Motivic Integration

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Motives were first introduced by Grothendieck in an attempt to find an explanation for the similarities among the étale ℓ-adic cohomology groups of an algebraic variety over a field of nonzero characteristic p, when ℓ varies over the primes $\ell \neq p$. In characteristic zero, for an algebraic variety X over $\mathbb{C}$, such an explanation is given by comparison isomorphisms

$$H^i(X(\mathbb{C}), \mathbb{Z}) \otimes \mathbb{Z}_\ell \xrightarrow{\cong} H^i_{\text{ét}}(X, \mathbb{Z}_\ell)$$

between the ("usual") integer valued topological cohomology groups $H^i(X(\mathbb{C}), \mathbb{Z})$ and the ℓ-adic étale cohomology groups $H^i_{\text{ét}}(X, \mathbb{Z}_\ell)$. The theory of motives would be an attempt to find a substitute for the $H^i(X(\mathbb{C}), \mathbb{Z})$ (which are not available in characteristic p).

Motivic integration was initially introduced by Kontsevich as an analogue of p-adic integration over the $k[[t]]$-rational points of an algebraic variety, where $k[[t]]$ the ring of formal power series in t over a field k of characteristic zero. Motivic integrals in Kontsevich's sense take values in a certain ring (of motives) constructed from the category of varieties over k, called the Grothendieck ring of algebraic varieties over k. Later, Denef and Loeser developed another version of motivic integration, "arithmetic" motivic integration, which, among other things, brings out the uniformities in p in the zeta functions of p-adic varieties.

Recently, Cluckers and Loeser gave a new version of motivic integration, subsuming Kontsevich's theory and the arithmetic theory of Denef and Loeser, and applied it to p-adic integrals to extend the classical transfer principle of Ax-Kochen-Ershov between the fields of p-adic numbers and Laurent series over prime finite fields. Their transfer theorems allow the transfer to p-adic fields of the "fundamental lemma" from Langland's program. Even more recently, Hrushovski and Kazhdan developed a theory of integration with a similar scope, but built on different foundations.

Nowadays, motivic integration is a flourishing subject at the intersection of algebraic geometry, number theory, and model theory (a branch of mathematical logic).

In this seminar, we will try to cover at least the following two pillars of the Cluckers-Loeser theory:

(1) *Constructible motivic functions and motivic integration,* Invent. Math. **173** (2008), no. 1, 23–121.

(2) *Constructible exponential functions, motivic Fourier transform and transfer principle,* Ann. of Math. (2) **171** (2010), no. 2, 1011–1065.

We will not assume any familiarity with model theory. An introduction to this subject tailored to the needs of the seminar will be provided.

Surveys on motivic integration can be found in the two volumes *Motivic Integration and its Interactions with Model Theory and Non-Archimedean Geometry,* vol. I and II, London Mathematical Society Lecture Note Series, nos. 383 and 384, Cambridge University Press, Cambridge, 2011.

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