JULIA'S EQUATION AND DIFFERENTIAL TRANSCENDENCE

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Abstract. We show that the iterative logarithm of each non-linear entire function is differentially transcendental over the ring of entire functions, and we give a sufficient criterion for such an iterative logarithm to be differentially transcendental over the ring of convergent power series. Our results apply, in particular, to the exponential generating function of a sequence arising from work of Shadrin and Zvonkine on Hurwitz numbers.

1. Introduction and Main Results

In 1871, Schröder [34] suggested to study the iteration of a meromorphic function f by using the functional equation

$$\phi(\lambda z) = f(\phi(z)) \tag{1.1}$$

that now bears his name. If f satisfies this equation, then the compositional iterates f^n of f satisfy $\phi(\lambda^n z) = f^n(\phi(z))$, so in principle we have an "explicit" expression for the iterates of f in terms of ϕ and its inverse function. Schröder gave various examples, e.g., $\phi(z) = \tanh z$, $\lambda = 2$ and $f(z) = 2z/(1+z^2)$, as well as Jacobian elliptic functions ϕ which satisfy (1.1) for certain rational functions f.

Kœnigs [25] considered the case that f is holomorphic in a neighborhood of a fixed point ξ and showed that if the *multiplier* $\lambda = f'(\xi)$ of f at ξ satisfies $\lambda \neq 0$ and $|\lambda| \neq 1$, then (1.1) has a unique solution ϕ holomorphic in a neighborhood of 0 such that $\phi(0) = \xi$ and $\phi'(0) = 1$. Poincaré [31, p. 318] observed that if $|\lambda| > 1$ and if f is rational, then ϕ extends to a function meromorphic in the plane, and if $|\lambda| > 1$ and f is entire, then ϕ is entire. Therefore the solution ϕ of (1.1) for $|\lambda| > 1$ is also called the *Poincaré function* of f at ξ.

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Schröder [34, p. 303] expressed the opinion that the functions f whose iterates can be determined using (1.1) are of rather special type. One may argue that the results of Kœnigs and Poincaré say the opposite, but support for Schröder's view is given by a result of Ritt [32] which implies that only very few Poincaré functions are elementary functions. In order to state Ritt's result, we recall that a holomorphic function or, more generally, a formal power series g, is said to be *differentially algebraic* if it satisfies an algebraic differential equation; that is, an equation of the form

$$P\big(z, g(z), g'(z), \dots, g^{(n)}(z)\big) = 0$$

where P is a non-zero polynomial in $2+n$ indeterminates (for some n) with constant coefficients; if g is not differentially algebraic, then g is called *differentially transcendental*. Ritt's result says that a polynomial with a differentially algebraic Poincaré function is conjugate to a monomial, a Chebychev polynomial, or the negative of a Chebychev polynomial, the corresponding Poincaré functions being the exponential or trigonometric functions. For rational functions there are additional cases arising from the multiplication theorems of elliptic functions. Poincaré functions of transcendental entire functions are always differentially transcendental [11].

A family of meromorphic functions is called *coherent* (or *uniformly differentially algebraic*) if there exists an algebraic differential equation which is satisfied by all functions in the family, and *incoherent* otherwise. Boshernitzan and Rubel [14, Theorem 6.1] showed that a Poincaré function of a rational or entire function f is differentially algebraic if and only if the family of iterates of f is coherent. Thus the above results about differential transcendence of Poincaré functions can be rephrased as results about incoherence of iterates.

We now turn to the case where the multiplier $\lambda = f'(\xi)$ of f at its fixed point ξ does not satisfy the conditions $\lambda \neq 0$ and $|\lambda| \neq 1$ required for Kœnigs' theorem. If $|\lambda| = 1$, but λ is not a root of unity, Schröder's equation still has a formal power series solution. The question whether this series converges is rather delicate and forms the subject matter of famous results of Siegel, Brjuno and Yoccoz; see [30, Section 11] for a discussion. However, regardless of whether the series converges or not, it is differentially transcendental whenever f is a non-linear rational or entire function [8].

If $\lambda = 0$, then instead of Schröder's equation one considers Böttcher's equation. Again the solutions are differentially transcendental except in special cases [8].

Suppose now that λ is a root of unity. In this case the fixed point ξ is also called *parabolic*. Passing to an iterate of f we may assume that $\lambda = 1$. Assuming without loss of generality that $\xi = 0$ we then write f in the form

$$f(z) = z + \sum_{k=p}^{\infty} f_k z^k \quad (p \geq 2,\ f_k \in \mathbb{C} \text{ for } k \geq 2,\ f_p \neq 0). \tag{1.2}$$

A basic result of complex dynamics, called the Leau-Fatou flower theorem [21, 27], says that there are $p-1$ domains $L_1, \ldots, L_{p-1}$, called *petals* of f, such that $f(L_j) \subseteq L_j$ and the restriction $f^n \restriction L_j \to 0 \in \partial L_j$ as $n \to \infty$, for $j = 1, \ldots, p-1$. (See [30, Section 10].) Moreover, the Abel functional equation $\phi(z+1) = f(\phi(z))$ has a holomorpic solution ϕ_j mapping the right half-plane to L_j. The functions ϕ_j are again differentially transcendental [8].

A way to describe the iteration of f not only in the petals but in a full neighborhood of 0 is based on the functional equation

$$\phi(f(z)) = f'(z)\phi(z) \tag{1.3}$$

which is named after Julia (e.g., in [26, Sections 3.5B and 8.5A]) or Jabotinsky (e.g., in [1]). It has a unique formal power series solution

$$\phi(z) = f_p z^p + \sum_{k=p+1}^{\infty} \phi_k z^k \quad (\phi_k \in \mathbb{C} \text{ for } k \geq p+1), \tag{1.4}$$

which is called the *iterative logarithm* of f and denoted here by $\operatorname{itlog}(f)$. The name iterative logarithm, introduced by Écalle (see [16, p. 8] or [17]), is explained by the identity

$$\operatorname{itlog}(f^n) = n \operatorname{itlog}(f) \qquad \text{valid for all } n \in \mathbb{N}.$$

The general solution of (1.3) is given by $\phi = \alpha \operatorname{itlog}(f)$ where $\alpha \in \mathbb{C}$.

The series in (1.4) converges only in exceptional cases. For example, a result of Erdős and Jabotinsky [19] in combination with results of Baker [4] and Szekeres [39] shows that the only functions f meromorphic in $\mathbb{C}$ and of the form (1.2) for which the series in (1.4) converges in some neighborhood of 0 are the functions $f(z) = z/(1-cz)$ where $c \in \mathbb{C}$, with $\operatorname{itlog}(f)(z) = cz^2$. (However, Écalle [18] has shown that the iterative logarithm of a function f holomorphic in a neighborhood of 0 satisfying (1.2) is always Borel summable.)

