On Numbers, Germs, and Transseries

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Three intimately related topics...

(surreal)
Numbers

Germs
(in HARDY fields)

Transseries

Germs

(in HARDY fields)

HARDY fields

Let $\mathscr{C}^1$ be the ring of germs at $+\infty$ of continuously differentiable functions $(a, \infty) \to \mathbb{R}$ ($a \in \mathbb{R}$).

We denote the germ at $+\infty$ of a function f also by f, relying on context.

Definition

A ***HARDY field*** *is a subring of $\mathscr{C}^1$ which is a field that contains with each germ of a function f also the germ of its derivative f' (where f' might be defined on a smaller interval than f).*

Examples

$\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}(x)$, $\mathbb{R}(x, \mathrm{e}^x)$, $\mathbb{R}(x, \mathrm{e}^x, \log x)$, $\mathbb{R}(x, \mathrm{e}^{x^2}, \operatorname{erf} x)$

Hardy fields capture the somewhat vague notion of functions with "**regular growth**" at infinity (Borel, du Bois-Reymond, ...):

Let H be a Hardy field and $f \in H$. Then

$$f \neq 0 \implies \frac{1}{f} \in H \implies \begin{cases} f(t) > 0, \text{ eventually, or} \\ f(t) < 0, \text{ eventually.} \end{cases}$$

Consequently,

- H carries an ordering making H an **ordered field**:

$$f > 0 \iff f(t) > 0 \text{ eventually;}$$

- f is **eventually monotonic**, and

$$\lim_{t \to +\infty} f(t) \in \mathbb{R} \cup \{\pm\infty\}.$$

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Transseries

The field $\mathbb{T}$ of transseries

$\mathbb{T} :=$ closure of $\mathbb{R} \cup \{x\}$ under $\exp$, $\log$ and infinite summation

$$e^{e^{x}+\cdots} - 3\,e^{x^2} + 5\,(\log x)^{\pi} + 42 + x^{-1} + \cdots + e^{-x}$$

Here one should think of x as a positive infinite indeterminate.

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The field $\mathbb{T}$ of transseries

$\mathbb{T} :=$ closure of $\mathbb{R} \cup \{x\}$ under exp, log and infinite summation

$$\sum_{\mathfrak{m}} f_{\mathfrak{m}} \mathfrak{m} = \mathrm{e}^{\mathrm{e}^{x} + \mathrm{e}^{x/2} + \cdots} - 3\,\mathrm{e}^{x^2} + 5\,(\log x)^{\pi} + 42 + x^{-1} + 2\,x^{-2} + 6\,x^{-3} + \cdots + \mathrm{e}^{-x}$$

x: positive infinite indeterminate

$f_{\mathfrak{m}}$: coefficent

$\mathfrak{m}$: transmonomial

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The formal definition of $\mathbb{T}$ is inductive and somewhat lengthy. For each transseries there is a finite bound on the "nesting" of exp and log among its transmonomials: the following "transseries" are **not** in $\mathbb{T}$:

$$\frac{1}{x} + \frac{1}{\mathrm{e}^{x}} + \frac{1}{\mathrm{e}^{\mathrm{e}^{x}}} + \frac{1}{\mathrm{e}^{\mathrm{e}^{\mathrm{e}^{x}}}} + \cdots \qquad \frac{1}{x} + \frac{1}{x \log x} + \frac{1}{x \log x \log \log x} + \cdots$$

$\mathbb{T}$ as an ordered differential field

- With the natural ordering of transseries (via the leading coefficient), $\mathbb{T}$ is a *real closed ordered field* extension of $\mathbb{R}$.
- Each $f \in \mathbb{T}$ can be *differentiated* term by term (with $x' = 1$):

$$\left(\sum_{n=0}^{\infty} n! \frac{\mathrm{e}^x}{x^n}\right)' = \sum_{n=0}^{\infty} n! \left(\frac{\mathrm{e}^x}{x^n}\right)' = \sum_{n=0}^{\infty} n! \left(\frac{\mathrm{e}^x}{x^n} - n\frac{\mathrm{e}^x}{x^{n+1}}\right) = \frac{\mathrm{e}^x}{x}$$

- This yields a *derivation* $f \mapsto f'$ on the field $\mathbb{T}$:

$$(f+g)' = f' + g', \qquad (f \cdot g)' = f' \cdot g + f \cdot g'$$

  Its constant field is $\{f \in \mathbb{T} : f' = 0\} = \mathbb{R}$.
- Given $f, g \in \mathbb{T}$, the equation $y' + fy = g$ admits a solution $y \neq 0$ in $\mathbb{T}$.

Surreal numbers

(surreal)
Numbers

Transseries

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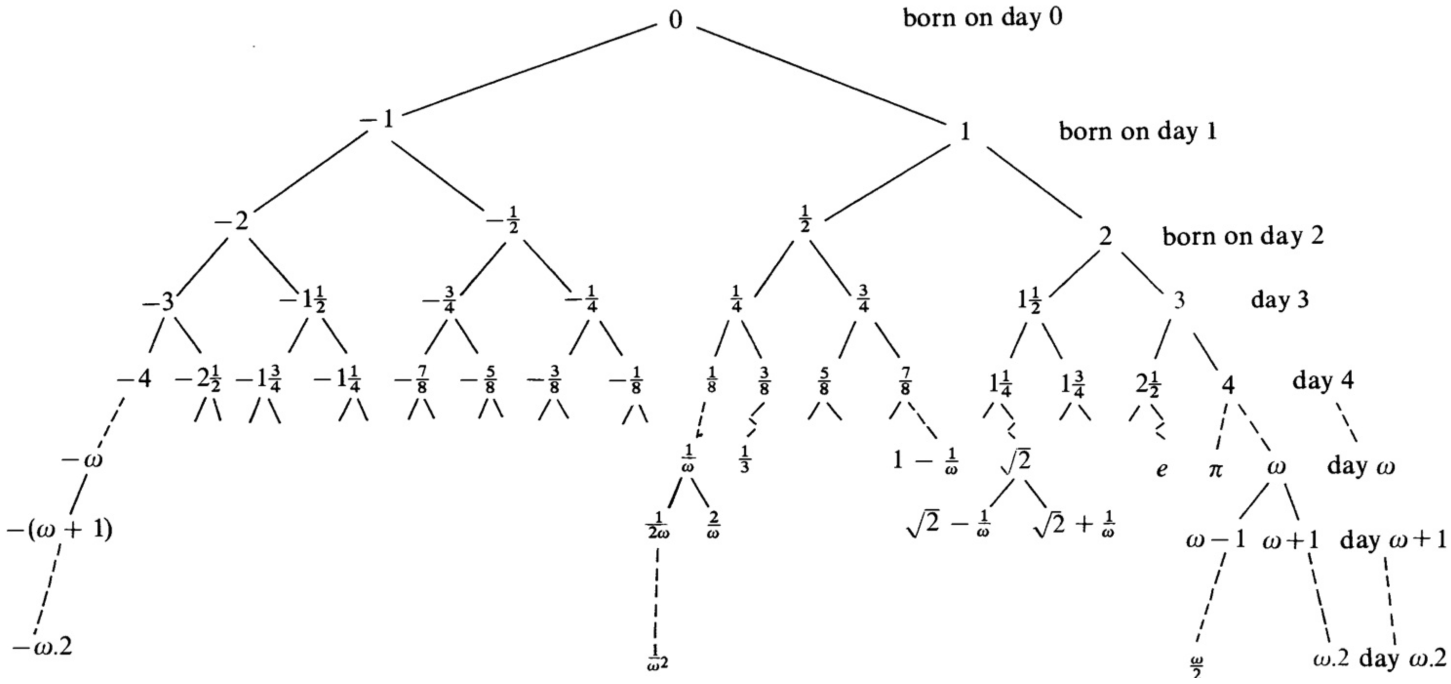
(surreal)

Numbers

Surreal numbers

These are simply strings of $+, -$ of arbitrary ordinal length. CONWAY turned the class **No** of surreals into a real closed field extension of $\mathbb{R}$.



