamma$. We put

$$\mathcal{O} := \{f \in K : vf \geqslant 0\} \quad \text{(the valuation ring of } v\text{)},$$

$$\mathfrak{m} := \{f \in K : vf > 0\} \quad \text{(the maximal ideal of } \mathcal{O}\text{)}.$$

Definition (Rosenlicht)**.** The valuation v is called a **differential valuation** of K (and the pair (K, v) a **differential-valued field**) if

(1) for all $f, g \in K^\times$ with $vf, vg \neq 0$: $vf \leqslant vg \iff v(f') \leqslant v(g')$;

(2) v is trivial on C, and $\mathcal{O} = C + \mathfrak{m}$.

Example. Suppose K is an H-field. The valuation with valuation ring $\mathcal{O}$ = the convex hull of C in K, is a differential valuation of K.

Example. Let C be a field of characteristic zero. Equip

$$K = C[[x^{\mathbb{Z}}]] = \text{the field of Laurent series in } x^{-1} \text{ over } C$$

with the derivation $D = \frac{d}{dx}$, with constant field C. For $f \in K$ written as

$$f = a_r x^r + a_{r-1} x^{r-1} + \cdots \qquad (a_r, a_{r-1}, \cdots \in C,\ a_r \neq 0,\ r \in \mathbb{Z})$$

put $vf := -r$. Then $v\colon K^\times \to \mathbb{Z}$ is a differential valuation of K.

Fact. (Rosenlicht.) *If K is a differential-valued field, then so is the algebraic closure K^{a} of K (with the unique extension of D to a derivation of K^{a} and any extension of v to a valuation of K^{a}).*

Let K be a differential-valued field. Sometimes it is useful to work with the **dominance relations** $\preccurlyeq$, $\prec$, $\asymp$, $\ldots$ on K associated to v, rather than with v directly:

$$f \preccurlyeq g \quad :\Longleftrightarrow \quad vf \geqslant vg \qquad (g \textbf{ dominates } f)$$

$$f \prec g \quad :\Longleftrightarrow \quad vf > vg \qquad (f \text{ can be } \textbf{neglected} \text{ with respect to } g)$$

$$f \asymp g \quad :\Longleftrightarrow \quad vf = vg \qquad (f \text{ and } g \text{ are } \textbf{asymptotic}).$$

Terminology:

$f \prec 1$: f is *infinitesimal*

$f \succ 1$: f is *infinite*

$f \preccurlyeq 1$: f is *finite* (or *bounded*).

Axiom (2) reads: $c \asymp 1$ for all $c \in C^{\times}$, and for every $f \preccurlyeq 1$ in K there exists $c \in C$ with $f - c \prec 1$.

By Axiom (1), for all $f \in K^\times$ with $vf \neq 0$, the value $v(f')$ *depends only on* vf. So the derivation of K induces a function

$$\psi\colon \Gamma^* = \Gamma \setminus \{0\} \to \Gamma, \qquad \psi(vf) := v(f^\dagger) = v(f') - v(f).$$

The pair (Γ, ψ) is called the **asymptotic couple** of K. (Rosenlicht.)

We say that the asymptotic couple (Γ, ψ) of K is **of H-type** if

$$0 < \alpha \leqslant \beta \Rightarrow \psi(\alpha) \geqslant \psi(\beta) \qquad \text{for all } \alpha, \beta \in \Gamma.$$

The asymptotic couple of an H-field is of H-type.

We say that K **preserves infinitesimals** if

$$f \prec 1 \Rightarrow f' \prec 1 \qquad \text{for all } f \in K.$$

(Can always achieved by replacing D by aD for suitable $a \in K^\times$.)

Other (less obvious) consequences of Axiom (1):

- for all $f, g \in K^\times$ with $vf, vg > 0$:

$$\psi(vf) < v(g') = (\mathrm{id} + \psi)(vg);$$

- there is at most one $\beta \in \Gamma$ with $\beta \notin (\mathrm{id} + \psi)(\Gamma^*)$;