It follows from the results in [11, 14, 32] that the iterative logarithm $\operatorname{itlog}(f)$ of a non-linear rational or entire function f is differentially transcendental; cf. the remarks at the end of section 2.2. This can be viewed as an indication that the coefficient sequence $(\phi_k)_{k>p}$ is very irregular: If a formal power series $y = \sum_k y_k z^k \in \mathbb{C}[[z]]$ is differentially algebraic, then the coefficient sequence (y_k) satisfies a certain (in general, non-linear) kind of recurrence relation [29, pp. 186–194]. Of particular importance in combinatorial enumeration is the class of *D-finite* (also called *holonomic*) power series [37, Chapter 6]. These are the formal power series whose coefficient sequence satisfies a homogeneous *linear* recurrence relation of finite degree with polynomial coefficients; equivalently [37, Proposition 6.4.3] those which satisfy a non-trivial *linear* differential equation over $\mathbb{C}[z]$.

In this paper, we show that for entire functions we have an even stronger irregularity result. To formulate this result we need some terminology: Given a subring R of the ring $\mathbb{C}[[z]]$ of formal power series over $\mathbb{C}$ which is closed under differentiation, we say that $\phi \in \mathbb{C}[[z]]$ is *differentially transcendental over* R if ϕ does not satisfy a non-trivial polynomial equation in ϕ and its derivatives with coefficients from R. (Thus "differentially transcendental" is synonymous with "differentially transcendental over $\mathbb{C}[z]$.")

Theorem 1. *Let* f *be a non-linear entire function of the form* (1.2)*. Then* $\operatorname{itlog}(f)$ *is differentially transcendental over the ring of entire functions.*

Under an additional hypothesis we can even show that $\operatorname{itlog}(f)$ is differentially transcendental over the ring $\mathbb{C}\{z\}$ of power series with positive radius of convergence. In order to state this hypothesis, for an entire function f we denote by $\operatorname{sing}(f^{-1})$ the set of singularities of the inverse function of f; see [10, Section 4.3] for a discussion of their role in complex dynamics. The set $\operatorname{sing}(f^{-1})$ coincides with the set of critical and (finite) asymptotic values of f. Here a point $w \in \mathbb{C}$ is called a *critical value* if there exists $\xi \in \mathbb{C}$ such that $f'(\xi) = 0$ and $f(\xi) = w$ while w is called an *asymptotic value* if there exists a curve $\gamma\colon [0,1) \to \mathbb{C}$ such that $\gamma(t) \to \infty$ and $f(\gamma(t)) \to w$ as $t \to 1$. If f is a polynomial, then we only have to consider critical values, since polynomials have no finite asymptotic values.

The *Speiser class* $\mathcal{S}$ consists of all non-linear entire functions f for which $\operatorname{sing}(f^{-1})$ is finite. It plays an important role in complex dynamics; cf. [10, 20].

The maximal domains U_j $(j = 1, \dots, p-1)$ containing the petals L_j such that $f(U_j) \subseteq U_j$ and $f^n \restriction U_j \to 0$ as $n \to \infty$ are called *Leau domains* of f. If $z \in U_j$, then $f^n(z) \in L_j$ for large n. A classical result of Fatou (see [10, Theorem 7] or [30, Theorem 10.15]) says that $U_j \cap \operatorname{sing}(f^{-1}) \neq \emptyset$ for all $j = 1, \dots, p-1$.

Theorem 2. *Let* $f \in \mathcal{S}$ *be of the form* (1.2)*. Denote by* $U_1, \dots, U_{p-1}$ *the associated Leau domains and suppose that*

$$\operatorname{sing}(f^{-1}) \subseteq \{0\} \cup \bigcup_{j=1}^{p-1} U_j. \tag{1.5}$$

Then $\operatorname{itlog}(f)$ *is differentially transcendental over* $\mathbb{C}\{z\}$.

Examples to which Theorem 2 applies are $f_1(z) = z + z^2$ and $f_2(z) = e^z - 1$. The function f_1 has only one critical point at $-1/2$ and $f(-1/2) = -1/4$ is the corresponding critical value. The function f_2 has the only asymptotic value -1 and no critical values. Thus $\operatorname{sing}(f_1^{-1}) = \{-1/4\}$ and $\operatorname{sing}(f_2^{-1}) = \{-1\}$. It follows from the result of Fatou mentioned above, or by direct computation, that f_1 and f_2 satisfy the hypothesis of Theorem 2.

Other examples are $f_3(z) = \sin z$ with $\operatorname{sing}(f_3^{-1}) = \{1, -1\}$ and two Leau domains at 0, one containing 1 and one containing -1, and $f_4(z) = ze^z$ with $\operatorname{sing}(f_4^{-1}) = \{0, -1/e\}$.

The results of [12, 15] imply that if $\operatorname{Re} a > 3/4$, then both critical points of $f(z) = z + z^2 + az^3$ are in the Leau domain at 0. Thus f satisfies the hypothesis of Theorem 2 if $\operatorname{Re} a > 3/4$. In fact, this even holds [15, p. 277] if $\operatorname{Re} a \geq 3/4 - 1/(2\log 3)$.

Theorem 2 suggests the following open question.

Question. Let f be any transcendental entire function of the form (1.2). Is $\operatorname{itlog}(f)$ differentially transcendental over $\mathbb{C}\{z\}$?

The iterative logarithm

$$\operatorname{itlog}(e^z - 1) = \frac{1}{2}z^2 - \frac{1}{12}z^3 + \frac{1}{48}z^4 - \frac{1}{180}z^5 + \frac{11}{8640}z^6 - \frac{1}{6720}z^7 + \cdots$$

of $f(z) = e^z - 1$ is of particular interest since it is the exponential generating function (egf) of a sequence

$$0, 0, 1, -\frac{1}{2}, \frac{1}{2}, -\frac{2}{3}, \frac{11}{12}, -\frac{3}{4}, -\frac{11}{6}, \frac{29}{4}, \frac{493}{12}, -\frac{2711}{6}, -\frac{12406}{15}, \frac{2636317}{60}, \ldots$$

of rational numbers which recently arose in a conjecture made by Shadrin and Zvonkine [36] (and proved in [2]) in connection with a generating series for Hurwitz numbers, and also in another context (ongoing joint work of the first-named author with van den Dries and van der Hoeven on asymptotic differential algebra [3]). By Theorem 2, its egf $\operatorname{itlog}(e^z - 1)$ is differentially transcendental over $\mathbb{C}\{z\}$. We do not know whether the *ordinary* generating function (ogf) of this sequence is differentially transcendental over $\mathbb{C}[z]$, let alone over $\mathbb{C}\{z\}$. (See [24] for some differential transcendence results over $\mathbb{C}\{z\}$ for ogf's of sequences of combinatorial origin.) We also do not know whether the coefficients ϕ_k of the power series $\phi = \operatorname{itlog}(e^z - 1) \in \mathbb{Q}[[z]]$ are non-zero for all $k \geq 3$. (A computation with MAPLE showed that $\phi_k \neq 0$ for $k = 3, \ldots, 300$.) Some general results about the coefficient sequence (ϕ_k) in the case where $\phi = \operatorname{itlog}(f) \in \mathbb{C}[[z]] \setminus \mathbb{C}\{z\}$ can be found in [23].