Addition of surreal numbers

- $x \in \mathbf{No}$ is **simpler** than $y \in \mathbf{No} \quad :\Longleftrightarrow \quad x$ is a prefix of y
- For subsets $L < R$ of **No**, let $\{L|R\}$ be the simplest $x \in \mathbf{No}$ with $L < x < R$.
- Any $x \in \mathbf{No}$ is of the form $x = \{L|R\}$ for suitable subsets $L < R$ of **No**.

Example

$0 = \{|\}, \ 1 = \{0|\}, \ 2 = \{0,1|\}, \ \frac{1}{2} = \{0|1\}, \ \omega = \{0,1,\ldots|\}$

Definition

If $x = \{x^L|x^R\}$ *and* $y = \{y^L|y^R\}$, *then*

$$x + y := \{x^L + y, x + y^L | x^R + y, x + y^R\}.$$

(Idea: we want $x^L + y < x + y < x^R + y, \ \ldots$*)*

Exponentiation and differentiation

- In the 1980s, GONSHOR (based on ideas of KRUSKAL) defined an exponential function $\exp\colon \mathbf{No} \rightarrow \mathbf{No}^{>0}$ that extends $x \mapsto \mathrm{e}^x$ on $\mathbb{R}$.

- In 2006, BERARDUCCI and MANTOVA (using ideas of VDH and SCHMELING) defined a derivation ∂_{BM} on $\mathbf{No}$ with

$$\ker \partial_{\mathrm{BM}} = \mathbb{R}, \quad \partial_{\mathrm{BM}}(\omega) = 1, \quad \partial_{\mathrm{BM}}(\exp(f)) = \partial_{\mathrm{BM}}(f) \cdot \exp(f) \text{ for } f \in \mathbf{No}.$$

  In a certain technical sense, it is the simplest such derivation that satisfies some natural further conditions.

- The BM-derivation on $\mathbf{No}$ behaves in many ways like the derivation on $\mathbb{T}$, with $\omega > \mathbb{R}$ in the role of $x > \mathbb{R}$. For instance, $\partial_{\mathrm{BM}}(\log \omega) = \frac{1}{\omega}$.

Towards a unified theory

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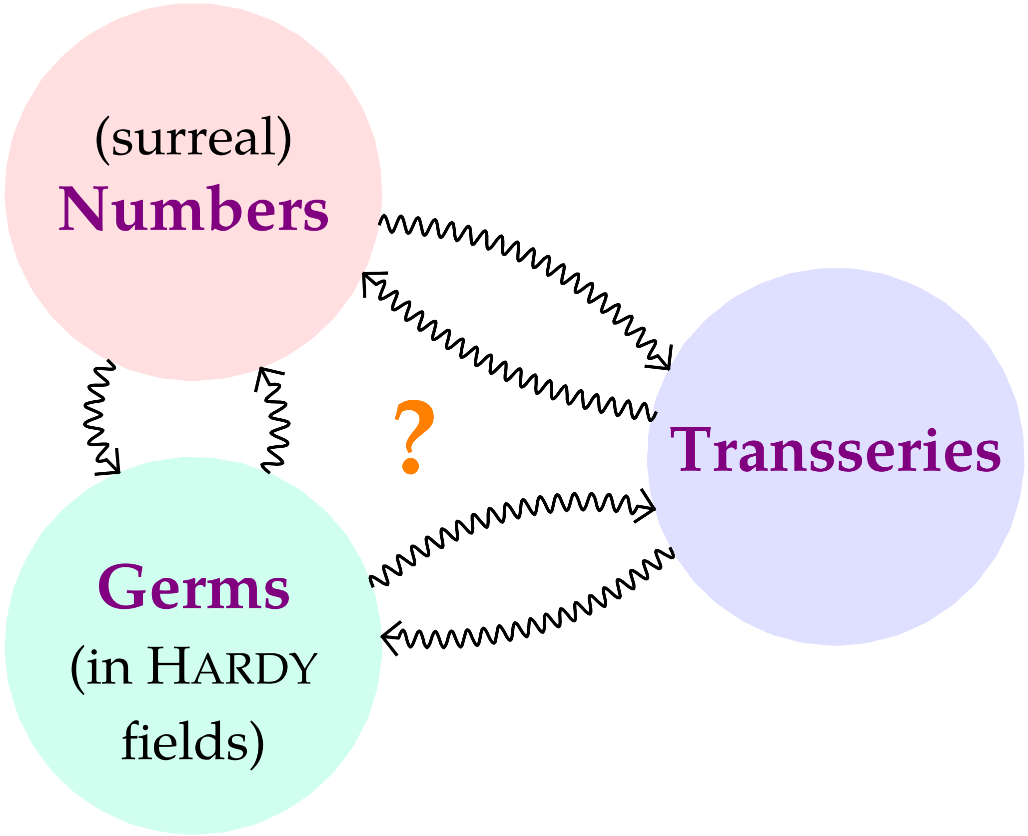
Towards a unified theory



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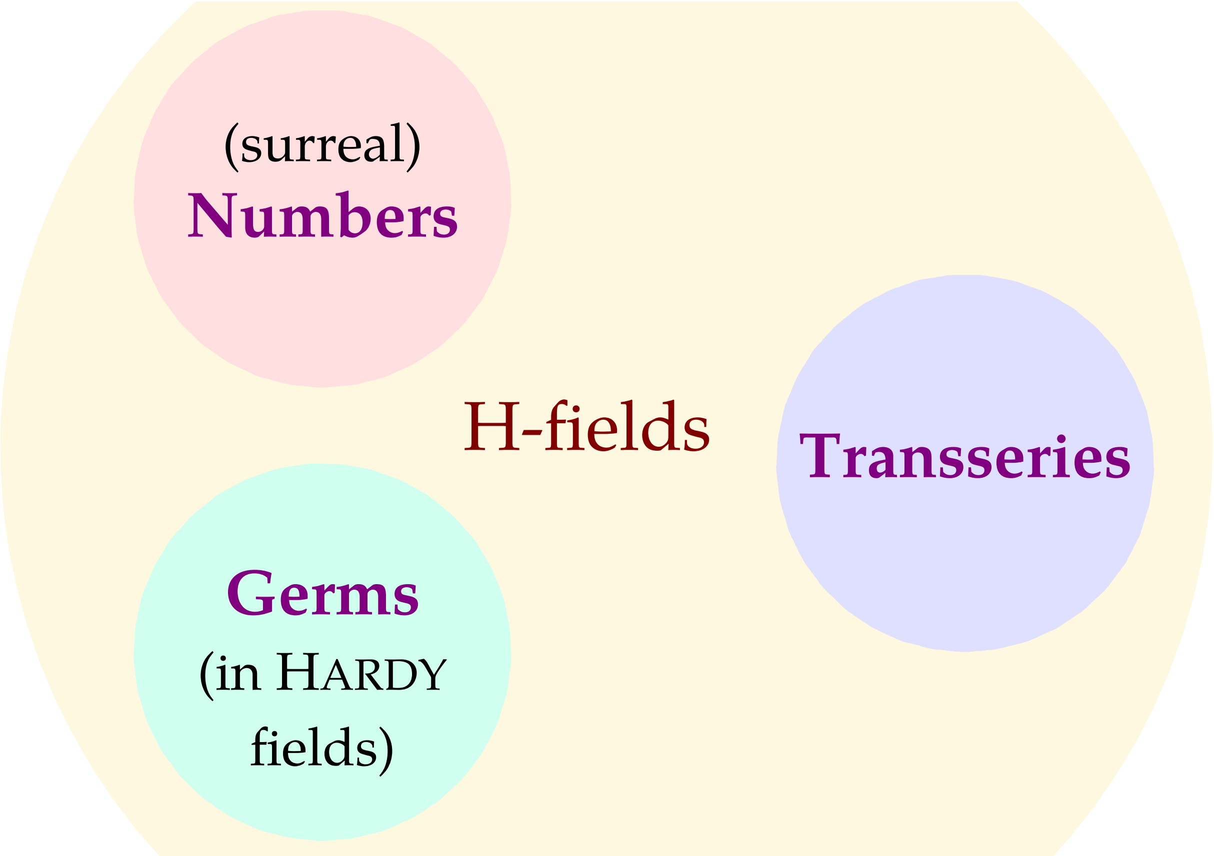