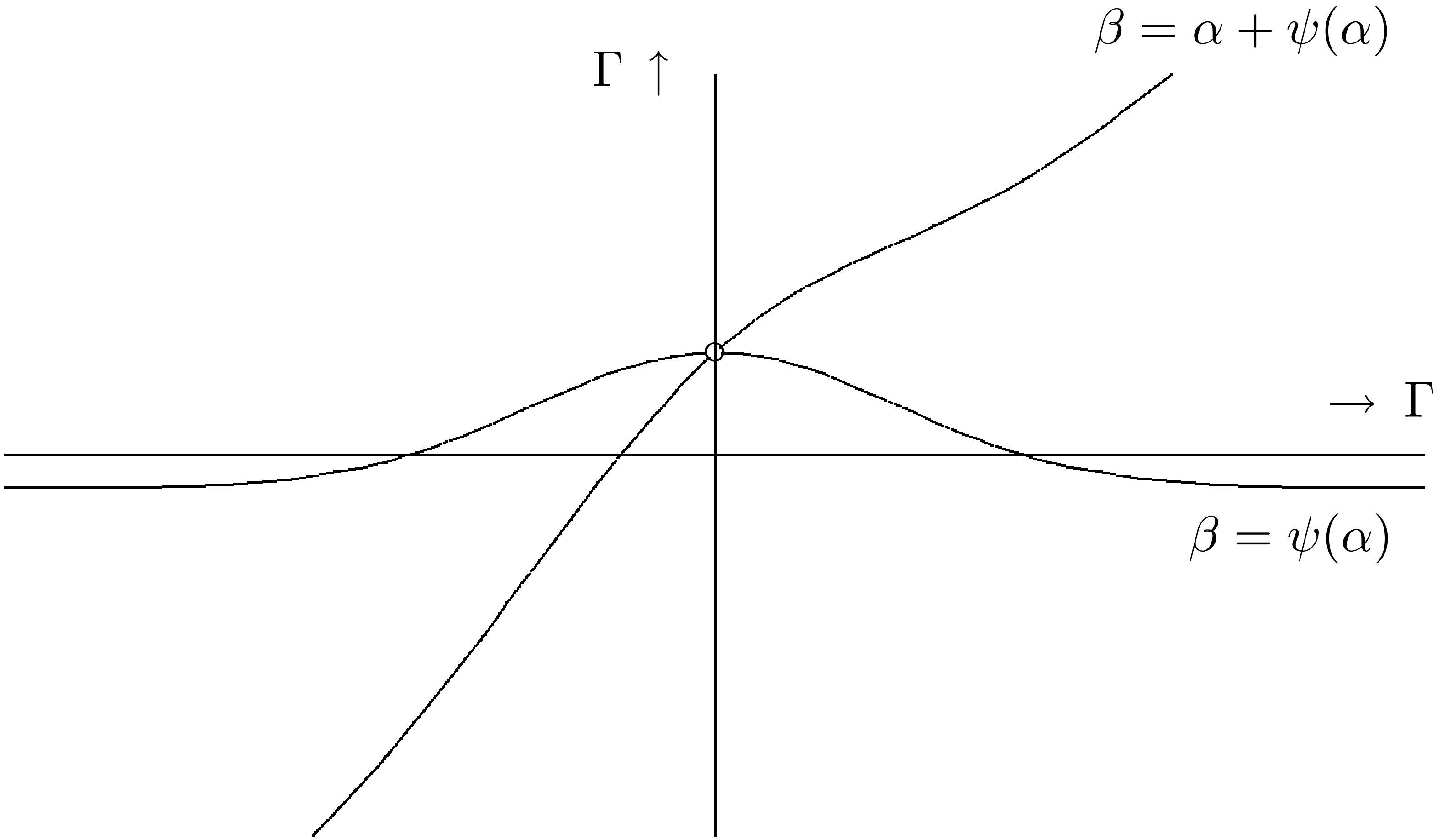
- if (Γ, ψ) is of H-type, then

$$\beta \in \Gamma \setminus (\mathrm{id} + \psi)(\Gamma^*) \iff \Psi < \beta < (\mathrm{id} + \psi)\big(\Gamma^{>0}\big), \text{ or } \beta = \max \Psi.$$

Here

$$\Psi := \big\{\psi(\gamma) : \gamma \in \Gamma^*\big\}.$$

Example. Suppose $K = C[[x^{\mathbb{Z}}]]$. Then $\Psi = \{-vx\}$.


$\beta = \alpha + \psi(\alpha)$
$\Gamma \uparrow$
$\rightarrow \Gamma$
$\beta = \psi(\alpha)$

3. Linear differential operators over differential-valued fields.

Let K be a differential-valued field whose asymptotic couple (Γ, ψ) is of H-type, with $\Gamma \neq \{0\}$. Let

$$A = a_0 + a_1 D + \cdots + a_n D^n \in K[D], \quad a_0, \ldots, a_n \in K, \quad a_n \neq 0.$$

We write

$$v(A) := \min_i v(a_i) \quad \text{(the Gauss valuation of } A\text{)}$$
$$\mu(A) := \min \big\{ i : v(a_i) = v(A) \big\}.$$

Fact. *For each $y \in K^\times$, $v(Ay)$ and $\mu(Ay)$ only depend on vy.*

Hence we get induced functions

$$vy \mapsto v_A(vy) := v(Ay) \colon \Gamma \to \Gamma,$$
$$vy \mapsto \mu_A(vy) := \mu(Ay) \colon \Gamma \to \{0, \ldots, n\}.$$

Some basic facts about v_A and μ_A:

Theorem.