The idea of the proof of Theorem 1 is as follows. Assuming that $\operatorname{itlog}(f)$ is differentially algebraic over $\mathbb{C}\{z\}$, we start with a differential equation which is "minimal" in a certain sense; cf. §2.1. We then use the functional equation of the iterative logarithm, i.e., equation (1.3), to obtain a differential equation with meromorphic coefficients which is satisfied by f and all its iterates. This implies that a Poincaré function ψ associated to f also satisfies such a differential equation. Using a result of Steinmetz (Theorem 4 in §2.3) we deduce that ψ actually satisfies an algebraic differential equation with constant coefficients. This contradicts results about differentially algebraic Poincaré functions due to Ritt and the second author; see §2.2.

The idea of the proof of Theorem 2 is to assume that $\operatorname{itlog}(f)$ satisfies a differential equation with coefficients analytic in a neighborhood of 0 and then use inverse branches of f to continue these coefficients analytically to the whole plane. The conclusion then follows from Theorem 1. Some further remarks on the proof of Theorem 2 are made immediately after the proof.

Conventions and notations. Throughout the paper, i, j, m, n, p range over the set $\mathbb{N} = \{0, 1, 2, \dots\}$ of natural numbers.

2. Preliminaries

In this section we first introduce some basic terminology concerning differential polynomials used later. We then recall more basic facts on repelling periodic points and Poincaré functions, in addition to the ones already appearing in the introduction. In the last part of this section we state a theorem of Steinmetz which is at the heart of the proof of Theorem 1.

2.1. **Algebraic differential equations.** Let R be a *differential ring,* that is, a commutative ring (with 1) equipped with a *derivation* of R, i.e., a map $f \mapsto f' \colon R \to R$ which is additive and satisfies the Leibniz Rule:

$$(f+g)' = f' + g', \quad (f \cdot g)' = f \cdot g' + f' \cdot g \qquad \text{for all } f, g \in R.$$

We let $f \mapsto f^{(n)}$ denote the nth compositional iterate of $f \mapsto f'$. A subring S of R which is closed under $f \mapsto f'$ is called a *differential subring* of R, and in this case R is called a *differential ring extension* of S. For any $(r+1)$-tuple $\underline{i} = (i_0, \dots, i_r)$ of natural numbers and an element y in a differential ring extension of R, put

$$y^{\underline{i}} := y^{i_0} (y')^{i_1} \cdots (y^{(r)})^{i_r}.$$

Let Y be a differential indeterminate over R. Then $R\{Y\}$ denotes the ring of differential polynomials in Y over R (not to be confused with the ring $\mathbb{C}\{z\}$ of convergent power series with complex coefficients in the indeterminate z). As ring, $R\{Y\}$ is just the polynomial ring $R[Y, Y', Y'', \dots]$ in the distinct indeterminates $Y^{(n)}$ over R, where as usual we write $Y = Y^{(0)}$, $Y' = Y^{(1)}$, $Y'' = Y^{(2)}$, etc. We consider $R\{Y\}$ as the differential ring whose derivation extends the derivation of R and satisfies $(Y^{(n)})' = Y^{(n+1)}$ for every n. For $P(Y) \in R\{Y\}$ and y an element of a differential ring extension of R, we let $P(y)$ be the element of that extension obtained by substituting $y, y', \dots$ for $Y, Y', \dots$ in P, respectively. We say that an element y of a differential ring extension of R is *differentially algebraic over* R if there is some $P \in R\{Y\}$, $P \neq 0$, such that $P(y) = 0$, and if y is not differentially algebraic over R, then y is said to be *differentially transcendental over* R.

For $P \in R\{Y\}$, the smallest $r \in \mathbb{N}$ such that $P \in R[Y, Y', \dots, Y^{(r)}]$ is called the *order* of the differential polynomial P. Let $P \in R\{Y\}$ have order r, and let $\underline{i} = (i_0, \dots, i_r)$ range over $\mathbb{N}^{1+r}$. We call $Y^{\underline{i}}$ a *monomial*, and denote by $P_{\underline{i}} \in R$ the coefficient of $Y^{\underline{i}}$ in P. Thus P can be uniquely written as

$$P = \sum_{\underline{i}} P_{\underline{i}} Y^{\underline{i}},$$

where the *support* of P, defined by

$$\operatorname{supp} P := \big\{ \underline{i} \colon P_{\underline{i}} \neq 0 \big\},$$

is finite. We say that a monomial $Y^{\underline{i}}$ *occurs* in P if $\underline{i} \in \operatorname{supp} P$. We set

$$|\underline{i}| := i_0 + \cdots + i_r, \qquad \|\underline{i}\| := i_1 + 2i_2 + \cdots + ri_r.$$

For $P \neq 0$ we call

$$\deg(P) = \max_{\underline{i} \in \operatorname{supp} P} |\underline{i}|, \qquad \operatorname{wt}(P) = \max_{\underline{i} \in \operatorname{supp} P} \|\underline{i}\|$$

the *degree* of P respectively the *weight* of P. We say that $P \neq 0$ is *homogeneous* if $|\underline{i}| = \deg(P)$ for every $\underline{i} \in \operatorname{supp} P$ and *isobaric* if $\|\underline{i}\| = \operatorname{wt}(P)$ for every $\underline{i} \in \operatorname{supp} P$.