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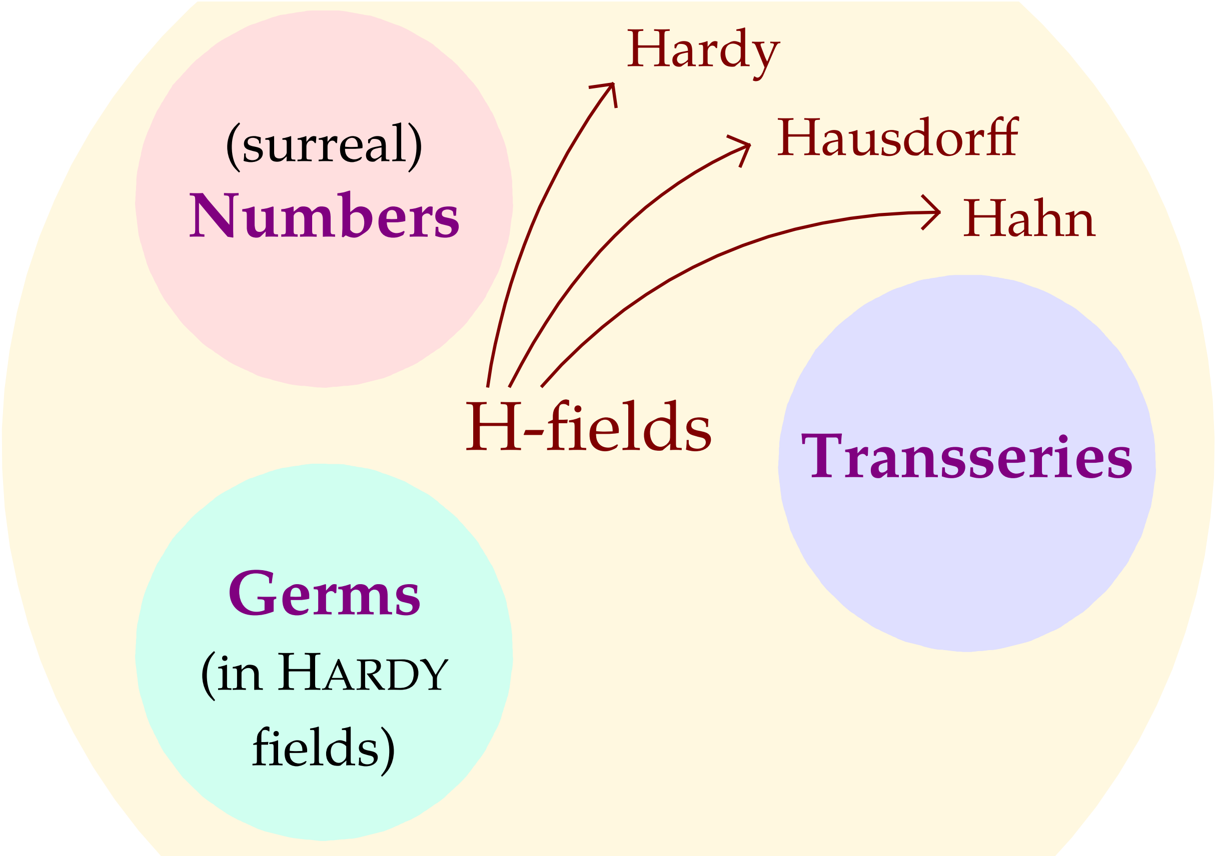


Asymptotic relations

Let K be an ordered differential field with constant field

$$C = \{f \in K : f' = 0\}.$$

We define

$$f \preccurlyeq g \;:\Longleftrightarrow\; |f| \leqslant c\,|g| \text{ for some } c \in C^{>0} \qquad (f \text{ is } \textit{dominated} \text{ by } g)$$

$$f \prec g \;:\Longleftrightarrow\; |f| \leqslant c\,|g| \text{ for all } c \in C^{>0} \qquad (f \text{ is } \textit{negligible} \text{ w.r.t. } g)$$

$$f \asymp g \;:\Longleftrightarrow\; f \preccurlyeq g \preccurlyeq f \qquad (f \text{ is } \textit{asymptotic} \text{ to } g)$$

$$f \sim g \;:\Longleftrightarrow\; f - g \prec g \qquad (f \text{ is } \textit{equivalent} \text{ to } g)$$

Example

In $\mathbb{T}$: $0 \prec \mathrm{e}^{-x} \prec x^{-10} \prec 1 \asymp 100 \prec \log x \prec x^{1/10} \prec \mathrm{e}^{x} \sim \mathrm{e}^{x} + x \prec \mathrm{e}^{\mathrm{e}^{x}}$

H-fields

Definition

We call K an ***H-field*** *if*

H1. $f > C \implies f' > 0$;

H2. $f \asymp 1 \implies f \sim c$ *for some* $c \in C$.

Examples

HARDY fields containing $\mathbb{R}$; ordered differential subfields of $\mathbb{T}$ or **No** that contain $\mathbb{R}$.

$\mathbb{T}$ admits further elementary properties in addition to being an H-field. It

- has **small derivation**, that is, $f \prec 1 \Longrightarrow f' \prec 1$; and
- is **LIOUVILLE closed**, that is, it is real closed and for all f, g, there is some $y \neq 0$ with $y' + fy = g$.