- *The map* $v_A\colon \Gamma \to \Gamma$ *is an order-preserving bijection.*
- *Suppose that* Ψ *has a supremum in* Γ. *Then* $\mu_A(\gamma) = 0$ *for all but finitely many* γ*; in fact,* $\sum_\gamma \mu_A(\gamma) \leqslant n$.

Note also that for $y \in K$ we have

$$Ay = A(y) + (\cdots)D + (\cdots)D^2 + \cdots + a_n D^n,$$

hence $v_A(vy) = v\big(A(y)\big) \iff \mu_A(vy) = 0$. Put

$$\mathscr{E}(A) := \big\{\gamma \in \Gamma : \mu_A(\gamma) > 0\big\}.$$

Note: $A(y) = 0 \Rightarrow vy \in \mathscr{E}(A)$, so $\dim_C \ker A \leqslant |\mathscr{E}(A)|$.

Ingredients in the proof: *Newton diagrams* and *Riccati polynomials*.

Newton diagrams. Suppose $P(Z) = a_0 + a_1 Z + \cdots + a_n Z^n \in K[Z]$ is an ordinary polynomial over K, $a_n \neq 0$. The **Newton diagram** of P is

$$\mathcal{N}(P) := \big\{\big(i, v(a_i)\big) : 0 \leqslant i \leqslant n, a_i \neq 0\big\} \subseteq \mathbb{Z} \times \Gamma.$$

An **approximate zero** of P is an element $z \in K$ such that

$$P(z) \prec a_i z^i \qquad \text{for all } i.$$

Studying how $\mathcal{N}(P)$ changes when passing from $P(Z)$ to

$$P(Z + \phi) = P_{+\phi}(Z),$$

where ϕ is an approximate zero of P, one obtains a piecewise uniform description of $z \mapsto v\big(P(z)\big)$ in terms of functions of the form

$$z \mapsto v(z - \theta), \qquad \theta \in K,$$

provided K is *henselian* as valued field.

Riccati polynomials. For every n there exists $R_n(Z) \in \mathbb{Q}\{Z\}$ such that

$$\frac{y^{(n)}}{y} = R_n(z) \qquad \textit{for } y \in K^\times,\ z = y^\dagger.$$

Examples. $R_0(Z) = 1$, $R_1(Z) = Z$, $R_2(Z) = Z^2 + Z'$, $\dots$

We associate to A its **Riccati polynomial**

$$\operatorname{Ri} A := a_0 R_0 + a_1 R_1 + \cdots + a_n R_n \in K\{Z\}$$

and its **Newton diagram** $\mathcal{N}(A) := \mathcal{N}(P)$ where

$$P(Z) := a_0 + a_1 Z + \cdots + a_n Z^n \in K[Z].$$

We have, for $y \in K^\times$, $z = y^\dagger$:

- $A(y)/y = (\mathrm{Ri}\,A)(z)$;
- $\mathrm{Ri}(Ay) = y\,\mathrm{Ri}(A)_{+z}$;
- $v(A) = v\big(\mathrm{Ri}(A)\big)$.

An element z of K is an **approximate zero** of $\mathrm{Ri}\,A$ if

$$(\mathrm{Ri}\,A)(z) \prec a_i R_i(z) \qquad \text{for all } i.$$

Fact. *For $z \succcurlyeq 1$, we have:*

z is an approximate zero of $\mathrm{Ri}\,A \iff z$ is an approximate zero of P.

This leads to a piecewise uniform description of $z \mapsto v\big((\mathrm{Ri}\,A)_{+z}\big)$ in terms of functions of the form

$$z \mapsto v(z-\theta), \qquad \theta \in K.$$

4. Factorization theorems for linear differential operators.

Let K be a differential-valued field. We say that

- K is 1-**maximal** if K is henselian, and whenever $A(f) = g$ with $A \in K[D]$ of order 1, $g \in K$, and f in an immediate differential-valued field extension of K, then $f \in K$;
- for $n \geqslant 2$, K is said to be n-**maximal** if K is $(n-1)$-maximal, and whenever $A(f) = 0$ with $A \in K[D]$ of order n and f in an immediate differential-valued field extension of K, then $f \in K$;
- K is ∞-maximal if K is n-maximal for all $n > 0$.