For $r, s \in \mathbb{N}$ with $r \leq s$ we identify each $\underline{i} = (i_0, \ldots, i_r) \in \mathbb{N}^{1+r}$ with the tuple $(i_0, \ldots, i_r, 0, \ldots, 0) \in \mathbb{N}^{1+s}$ and thus view $\mathbb{N}^{1+r}$ as a subset of $\mathbb{N}^{1+s}$. We set $\mathbb{N}^* := \bigcup_{r \in \mathbb{N}} \mathbb{N}^{1+r}$ and order $\mathbb{N}^*$ anti-lexicographically; that is, for $\underline{i} = (i_0, i_1, \ldots)$ and $\underline{j} = (j_0, j_1, \ldots) \in \mathbb{N}^*$ we set

$$\underline{i} < \underline{j} \quad :\Longleftrightarrow \quad \text{there is some } k \in \mathbb{N} \text{ with } i_k < j_k \text{ and } i_l = j_l \text{ for } l \geq k+1,$$

and we set $\underline{i} \leq \underline{j} :\Longleftrightarrow \underline{i} < \underline{j}$ or $\underline{i} = \underline{j}$. It is easy to verify that $\leq$ is a well-ordering of $\mathbb{N}^*$, that is, $\leq$ is a linear ordering of $\mathbb{N}^*$, and every non-empty subset of $\mathbb{N}^*$ has a smallest element with respect to $\leq$. For $P \neq 0$ we let the *rank* $\underline{r} = \underline{r}(P)$ of P be the largest element of $\operatorname{supp} P$ with respect to $\leq$. Below $\underline{i}$, $\underline{j}$ range over $\mathbb{N}^*$. For $\underline{i} = (i_0, i_1, \ldots)$ and $\underline{j} = (j_0, j_1, \ldots)$ we put $\underline{i} + \underline{j} = (i_0 + j_0, i_1 + j_1, \ldots)$.

We view the ring $\mathbb{C}[[z]]$ of formal power series over $\mathbb{C}$ as a differential ring in the usual way (with derivation $\frac{d}{dz}$), and we work with two differential subrings of $\mathbb{C}[[z]]$: the differential subring $\mathbb{C}\{z\}$ of $\mathbb{C}[[z]]$ consisting of the convergent power series, and the smaller differential subring $\mathbb{C}\{z\}_\infty$ of $\mathbb{C}[[z]]$ consisting of the (Taylor series at 0 of) entire functions. Theorem 1 says that the iterative logarithm of a non-linear entire function is differentially transcendental over $\mathbb{C}\{z\}_\infty$, while Theorem 2 says that—under the hypotheses made on f—the iterative logarithm of f is differentially transcendental over $\mathbb{C}\{z\}$.

A differential field is a differential ring whose underlying ring happens to be a field. Sometimes we find it convenient to work in the differential field of meromorphic functions (which may be naturally identified with the fraction field of $\mathbb{C}\{z\}_\infty$, equipped with the unique derivation extending that of $\mathbb{C}\{z\}_\infty$). We note that a function is differentially algebraic over the field of meromorphic functions if and only if it is differentially algebraic over the ring of entire functions.

2.2. **Repelling periodic points and Poincaré functions.** Let f be a non-linear entire (or rational) function. A point $\xi \in \mathbb{C}$ is called a *periodic* point of f if there exists some $p \geq 1$ such that $f^p(\xi) = \xi$; the smallest such p is called the *period* of ξ. One calls a periodic point ξ of f with period p *repelling* if the *multiplier* $\lambda = (f^p)'(\xi)$ of f at ξ satisfies $|\lambda| > 1$.

The *Julia set* $J(f)$ of f is the of all points in the plane (or Riemann sphere) where the iterates of f do not form a normal family. A standard result of complex dynamics says that $J(f)$ is the closure of the set of repelling periodic points of f. For rational functions this was already proved by Fatou and Julia, by different methods (see [30, Section 14] for an exposition of both proofs), for transcendental entire functions it is due to Baker [5] (see [6, 13, 35] for simpler proofs). The Julia set of f is always non-empty (in fact, a perfect set).

As mentioned in the introduction, results of Kœnigs and Poincaré say that if ξ is a repelling periodic point of f with period p and multiplier λ, then there exists a function ψ holomorphic in a neighborhood of 0 such that $\psi(0) = \xi$, $\psi'(0) = 1$ and $\psi(\lambda z) = f^p(\psi(z))$, called the Poincaré function of f at ξ. If f is rational, then ψ is meromorphic in the plane, and if f is entire, then so is ψ. Moreover, ψ is given by (cf. [32, p. 670])

$$\psi(z) = \lim_{n\to\infty} f^{np}(\xi + \lambda^{-n} z). \tag{2.1}$$

Differentiating (2.1) we also obtain

$$\psi^{(m)}(z) = \lim_{n\to\infty} \lambda^{-mn} (f^{np})^{(m)}(\xi + \lambda^{-n} z) \quad \text{for each } m,$$

hence

$$\psi^{\underline{i}}(z) = \lim_{n\to\infty} \lambda^{-\|\underline{i}\| n} (f^{np})^{\underline{i}}(\xi + \lambda^{-n} z) \quad \text{for each } \underline{i} \tag{2.2}$$

and thus

$$(\psi')^{\underline{i}}(z) = \lim_{n\to\infty} \lambda^{-(|\underline{i}|+\|\underline{i}\|)n} \big((f^{np})'\big)^{\underline{i}}(\xi + \lambda^{-n} z) \quad \text{for each } \underline{i}. \tag{2.3}$$

As mentioned in the introduction, Ritt [32] determined all differentially algebraic Poincaré functions of rational functions. His result shows in particular that rational functions with differentially algebraic Poincaré functions have no parabolic fixed points, so there is no iterative logarithm associated to these functions. Moreover, it was shown in [11] that Poincaré functions to transcendental entire functions are differentially transcendental. Combining this with Ritt's result we obtain the following.

Theorem 3. *Let f be a non-linear rational or entire function with a parabolic fixed point. Then the Poincaré functions associated to the repelling fixed points of f are all differentially transcendental.*

Together with the results of Boshernitzan and Rubel [14] quoted earlier this implies the following result already mentioned in the introduction:

Corollary 1. *Let f be a non-linear rational or entire function. Then the iterative logarithm of f at each parabolic fixed point of f is differentially transcendental.*

Proof. Suppose 0 is a parabolic fixed point of f; it is enough to show that then $\operatorname{itlog}(f)$ is differentially transcendental. Assume otherwise. Then by [14, Theorem 6.4] (see also [2, Corollary 6.3]) there is a nonzero differential polynomial $P \in \mathbb{C}[z]\{Y\}$ such that $P(f^n) = 0$ for all n. Let $\zeta \in \mathbb{C}$ be a repelling periodic point of f, with period p. Replacing f by f^p we may assume that $p = 1$, so $f(\zeta) = \zeta$. Let $g(z) := f(z+\zeta) - \zeta$; then 0 is a repelling fixed point of g, and with $Q := P(Y + \zeta)$ we have $Q(g^n) = 0$ for each n. Let ψ be the Poincaré function of g at 0. By [14, Theorem 6.1], ψ^{-1} is differentially algebraic, hence so is ψ, contradicting Theorem 3. □

2.3. **A result of Steinmetz.** The following result is due to Steinmetz [38, Satz 1]. We denote by $T(r, f)$ the Nevanlinna characteristic of a meromorphic function f, and as usual in Nevanlinna theory, $S(r, f)$ denotes any term satisfying $S(r, f) = o\big(T(r, f)\big)$ as $r \to \infty$ outside some exceptional set of finite measure. See [22] as a reference for Nevanlinna theory.