One of our main results

We view $\mathbb{T}$ model-theoretically as a structure with the primitives

$$0, \quad 1, \quad +, \quad \times, \quad \partial \text{ (derivation)}, \quad \leqslant \text{ (ordering)}.$$

Theorem (Ann. of Math. Stud. vol. 195)

The elementary theory of $\mathbb{T}$ is completely axiomatized by:

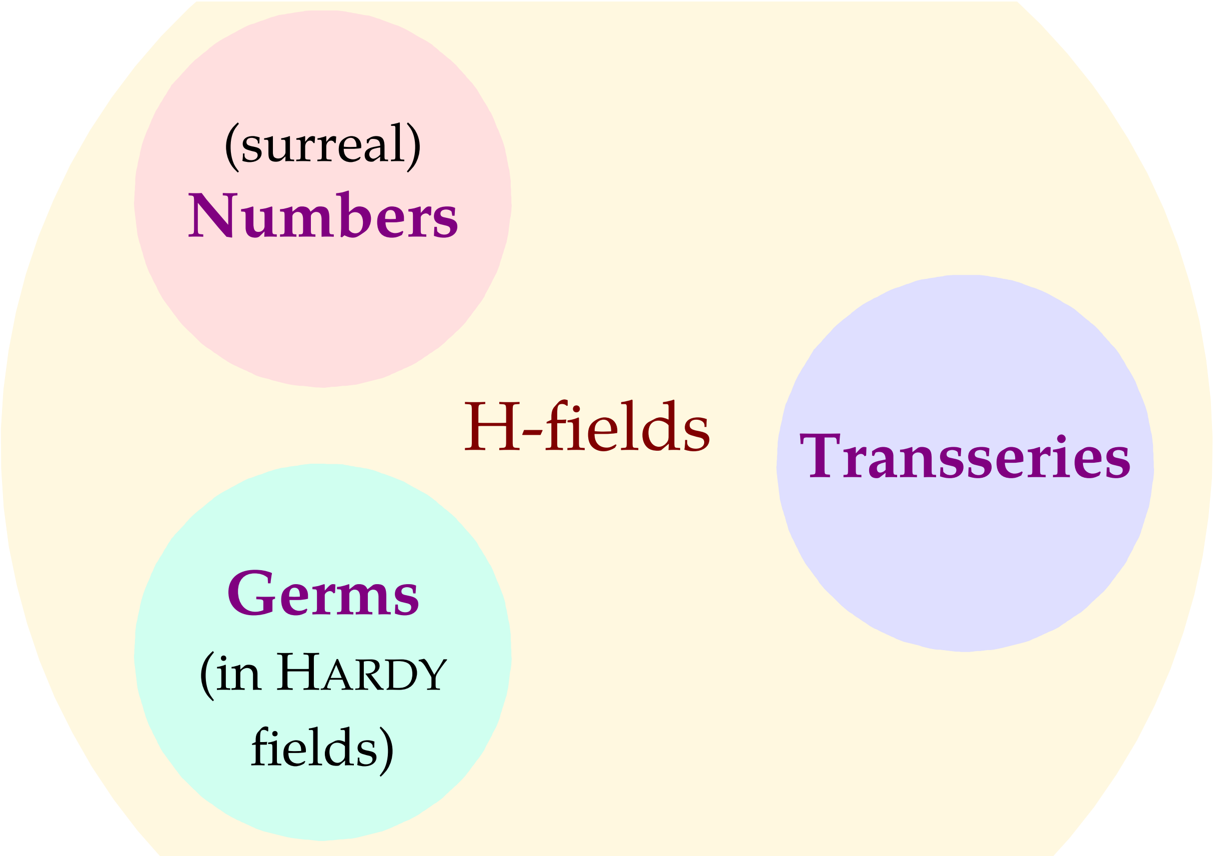
1. *$\mathbb{T}$ is a* LIOUVILLE *closed H-field with small derivation;*
2. *$\mathbb{T}$ satisfies the intermediate value property for differential polynomials.*

Actually 2 is a bit of an afterthought.

A corollary of the theorem: the theory of $\mathbb{T}$ is decidable.

We also prove a quantifier elimination result for $\mathbb{T}$ in a natural expansion of the above language.

H-field elements as germs



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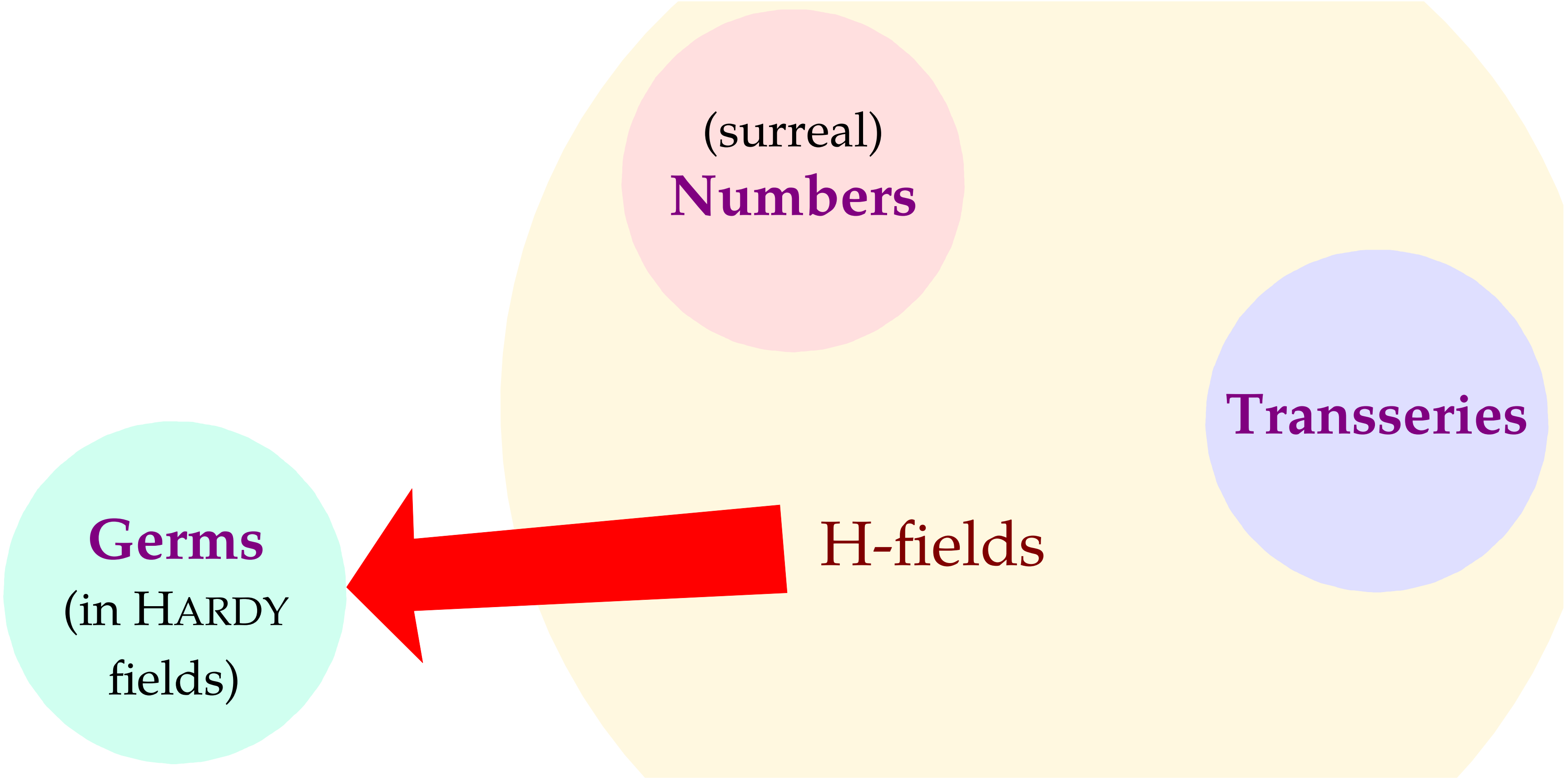