By Zorn, K has an immediate differential-valued field extension L which has no proper immediate differential-valued field extension; such an L is ∞-maximal. In particular, maximally valued $\Rightarrow$ ∞-maximal; e.g., $\mathbb{R}[[x^{\mathbb{Z}}]]$ and $\mathbb{C}[[x^{\mathbb{Z}}]]$ are ∞-maximal.

The differential-valued subfield $\mathbb{R}\{\{x^{\mathbb{Z}}\}\}$ of $\mathbb{R}[[x^{\mathbb{Z}}]]$ is *not* 1-maximal.

From now on assume that the asymptotic couple (Γ, ψ) of K is of H-type, with $\Gamma \neq \{0\}$.

Theorem. *Suppose K is algebraically closed and n-maximal, $n > 0$, and Ψ has a supremum in Γ. Then each $A \in K[D]$ of order n is a product $A = A_1 \cdots A_n$ with all $A_i \in K[D]$ of order 1, and*

$$\dim_C \ker A = \dim_C K/A(K).$$

Main problem in the proof. We do not know whether K has an immediate differential-valued field extension that is maximally valued.

Corollary. *Suppose K is real closed, its algebraic closure is n-maximal, $n > 0$, and Ψ has a supremum in Γ. Then each $A \in K[D]$ of order n is a product $A_1 \cdots A_m$ with all $A_i \in K[D]$ irreducible of order 1 or order 2.*

5. Complete solution of $A(y) = g$.

Let K be a differential-valued field whose asymptotic couple (Γ, ψ) is of H-type, $\Gamma \neq \{0\}$, and $\sup \Psi = 0$. Then there is no $y \in K$ with $y' = 1$. However, we can adjoin a solution of this equation to K:

Let $K(x)$ be a field extension of K with x transcendental over K. There is a unique pair consisting of a derivation of $K(x)$ and a valuation ring of $K(x)$ that makes $K(x)$ a differential-valued field extension of K such that $x' = 1$ and $x \succ 1$.

Suppose now that K is algebraically closed and ∞-maximal, and let $A \in K[D]$ have order $n > 0$.

Theorem. *There exists a C-linear operator $A^{-1}\colon K[x] \to K[x]$ such that for all $h \in K[x]$:*

$$A\big(A^{-1}(h)\big) = h, \quad \deg_x A^{-1}(h) \leqslant \deg_x h + \sum_{\gamma} \mu_A(\gamma).$$

As to solving the homogeneous equation $A(y) = 0$ in $K[x]$, we have:

Theorem. *Let $\alpha_1 > \cdots > \alpha_r$ be the distinct elements of $\mathscr{E}(A)$. Then there are $h_{ij} \in K[x]$ for $1 \leqslant i \leqslant r$ and $0 \leqslant j < \mu_A(\alpha_i)$ such that*

$$A(h_{ij}) = 0, \quad v(h_{ij}) = \alpha_i + j \cdot vx, \quad \deg_x h_{ij} < \sum_\gamma \mu_A(\gamma).$$

Each such family (h_{ij}) is a basis of the C-linear space

$$\ker_x A := \{h \in K[x] : A(h) = 0\}.$$

In particular

$$\sum_\gamma \mu_A(\gamma) = \mu_A(\alpha_1) + \cdots + \mu_A(\alpha_r) = \dim_C \ker_x A.$$

Let

$$\mathscr{L} := \{y^\dagger : y \in K^\times\} \qquad \text{(a $\mathbb{Q}$-linear subspace of K)}$$

and let Q be a $\mathbb{Q}$-linear subspace Q of K such that $K = \mathscr{L} \oplus Q$. Let

$$q \mapsto \mathrm{e}(q) \colon Q \xrightarrow{\cong} \mathrm{e}(Q)$$

be a multiplicatively written copy of Q, and let

$$x^{\mathbb{N}} := \{x^n : n \in \mathbb{N}\} \subseteq K(x).$$

Equip

$$U := K\big[\,\mathrm{e}(Q) \cdot x^{\mathbb{N}}\big]$$

with the unique derivation extending the one on $K[x]$ and satisfying $\mathrm{e}(q)' = q\,\mathrm{e}(q)$ for all $q \in Q$. (Think of $\mathrm{e}(q)$ as $\exp(\int q)$.)

Proposition. *There are C-linearly independent $h_1, \dots, h_n \in U$ with*

$$A(h_i) = 0, \quad \deg_x h_i < n \qquad \text{for } i = 1, \dots, n.$$

6. Examples.

Let K be a differential-valued field with asymptotic couple of H-type.