Theorem 4. *Let $F_0, F_1, \ldots, F_m$ and $h_0, h_1, \ldots, h_m$ be not identically vanishing meromorphic functions and let g be a nonconstant entire function such that*

$$F_0(g)h_0 + F_1(g)h_1 + \cdots + F_m(g)h_m = 0.$$

Suppose that there exists a positive $K \in \mathbb{R}$ such that

$$\sum_{j=0}^{m} T(r, h_j) \le K\, T(r, g) + S(r, g).$$

Then there exist polynomials $P_0, P_1, \ldots, P_m$ with constant coefficients, not all zero, such that

$$P_0(g)h_0 + P_1(g)h_1 + \cdots + P_m(g)h_m = 0.$$

3. Proof of Theorem 1

Let f be a non-linear entire function as in (1.2), with iterative logarithm $\phi = \operatorname{itlog}(f)$. Differentiation of (1.3) yields

$$\phi'(f) \cdot f' = f'' \cdot \phi + f' \cdot \phi' = A_{01}(f') \cdot \phi + A_{11}(f') \cdot \phi' \tag{3.1}$$

with $A_{01}(X) = X'$ and $A_{11}(X) = X$. Differentiating this equation, multiplying by f' and substituting (3.1), one obtains

$$\begin{aligned} \phi''(f) \cdot (f')^3 &= \big(f'''f' - (f'')^2\big) \cdot \phi + f''f' \cdot \phi' + (f')^2 \cdot \phi'' \\ &= A_{02}(f') \cdot \phi + A_{12}(f') \cdot \phi' + A_{22}(f') \cdot \phi'' \end{aligned}$$

with $A_{02}(X) = X''X - (X')^2$, $A_{12}(X) = X'X$ and $A_{22}(X) = X^2$. Induction yields the existence of differential polynomials $A_{ij} \in \mathbb{Z}\{X\}$ $(i \le j)$ in a differential indeterminate X, independent of f, such that

$$\phi^{(j)}(f) \cdot (f')^{2j-1} = \sum_{i=0}^{j} A_{ij}(f') \cdot \phi^{(i)}.$$

Each A_{ij} is homogeneous and isobaric, and if non-zero, of degree j and weight $j - i$. (See [2, Section 6.5], where $H_{ij}(X) = A_{ij}(X')$.) Moreover,

$A_{jj} = X^j$ and the monomial of highest rank occurring in A_{0j} is $X^{(j)}X^{j-1}$. For $\underline{j} \in \mathbb{N}^*$ this yields

$$\phi^{\underline{j}}(f) \cdot (f')^{2\|\underline{j}\|-|\underline{j}|} = \sum_{\substack{\underline{i}\leq \underline{j} \\ |\underline{i}|=|\underline{j}|}} B_{\underline{i},\underline{j}}(f') \cdot \phi^{\underline{i}} \tag{3.2}$$

with differential polynomials $B_{\underline{i},\underline{j}} \in \mathbb{Z}\{X\}$ ($\underline{i} \leq \underline{j}$, $|\underline{i}| = |\underline{j}|$), independent of f. For $\underline{j} = (j_0, \dots, j_r) \in \mathbb{N}^{1+r}$, we have

$$B_{\underline{j},\underline{j}} = \prod_{k=0}^{r} (A_{kk})^{j_k} = X^{\|\underline{j}\|}$$

and

$$B_{(|\underline{j}|),\underline{j}} = \prod_{k=0}^{r} (A_{0k})^{j_k},$$

so $B_{(|\underline{j}|),\underline{j}}$ is homogeneous of degree $\|\underline{j}\|$ and isobaric of weight $\|\underline{j}\|$, and the monomial of highest rank occurring in $B_{(|\underline{j}|),\underline{j}}$ is $X^{\underline{j}}X^{\|\underline{j}\|-|\underline{j}|}$. Note that for each n, (1.3) also holds with f replaced by the iterate $F = f^n$ of f, that is,

$$\phi(F(z)) = F'(z)\phi(z) \qquad \text{where } F = f^n, \tag{3.3}$$

and so (3.2) also holds with f replaced by F.

Towards a contradiction assume now that $\phi = \operatorname{itlog}(f)$ is differentially algebraic over $\mathbb{C}\{z\}_\infty$, that is, ϕ satisfies an equation

$$P(\phi) = \sum_{\underline{i}} P_{\underline{i}}\phi^{\underline{i}} = 0 \tag{3.4}$$

where $P = \sum_{\underline{i}} P_{\underline{i}} Y^{\underline{i}}$ is a non-zero differential polynomial with entire coefficients $P_{\underline{i}} = P_{\underline{i}}(z)$. We assume that P is chosen so that its rank $\underline{r} = \underline{r}(P)$ is minimal. Note that $\|\underline{r}\| > 0$, since otherwise (3.4) would show that ϕ is algebraic over $\mathbb{C}\{z\}$ and hence in $\mathbb{C}\{z\}$ (since $\mathbb{C}\{z\}$ is algebraically closed in $\mathbb{C}[[z]]$ by Puiseux's Theorem, see [33, III, §4]), contrary to the results in [4, 19, 39] already quoted in the introduction, which say that the only functions f meromorphic in $\mathbb{C}$ for which $\operatorname{itlog}(f) \in \mathbb{C}\{z\}$ are those of the form $f(z) = z/(1 - cz)$. Allowing the coefficients $P_{\underline{i}}$ to be meromorphic, we may also assume that $P_{\underline{r}} = 1$.