Closure properties of HARDY fields

Theorem (HARDY 1910, BOURBAKI 1951)

Any HARDY *field has a smallest* LIOUVILLE *closed* HARDY *field extension.*

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Theorem (SINGER 1975)

Let H *be a* HARDY *field and* $\Phi \in H(Y)$ *be a rational function. If* $y \in \mathscr{C}^1$ *satisfies the differential equation* $y' = \Phi(y)$, *then* $H\langle y\rangle = H(y, y')$ *is a* HARDY *field.*

Closure properties of Hardy fields

Theorem (Hardy 1910, Bourbaki 1951)

Any Hardy *field has a smallest* Liouville *closed* Hardy *field extension.*

Theorem (Singer 1975)

Let H be a Hardy *field and $\Phi \in H(Y)$ be a rational function. If $y \in \mathscr{C}^1$ satisfies the differential equation $y' = \Phi(y)$, then $H\langle y\rangle = H(y, y')$ is a* Hardy *field.*

Remark (Boshernitzan 1987)

Any solution $y \in \mathscr{C}^1$ to

$$y'' + y = \mathrm{e}^{x^2}$$

lies in a Hardy field, but any Hardy field contains at most one solution.

Conjectural closure properties

Conjecture

Let H be a maximal HARDY *field. Then*

- Ⓐ *H satisfies the differential intermediate value property.*
- Ⓑ *For countable subsets $A < B$ of H, there exists an $h \in H$ with $A < h < B$.*

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Corollary

- Ⓐ *H is elementarily equivalent to $\mathbb{T}$ as an ordered differential field.*
- Ⓑ *Under* CH*, all maximal* HARDY *fields are isomorphic.*

Work in progress on Conjecture A

Theorem (van der Hoeven 2009)

The subfield $\mathbb{T}^{\mathrm{da}}$ of transseries that satisfy an algebraic differential equation over $\mathbb{R}$ can be embedded (as an ordered differential field) in a Hardy *field.*

Work in progress on Conjecture A

Theorem (van der Hoeven 2009)

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$$y'' = \mathrm{e}^{-\mathrm{e}^{x}} + y^2$$

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Theorem in progress

✓ *Any* HARDY *field has an ω-free* HARDY *field extension.*

… *Any ω-free* HARDY *field has a newtonian differentially algebraic* HARDY *field extension.*

Work in progress on Conjecture B

Theorem (Borel 1895)

Any $y \in \mathbb{R}[[x^{-1}]]$ *is the asymptotic expansion of a germ* $\tilde{y}$ *in* $\mathscr{C}^{\infty}$.

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Example

$y = x^{-1} + 2!!\, x^{-2} + 3!!\, x^{-3} + \cdots$ is differentially transcendental over $\mathbb{R}$

$\Longrightarrow$ the differential field $\mathbb{R}\langle\tilde{y}\rangle = \mathbb{R}(\tilde{y}, \tilde{y}', \tilde{y}'', \ldots)$ is a Hardy field.

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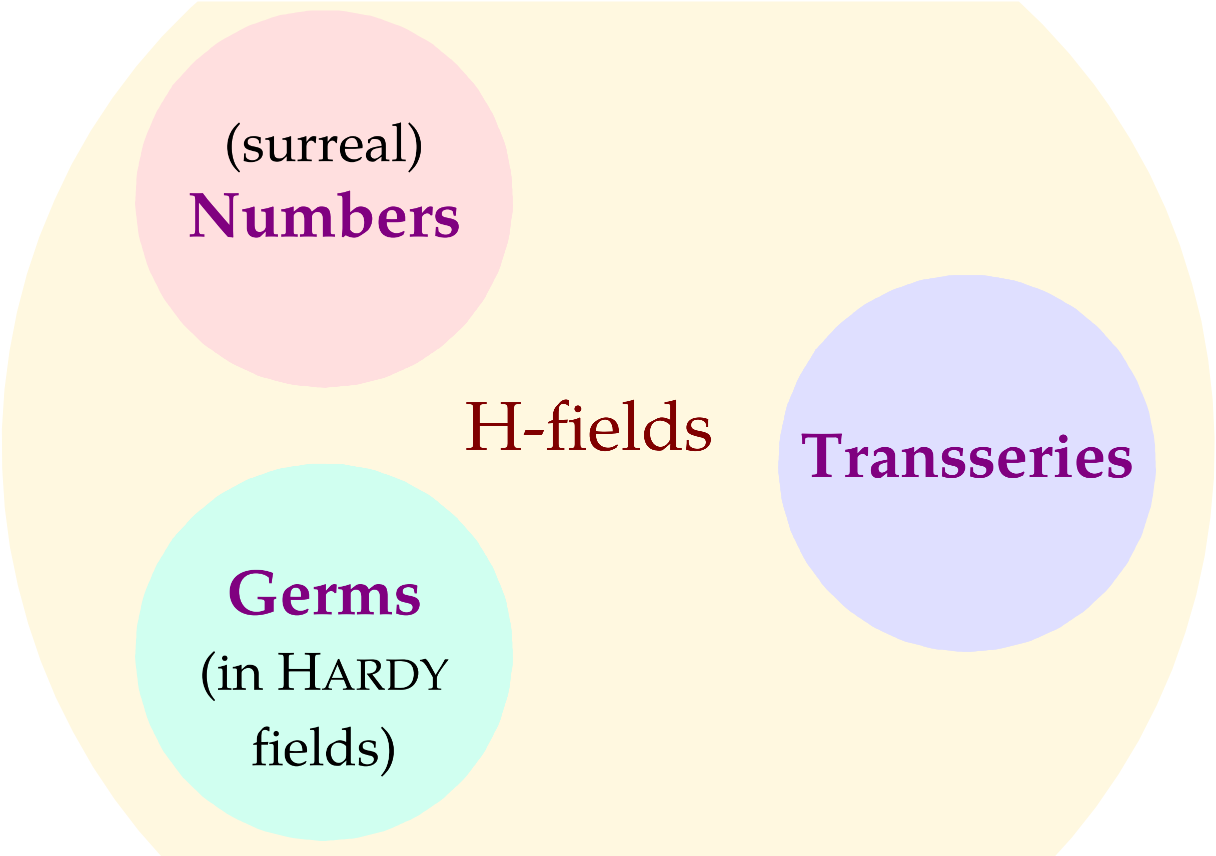
Theorem in progress

✓ *Every pseudocauchy sequence (y_n) in a* HARDY *field H has a pseudolimit in some* HARDY *field extension of H.*

✓ *Conjecture* Ⓑ *for countable A and $B = \emptyset$* (SJÖDIN 1970).