Corollary (to the factorization theorem)**.** *Suppose K is a directed union of maximally valued differential-valued subfields F whose Ψ_F has a supremum in Γ_F.*

- *If K is algebraically closed, then every $A \in K[D]$ of degree $n > 0$ is a product $A = A_1 \cdots A_n$ with all $A_i \in K[D]$ of degree 1.*
- *If K is real closed, then every $A \in K[D]$ of positive degree is a product $A = A_1 \cdots A_m$ with all $A_i \in K[D]$ of degree 1 or degree 2.*

In particular, for $K = \mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$, we obtain: every $A \in K[D]$ of positive degree is a product $A = A_1 \cdots A_m$ with all $A_i \in K[D]$ of degree 1 or degree 2. The $\mathbb{R}$-linear map $y \mapsto A(y) \colon K \to K$ is surjective.

The following differential-valued fields are ∞-maximal:

- $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$, as well as its algebraic closure $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}(i)$;
- the real closure $\mathrm{P}(\mathbb{R})$ of $\mathbb{R}[[x^{\mathbb{Z}}]]$ (= the field of Puiseux series in x^{-1} with real coefficients);
- the algebraic closure $\mathrm{P}(\mathbb{C})$ of $\mathbb{C}[[x^{\mathbb{Z}}]]$ (= the field of Puiseux series in x^{-1} with complex coefficients);
- every existentially closed H-field.

(If K is henselian, and K has a differential-valued field extension, algebraic over K, which is n-maximal, $n > 0$, then K is n-maximal.)

7. Uniqueness questions.

Let K be a differential-valued field with asymptotic couple (Γ, ψ) of H-type, $\Gamma \neq \{0\}$, and $\sup \Psi = 0$.

Question. *Let $n > 0$. Is there an immediate differential-valued field extension M of K which is n-maximal, and such that for every immediate n-maximal differential-valued field extension L of K there exists an embedding $M \to L$ which is the identity on K?*

The answer is "no" even for $n = 1$:

Example. Suppose $K =$ the real closure of the H-subfield $\mathbb{R}(e^x, e^{e^x})$ of $\mathbb{R}[[x^{\mathbb{R}}]]^{\mathrm{LE}}$. Then $a' - e^{e^x} \succ 1$ for all $a \in K$. For every 1-maximal differential-valued field extension M of K there exists an immediate 1-maximal differential-valued field extension L of K such that there is *no* embedding $M \to L$ which is the identity on K.

We say that K is **closed under logarithmic integration** if for all $s \in K$ there is $y \in K^\times$ with $y^\dagger = s$. For $n > 0$, we say that K is **strongly n-maximal** if the algebraic closure of K is n-maximal.

Suppose that $\sup \Psi = 0$, and K is equipped with an ordering making it a real closed H-field, with algebraic closure $K(i)$, $i^2 = -1$.

Theorem. *Suppose* $\max \Psi = 0$. *There exists an H-field extension M of K with the following properties:*

(1) $\max \Psi_M = 0$ *and* $C_M = C$;

(2) *M is real closed, strongly* 1*-maximal, and closed under logarithmic integration;*

(3) *no proper real closed H-subfield of M contains K and is strongly* 1*-maximal and closed under logarithmic integration.*

For each M with these properties and each existentially closed H-field extension E of K there is an embedding $M \to E$ that is the identity on K.

The theorem remains true if "strongly 1-maximal" is replaced by "1-maximal", and "existentially closed" by "Liouville closed."

The H-field K is strongly 1*-maximal if and only if*

(1) *for each $\varepsilon \prec 1$ in K there are $y, z \prec 1$ in K with*

$$y' = \varepsilon, \qquad (1+z)^\dagger = \varepsilon;$$

(2) *for all $\varepsilon \prec 1$ in K there are $y_1, y_2 \prec 1$ in K with*

$$(1 + y_1 + y_2 i)^\dagger = \varepsilon i$$

(think of $y_1 = -1 + \cos \int \varepsilon$ and $y_2 = \sin \int \varepsilon$);

(3) *for every $g \in K$ there is $a \in K$ with $a' - g \prec 1$;*

(4) *for every $A \in K^{\mathrm{a}}[D]$ of degree 1 with $\mathscr{E}(A) = \emptyset$ and every $g \in K$ there exists $f \in K^{\mathrm{a}}$ with $A(f) = g$.*

Also: K 2*-maximal* $\Rightarrow$ K *strongly* 1*-maximal* $\Rightarrow$ K 1*-maximal.*