It follows from (3.4) and (3.2) that

$$\begin{aligned} 0 &= \sum_{\underline{j}\leq \underline{r}} P_{\underline{j}}(f) \cdot \phi^{\underline{j}}(f) \cdot (f')^{2\|\underline{r}\|-|\underline{r}|} \\ &= \sum_{\underline{j}\leq \underline{r}} P_{\underline{j}}(f) \cdot (f')^{2\|\underline{r}\|-2\|\underline{j}\|-|\underline{r}|+|\underline{j}|} \sum_{\substack{\underline{i}\leq \underline{j} \\ |\underline{i}|=|\underline{j}|}} B_{\underline{i},\underline{j}}(f') \cdot \phi^{\underline{i}} \end{aligned}$$

so that

$$\sum_{\underline{i}} \left(\sum_{\substack{\underline{i} \le \underline{j} \le \underline{r} \\ |\underline{i}| = |\underline{j}|}} P_{\underline{j}}(f) \cdot (f')^{2\|\underline{r}\| - 2\|\underline{j}\| - |\underline{r}| + |\underline{j}|} \cdot B_{\underline{i},\underline{j}}(f') \right) \phi^{\underline{i}} = 0. \tag{3.5}$$

It also follows from (3.4) that

$$\sum_{\underline{i}} P_{\underline{i}} \cdot (f')^{\|\underline{r}\|} \cdot \phi^{\underline{i}} = 0. \tag{3.6}$$

In the last two equations, the coefficient of $\phi^{\underline{r}}$ is $(f')^{\|\underline{r}\|}$. By the minimality of $\underline{r}$ the two equations are thus equal. (We note that the exponent of f' might actually be negative for some terms on the left hand side of (3.5), but this does not affect the argument, since we may multiply both equations by a sufficiently high power of f'. Similar adjustments are tacitly made in what follows.)

Equating coefficients in (3.5) and (3.6) we obtain, for all $\underline{i} < \underline{r}$, a (possibly trivial) differential equation for f with meromorphic coefficients. We shall only consider the case that $\underline{i} = (|r|)$ and we shall see, that then the resulting differential equation for f' is non-trivial. So we compare the coefficients of $\phi^{\underline{i}}$ in (3.5) and (3.6) for $\underline{i} = (|r|)$ and, putting $a = P_{(|\underline{r}|)}$, we obtain

$$\sum_{\substack{(|\underline{r}|) \le \underline{j} \le \underline{r} \\ |\underline{j}| = |\underline{r}|}} P_{\underline{j}}(f) \cdot (f')^{2\|\underline{r}\| - 2\|\underline{j}\|} \cdot B_{(|\underline{j}|),\underline{j}}(f') = a \cdot (f')^{\|\underline{r}\|}. \tag{3.7}$$

As noted before, $X^{\|\underline{j}\| - |\underline{j}|} X^{\underline{j}}$ is the monomial of highest rank occurring in $B_{(|\underline{j}|),\underline{j}}$, and thus the monomial of highest rank in $X^{2\|\underline{r}\| - 2\|\underline{j}\|} \cdot B_{(|\underline{j}|),\underline{j}}$ is $X^{2\|\underline{r}\| - \|\underline{j}\| - |\underline{j}|} X^{\underline{j}}$. Hence among the monomials occurring in the differential polynomials $X^{2\|\underline{r}\| - \|\underline{j}\|} B_{(|\underline{j}|),\underline{j}}$ on the left side of (3.7) the one of maximal rank given by $X^{\|\underline{r}\| - |\underline{r}|} X^{\underline{r}}$, and it is contributed only by the term corresponding to $\underline{j} = \underline{r}$. Since $P_{\underline{r}} = 1 \ne 0$, we conclude that the differential equation (3.7) is non-trivial.

Also, since $B_{(|\underline{j}|),\underline{j}}$ is homogeneous of degree $\|\underline{j}\|$ and isobaric of weight $\|\underline{j}\|$, each $\underline{i} \in \operatorname{supp} X^{2\|\underline{r}\| - 2\|\underline{j}\|} B_{(|\underline{j}|),\underline{j}}$ satisfies $|\underline{i}| + \|\underline{i}\| = 2\|\underline{r}\|$. Thus, incorporating the terms $X^{2\|\underline{r}\| - 2\|\underline{j}\|}$ into the monomials occurring in $B_{(|\underline{j}|),\underline{j}}$, equation (3.7) takes the form

$$\sum_{\substack{\underline{i} \le (\|\underline{r}\| - |\underline{r}|) + \underline{r} \\ |\underline{i}| + \|\underline{i}\| = 2\|\underline{r}\|}} b_{\underline{i}}(f) \cdot (f')^{\underline{i}} = a \cdot (f')^{\|\underline{r}\|} \tag{3.8}$$

with meromorphic functions $b_{\underline{i}}$, and $b_{(\|\underline{r}\| - |\underline{r}|) + \underline{r}} = 1$. Let

$$I = \left\{ \underline{i} : \underline{i} \le (\|\underline{r}\| - |\underline{r}|) + \underline{r},\ |\underline{i}| + \|\underline{i}\| = 2\|\underline{r}\| \right\}.$$

By the remarks following (3.2), equation (3.8) also holds for f replaced by $F = f^n$, for each n, so

$$\sum_{\underline{i}\in I} b_{\underline{i}}\big(F(z)\big)\cdot\big(F'(z)\big)^{\underline{i}} = a\cdot\big(F'(z)\big)^{\|\underline{r}\|} \qquad \text{where } F = f^n. \tag{3.9}$$

As noted in Section 2.2, f has repelling periodic points. (Actually it was shown in [9] that every iterate of f apart possibly from f itself has repelling fixed points.) Replacing f by an iterate, we may in fact assume that f has a repelling fixed point ξ. Moreover, we may assume that ξ is not a pole of a. With $\lambda = f'(\xi)$ we define the Poincaré function ψ by (2.1). From (2.2) and (2.3) recall that

$$(\psi')^k(z) = \lim_{n\to\infty}\lambda^{-kn}\big((f^n)'\big)^k(\xi+\lambda^{-n}z),$$
$$(\psi')^{\underline{i}}(z) = \lim_{n\to\infty}\lambda^{-(|\underline{i}|+\|\underline{i}\|)n}\big((f^n)'\big)^{\underline{i}}(\xi+\lambda^{-n}z) \quad \text{for each } \underline{i}.$$

We substitute $\xi+\lambda^{-n}z$ for z in (3.9), multiply both sides of the equation by $\lambda^{-2\|\underline{r}\|n}$, take the limit as $n\to\infty$, and using $\|\underline{r}\|>0$, obtain

$$\sum_{\underline{i}\in I} b_{\underline{i}}(\psi)\cdot(\psi')^{\underline{i}} = 0.$$

It is a standard result of Nevanlinna theory [22, p. 56] that

$$T\big(r,\psi^{(k)}\big) \le T(r,\psi) + S(r,\psi)$$

for each k. This implies that

$$\sum_{\underline{i}\in I} T\big(r,(\psi')^{\underline{i}}\big) \le K\,T(r,\psi) + S(r,\psi)$$

for some constant $K\in\mathbb{R}$. Theorem 4 now implies that there exist polynomials $Q_{\underline{i}}$ ($\underline{i}\in I$) with constant coefficients, not all zero, such that

$$\sum_{\underline{i}\in I} Q_{\underline{i}}(\psi)\cdot(\psi')^{\underline{i}} = 0.$$

Thus ψ satisfies an algebraic differential equation with constants coefficients, contradicting Theorem 3. □

4. Proof of Theorem 2

Suppose that $\phi = \operatorname{itlog}(f)$ satisfies an equation of the form (3.4) whose coefficients $P_{\underline{i}}$ are in $\mathbb{C}\{z\}$. Again we assume the rank $\underline{r} = \underline{r}(P)$ of P to be minimal. We choose $\rho>0$ such that all $P_{\underline{i}}$ are holomorphic in $D_\rho = \{z\colon |z|<\rho\}$. Allowing the coefficients $P_{\underline{i}}$ to be meromorphic in D_ρ we may assume that $P_{\underline{r}} = 1$. We want to show that the $P_{\underline{i}}$ are actually meromorphic in $\mathbb{C}$, thereby obtaining a contradiction to Theorem 1.