… *General case.*

H-field elements as surreal numbers



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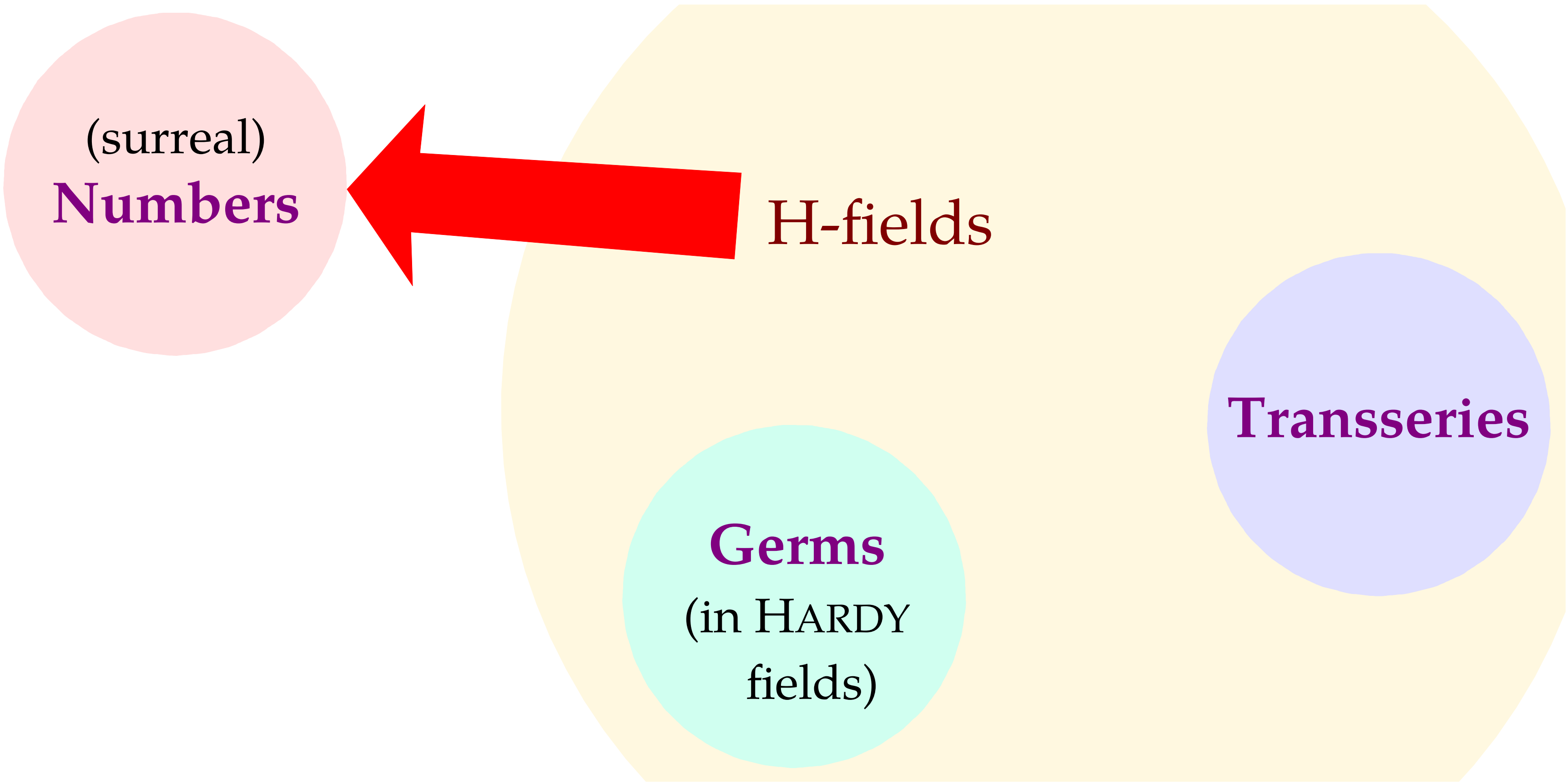

Embedding H-fields into the surreals

Theorem (to appear in JEMS)

Every H-field with small derivation and constant field $\mathbb{R}$ can be embedded as an ordered differential field into **No**.

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Let κ be an uncountable cardinal. The field $\mathbf{No}(\kappa)$ of surreal numbers of length $<\kappa$ is an elementary submodel of $\mathbf{No}$.

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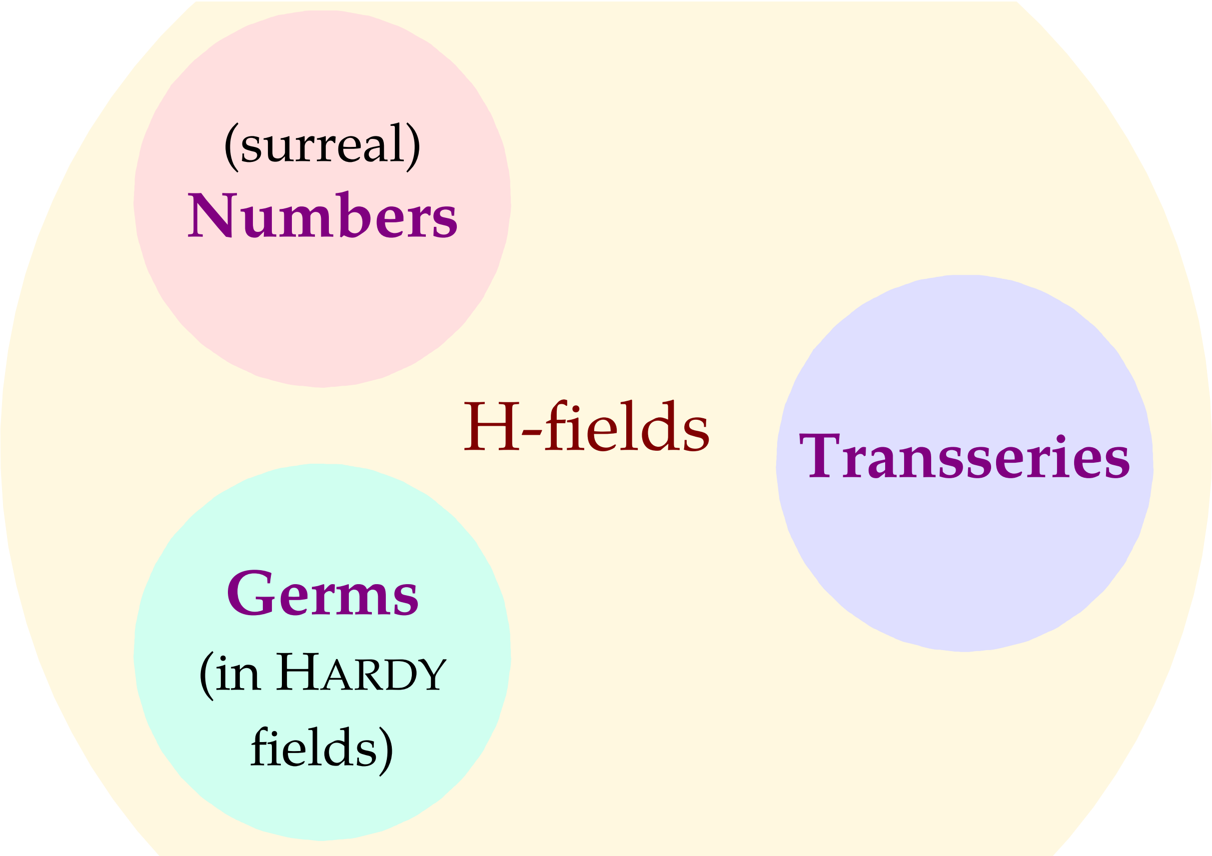
Corollary in progress

Under CH all maximal Hardy *fields are isomorphic to* $\mathbf{No}(\omega_1)$.

H-field elements as transseries



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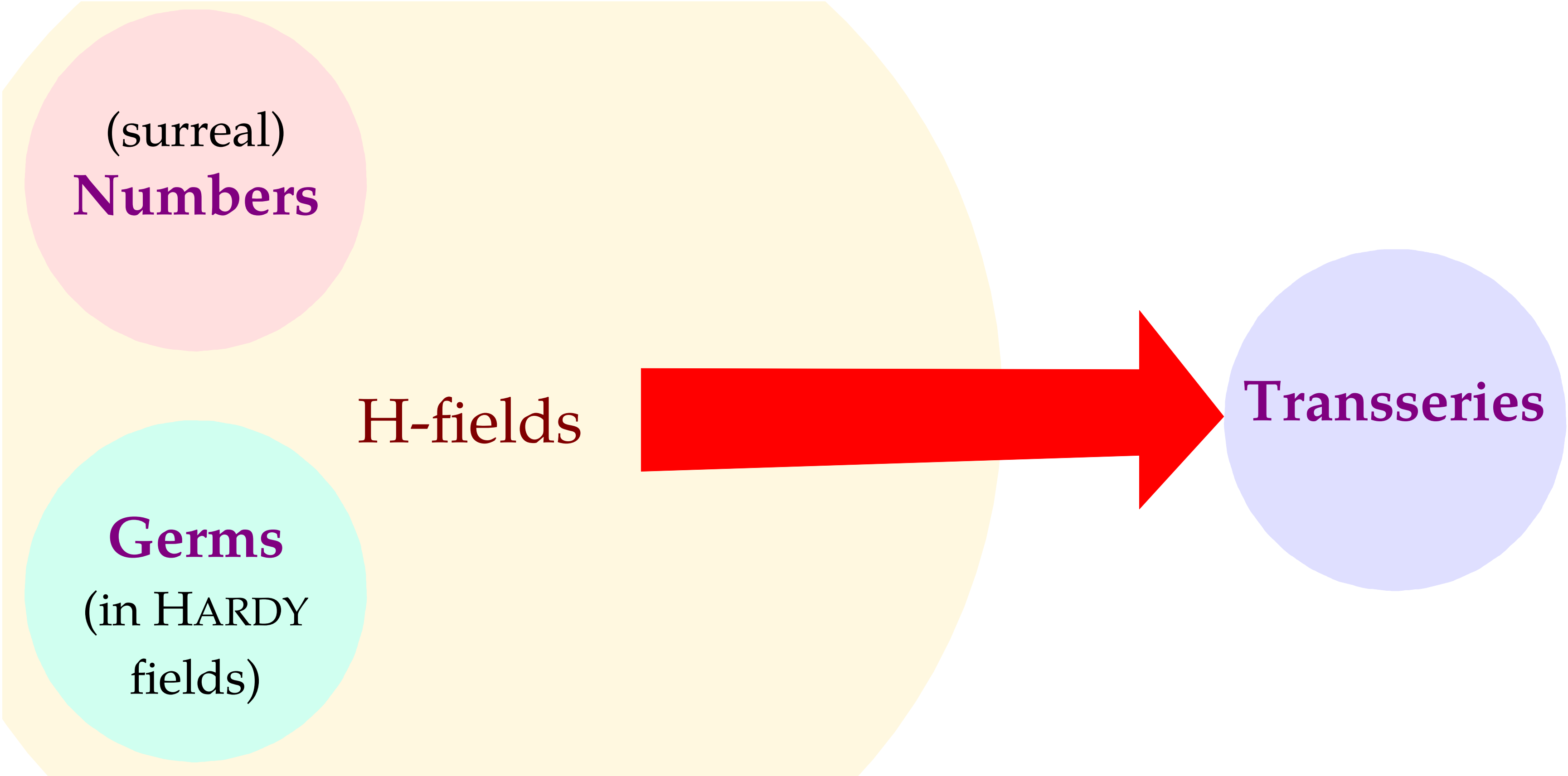

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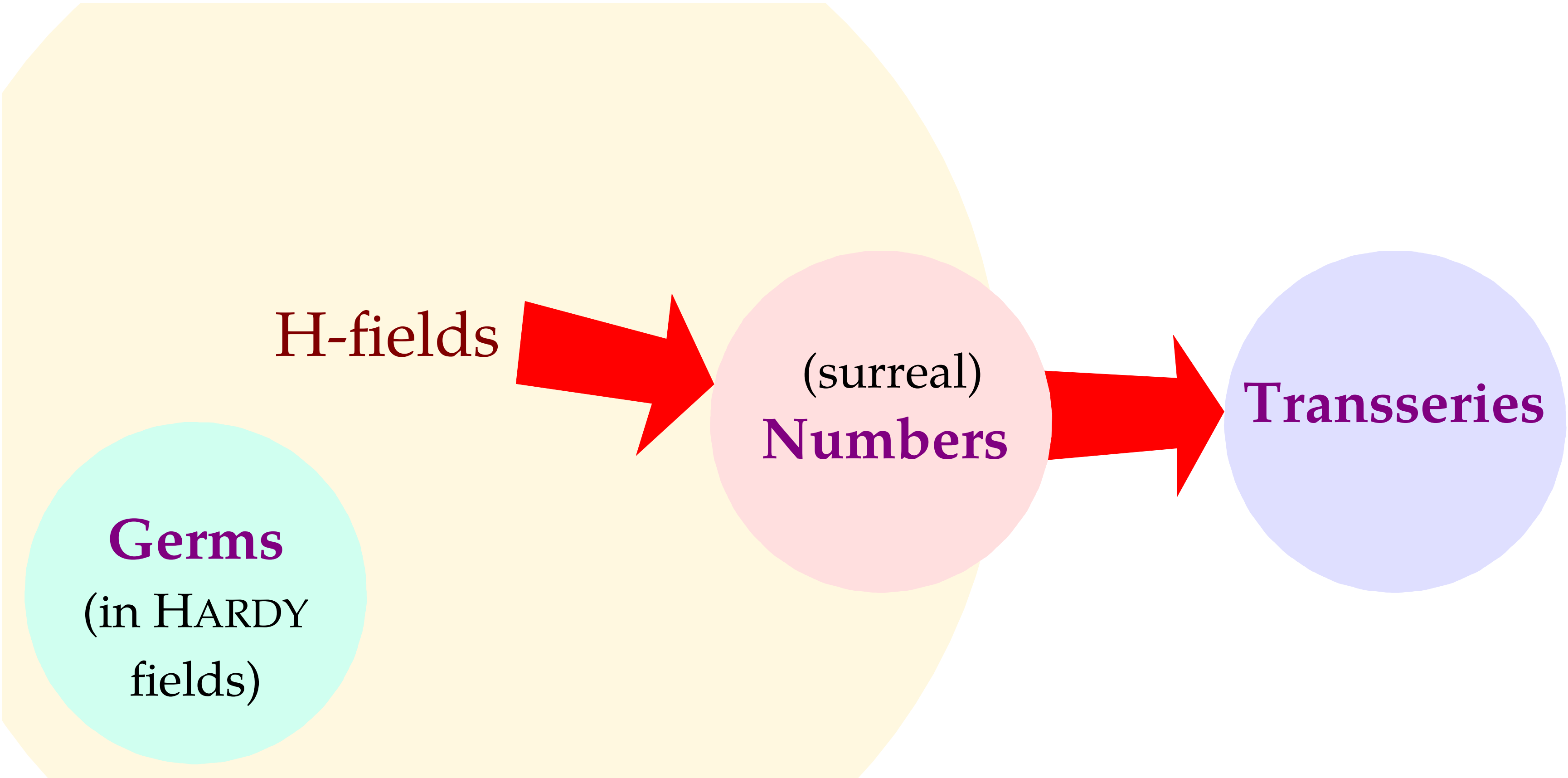