Let $L_1,\dots,L_{p-1}$ be petals of f associated to the fixed point 0 as stipulated in the Leau-Fatou theorem. These petals L_j can be chosen arbitrarily small, and thus we may assume that their closures are contained in D_ρ. As $f'(0) = 1$, there exists a branch ψ of the inverse function of f defined in a neighborhood of 0 such that $\psi(0) = 0$ and $\psi'(0) = 1$. The Leau-Fatou theorem may also be applied to ψ. We denote by $L'_1, \dots, L'_{p-1}$ the petals for ψ. (These petals are also called repelling petals for f.)

By (1.5) there exists n such that

$$f^n\big(\operatorname{sing}(f^{-1})\big) \subseteq \{0\} \cup \bigcup_{j=0}^{p-1} L_j \subseteq D_\rho \tag{4.1}$$

and thus

$$f^m\big(\operatorname{sing}(f^{-1})\big) \subseteq \{0\} \cup \bigcup_{j=0}^{p-1} L_j \subseteq D_\rho \quad \text{for all } m \geq n. \tag{4.2}$$

Again we put $F = f^n$ and, proceeding as in the proof of Theorem 1, we find that the equations (3.5) and (3.6) are equal. The coefficients of (3.6) are defined in D_ρ while the coefficients of (3.5), with f replaced by F, are defined in the component of $F^{-1}(D_\rho)$ that contains 0. We denote this component by V. So the germs of the coefficients $P_{\underline{i}}$ at 0 can be continued meromorphically to both D_ρ and V.

Actually, by passing to slightly smaller domains L_j and D_ρ if necessary, we may assume that these germs can be continued meromorphically to a region containing the closure $\overline{V}$ of V. Moreover, we may assume that $f^k\big(\operatorname{sing}(f^{-1})\big) \cap \partial D_\rho = \emptyset$ for all k, which implies that ∂V consists of analytic curves.

By the choice of n we have $\operatorname{sing}(f^{-1}) \subseteq V$ and in fact

$$f^m\big(\operatorname{sing}(f^{-1})\big) \subseteq V \quad \text{for all } m. \tag{4.3}$$

Also, we may choose the petals L_j and L'_j so small that $\overline{L_j} \subseteq V$ and $\overline{L'_j} \subseteq V$ for $j = 1, \dots, p-1$.

As mentioned, we want to show that the germs of the coefficients $P_{\underline{i}}$ at 0 can be continued to functions meromorphic in $\mathbb{C}$. By the Monodromy Theorem, it suffices to show that the germs can be continued meromorphically along any curve in $\mathbb{C}$ starting in 0. We may restrict here to curves which intersect ∂V only finitely often. For example, this follows since it suffices to consider continuation along polygonal paths and since ∂V consists of analytic curves.

We now show that it suffices to consider continuation along those curves $\gamma\colon [0,1] \to \mathbb{C}$ for which there exists $t_1 \in (0,1)$ such that $\gamma\big([0,t_1]\big) \subseteq \overline{V}$ while $\gamma\big([t_1,1]\big) \subseteq \mathbb{C} \setminus V$. In fact, suppose that continuation along such curves is possible and let $\sigma\colon [0,1] \to \mathbb{C}$ be a curve such that

$$\sigma\big([0,s_1]\big) \subseteq \overline{V},\ \sigma\big((s_1,s_2)\big) \subseteq \mathbb{C} \setminus \overline{V},\ \sigma\big([s_2,s_3]\big) \subseteq \overline{V} \quad (0 < s_1 < s_2 \leq s_3 \leq 1).$$

Then there exists a curve $\tau\colon [s_1, s_2] \to \partial V$ satisfying $\tau(s_1) = \sigma(s_1)$ and $\tau(s_2) = \sigma(s_2)$ which is homotopic to $\sigma \restriction [s_1, s_2]$ in $\mathbb{C} \setminus V$; that is, there exists a continuous function $\Gamma\colon [s_1, s_2] \times [0, 1] \to \mathbb{C} \setminus V$ such that $\Gamma(s, 0) = \sigma(s)$ and $\Gamma(s, 1) = \tau(s)$ for all $s \in [s_1, s_2]$ and $\Gamma(s_1, t) = \sigma(s_1)$ and $\Gamma(s_2, t) = \sigma(s_2)$ for all $t \in [0, 1]$. Let $\sigma^*\colon [0, 1] \to \mathbb{C}$ be defined by

$$\sigma^*(t) = \begin{cases} \sigma(t) & \text{for } t \in [0,1] \setminus [s_1, s_2], \\ \tau(t) & \text{for } t \in [s_1, s_2]. \end{cases}$$

By our assumption, meromorphic continuation is possible along both $\sigma \restriction [0, s_2]$ and $\sigma^* \restriction [0, s_2]$. Moreover, Γ yields a homotopy from $\sigma \restriction [0, s_2]$ to $\sigma^* \restriction [0, s_2]$ with the property that meromorphic continuation is possible along all curves in the homotopy. Thus, by the Monodromy Theorem, meromorphic continuation along $\sigma \restriction [0, s_2]$ and $\sigma^* \restriction [0, s_2]$ leads to the same result. Since $\sigma \restriction [s_2, s_3] = \sigma^* \restriction [s_2, s_3]$, meromorphic continuation along $\sigma \restriction [0, s_3]$ and $\sigma^* \restriction [0, s_3]$ also leads to the same result.