Definition (van der Hoeven 2000, Schmeling 2001)

A field $\mathbf{T} = \mathbb{R}[[\mathfrak{M}]]$ *with partial* $\log : \mathbf{T} \dashrightarrow \mathbf{T}$ *is a* ***field of transseries*** *if*

T1. *The domain of* $\log$ *is* $\mathbf{T}^{>0}$;

T2. *for each* $\mathfrak{m} \in \mathfrak{M}$ *and* $\mathfrak{n} \in \operatorname{supp} \log \mathfrak{m}$, *we have* $\mathfrak{n} \succ 1$;

T3. $\log(1+\varepsilon) = \varepsilon - \frac{1}{2}\varepsilon^2 + \frac{1}{3}\varepsilon^3 + \cdots$, *for all* $\varepsilon \in \mathbf{T}$ *with* $\varepsilon \prec 1$;

T4. *for every sequence* $(\mathfrak{m}_n) \in \mathfrak{M}^{\mathbb{N}}$ *with* $\mathfrak{m}_{n+1} \in \operatorname{supp} \log \mathfrak{m}_n$ *for all* n, *there exists an index* n_0 *such that for all* $n > n_0$ *and all* $\mathfrak{n} \in \operatorname{supp} \log \mathfrak{m}_n$, *we have* $\mathfrak{n} \succcurlyeq \mathfrak{m}_{n+1}$ *and* $(\log \mathfrak{m}_n)_{\mathfrak{m}_{n+1}} = \pm 1$.

How to generalize transseries

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Surreal numbers as transseries

Definition

*A **transserial derivation** on* $\mathbf{T}$ *is a derivation* $\partial : \mathbf{T} \to \mathbf{T}$ *such that*

TD1. ∂ *is strong (i.e., it preserves infinite summation);*

TD2. $\partial \log f = \partial f / f$ *for all* $f \in \mathbf{T}^{>0}$*;*

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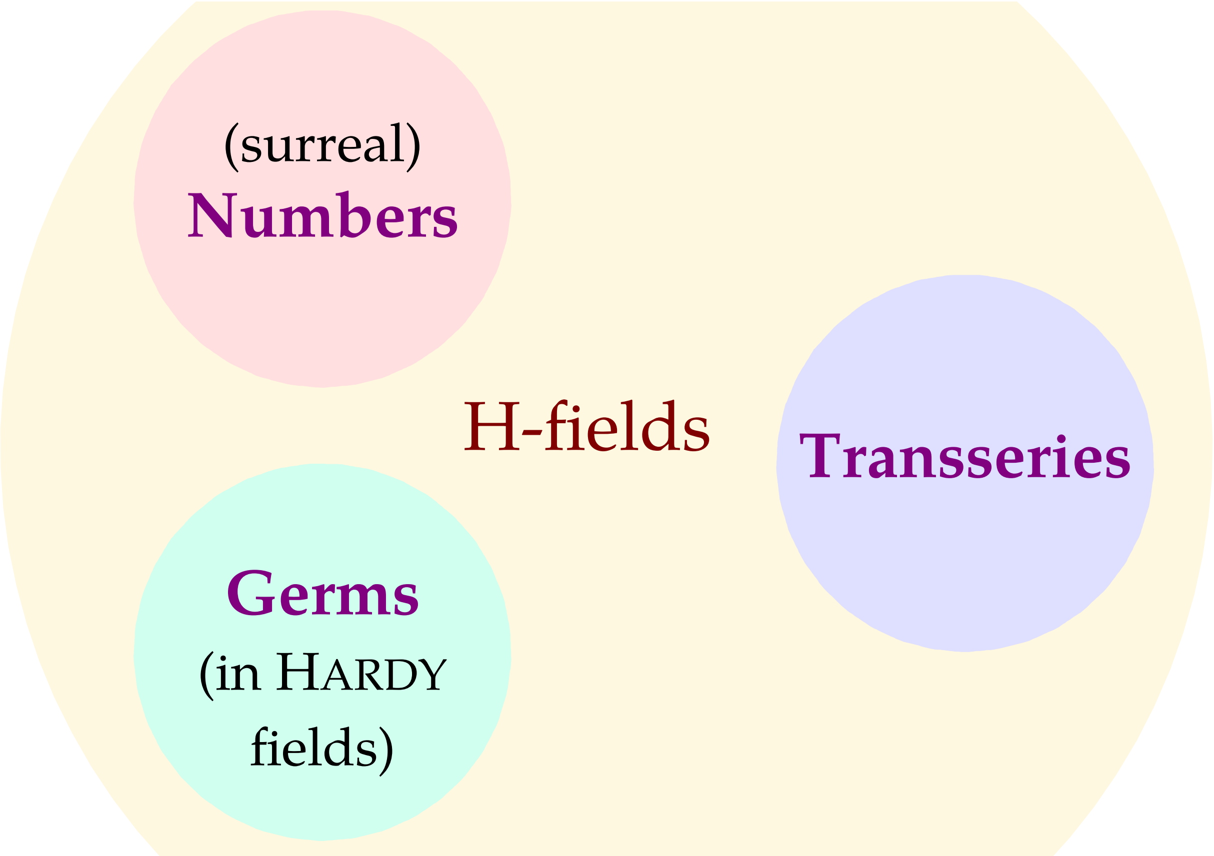
Corollary

Any H-field with constant field $\mathbb{R}$ *can be embedded in a field of transseries with a transserial derivation.*

What next?


(surreal)
Numbers
H-fields
Transseries
Germs
(in HARDY
fields)

What next?

(surreal)
Numbers

beyond H-fields

Transseries

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What next?

(surreal)
Numbers

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Hyperseries

Germs
(in HARDY fields)

Solving functional equations $\rightsquigarrow$ lack of hyperlogarithms

$$\log_\omega \log x \;=\; \log_\omega x - 1$$

Solving functional equations $\rightsquigarrow$ lack of hyperlogarithms

$$\begin{aligned}\log_\omega \log x &= \log_\omega x - 1\\ \log_\omega x &= \int \frac{1}{x \log x \log \log x \cdots}\end{aligned}$$

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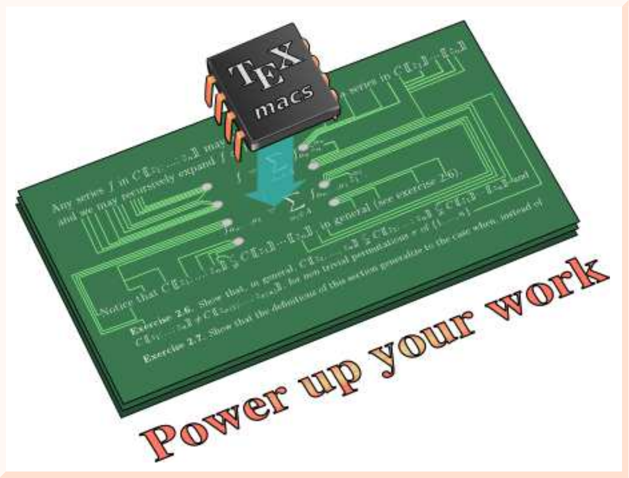
Conjecture

For a suitable definition of the class **Hy** *of hyperseries (including the nested ones), we have* **No** $\cong$ **Hy** *for the map* $\phi\colon \mathbf{Hy} \longrightarrow \mathbf{No}; f \longmapsto f(\omega)$.

Thank you!



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