Now σ^* has the property that $\sigma^*\big([0, s_3]\big) \subset \overline{V}$. Starting with a path $\sigma\colon [0, 1] \to \mathbb{C}$ which intersects ∂V in finitely many points $\sigma(s_1), \dots, \sigma(s_n)$, iteration of the above procedure yields a path $\gamma\colon [0, 1] \to \mathbb{C}$ such that continuation along σ and γ leads to the same result, and γ has the additional property that unless $\gamma\big([0, 1]\big) \subseteq \overline{V}$, there exists $t_1 \in (0, 1)$ such that $\gamma\big([0, t_1]\big) \subseteq \overline{V}$ and $\gamma\big([t_1, 1]\big) \subseteq \mathbb{C} \setminus V$. It thus suffices to consider curves $\gamma\colon [0, 1] \to \mathbb{C}$ with $\gamma(0) = 0$ for which such a t_1 exists. We may also assume that $\gamma \restriction [0, t_1]$ is injective.

Let now γ be such a curve. We have to show that the germs of the $P_{\underline{i}}$ at 0 can be continued meromorphically along γ. In order to do so, we may deform the part of γ which is in V, as long as it stays in V. Thus we may choose γ such that $\gamma(t) \in L_1'$ for $t \in (0, t_0]$ with some $t_0 \in (0, t_1)$, but

$$\gamma\big([0, 1]\big) \cap \bigcup_{j=0}^{p-1} L_j = \emptyset. \tag{4.4}$$

Using (4.2), (4.3) and (4.4) and noting that $\operatorname{sing}(f^{-1})$ is finite by hypothesis, we may in fact assume that

$$\gamma\big((0, 1]\big) \cap \overline{\bigcup_{l=0}^{\infty} f^l\big(\operatorname{sing}(f^{-1})\big)} = \emptyset. \tag{4.5}$$

This implies that branches of the inverse functions of the iterates of f defined in a neighborhood of 0 can be continued analytically along γ. In particular, for each m we may continue along γ the branch $\psi_{m,0}$ of the inverse function of f^m, defined in some neigborhood U_0 of 0, which is given by $\psi_{m,0}(0) = 0$. Thus for each $t \in (0, 1]$, there exists a neighborhood U_t of $\gamma(t)$, a holomorphic function $\psi_{m,t}\colon U_t \to \mathbb{C}$ and $\delta > 0$ such that whenever $|s - t| < \delta$, then $\gamma(s) \in U_t$ and $\psi_{m,s}(z) = \psi_{m,t}(z)$ for all z in some neighborhood of $\gamma(s)$.

Moreover, while U_0 depends on m, it follows from (4.5) that the domains U_t may be chosen independent of m for $t \in (0,1]$. For example, we may choose U_t as the largest disk around $\gamma(t)$ which does not intersect $\overline{\bigcup_{l=0}^{\infty} f^l\big(\operatorname{sing}(f^{-1})\big)}$.

It also follows from (4.5) that there exists a simply connected domain Ω containing $\gamma\big((0,t_0)\big)$ such that

$$\Omega \cap \overline{\bigcup_{l=0}^{\infty} f^l\big(\operatorname{sing}(f^{-1})\big)} = \emptyset.$$

Thus all $\psi_{m,0}$ may be continued to functions holomorphic in Ω and we may in fact assume that $\Omega \subseteq U_0$ for all m. By [7, Theorem 9.2.1] the $\psi_{m,0}$ form a normal family in Ω. Since $\psi_{m,0} \restriction (L_1' \cap \Omega) \to 0$ as $m \to \infty$ we deduce that in fact $\psi_{m,0} \restriction \Omega \to 0$ as $m \to \infty$. This implies that $\psi_{m,t} \to 0$ as $m \to \infty$ for all $t \in (0,1]$, locally uniformly in the domains U_t where they are defined. Altogether we see that if m is sufficiently large, then $\psi_{m,t}\big(\gamma(t)\big) \in D_\rho$ for all $t \in [0,1]$. For the curve $\sigma\colon [0,1] \to \mathbb{C}$ defined by $\sigma(t) = \psi_{m,t}\big(\gamma(t)\big)$ we thus have $\sigma\big([0,1]\big) \subseteq D_\rho$.

Next we note that (3.3) also holds for negative n, with a negative exponent standing for the branch of the inverse function of the appropriate iterate of f which fixes 0. Thus

$$\phi\big(\psi_{m,0}(z)\big) = \psi_{m,0}'(z)\, \phi(z)$$

for z in a neighborhood of 0. Using this instead of (3.3) we obtain (3.5) and (3.6) with f replaced by $\psi_{m,0}$. As before we find that these equations are equal. For the equation corresponding to (3.6) the coefficients are meromorphic in D_ρ. In particular, since $\sigma\big([0,1]\big) \subseteq D_\rho$, the germs of the $P_{\underline{i}}$ at 0 can be continued meromorphically along σ. Noting that $\sigma(t) = \psi_{m,t}\big(\gamma(t)\big)$, we deduce that the germs of the functions $P_{\underline{i}}(\psi_{m,0})$ at 0 can be continued meromorphically along γ. Since the coefficients of the equation corresponding to (3.5) are built from the $P_{\underline{i}}(\psi_{m,0})$ and from differential polynomials in $\psi_{m,0}$, these coefficients can also be continued meromorphically along γ. As the equations corresponding to (3.5) and (3.6) are equal, we see that the $P_{\underline{i}}$ can be continued meromorphically along γ. □

The basic idea of the above proof appears in a paper of Lewin [28] who proved that $\operatorname{itlog}(f) \notin \mathbb{C}\{z\}$ for $f = e^z - 1$. Assuming that $\operatorname{itlog}(f)$ is holomorphic in D_ρ but has a singularity $\zeta \in \partial D_\rho$, it is shown there by elementary estimates that $w_1 = f(\zeta) \in D_\rho$ or that there exists $w_2 \in D_\rho$ with $f(w_2) = \zeta$. These points w_j are also singularities of $\operatorname{itlog}(f)$, leading to a contradiction. Note that $w_2 = \psi(\zeta)$ for some branch ψ of the inverse of f. The proof of Theorem 2 also uses the idea that given $\zeta \in \partial D_\rho$ there exists m such that $f^m(\zeta) \in D_\rho$ or $\psi_m(\zeta) \in D_\rho$ for some branch ψ_m of the inverse function of f^m. However, in this more general setting we have to be careful about the domain where this branch of the inverse can be defined.

Remark. The Eremenko-Lyubich class $\mathcal{B}$ is defined as the class of all non-linear entire functions for which $\operatorname{sing}(f^{-1})$ is bounded. The proof of Theorem 2 shows that instead of demanding that $f \in \mathcal{S}$ it suffices to assume that $f \in \mathcal{B}$ and that there exists n such that (4.1) holds. This is equivalent to saying that on $\operatorname{sing}(f^{-1})$ the iterates of f converge uniformly to 0. An example of a function to which this remark applies is given by $f(z) = (\sin^2 z)/z$.

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