Tutorial: Model Theory of Transseries

Lecture 2

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Overview

I. Asymptotic Differential Algebra

In the previous lecture, JORIS introduced a variety of interesting differential fields (of transseries, germs of functions, ...) equipped with asymptotic structure, such as ordering and dominance. We will now introduce an algebraic framework to unify these examples and to study their general properties.

II. The Main Results

We'll give statements of our main theorems, though leaving some definitions as black boxes for JORIS' next lecture.

III. The Next Lectures

I. Asymptotic Differential Algebra

Let K be a differential field (always of characteristic 0), with derivation ∂. As usual

$$f' = \partial(f),\ f'' = \partial^2(f),\ \ldots,\ f^{(n)} = \partial^n(f),\ \ldots$$

The **constant field** of K is $C = C_K = \{f \in K : f' = 0\}$.

For $f \neq 0$ let $f^\dagger := f'/f$ be the **logarithmic derivative** of f. Note

$$(f \cdot g)^\dagger = f^\dagger + g^\dagger \qquad \text{for } f, g \neq 0.$$

The ring of **differential polynomials** (= d-polynomials) in $Y_1, \ldots, Y_n$ with coefficients in K is denoted by $K\{Y_1, \ldots, Y_n\}$.

For $\phi \neq 0$, we denote by K^ϕ the **compositional conjugate** of K by ϕ: the field K equipped with the derivation $\phi^{-1}\partial$.

For every $P \in K\{Y\}$ there is a $P^\phi \in K^\phi\{Y\}$ with $P(y) = P^\phi(y)$ for all y. (This will play an important role later.)

A **valued differential field** is a differential field K equipped with a valuation $v\colon K^\times \to \Gamma = \Gamma_K$, extended by $v(0) := \infty > \Gamma$. Put

$$\mathcal{O} := \{f : vf \geqslant 0\}, \quad \mathcal{o} := \{f : vf > 0\}, \quad \mathrm{res}(K) := \mathcal{O}/\mathcal{o}.$$

In our context it is often more natural to encode v in terms of its associated **dominance relation**:

$$f \preccurlyeq g \quad :\Longleftrightarrow \quad vf \geqslant vg \qquad \text{“}g \text{ dominates } f\text{”}.$$

We also use:

$$\begin{array}{llll} f \prec g & :\Longleftrightarrow & f \preccurlyeq g \ \&\ g \not\preccurlyeq f & \text{“}g \text{ strictly dominates } f\text{”} \\ f \asymp g & :\Longleftrightarrow & f \preccurlyeq g \ \&\ g \preccurlyeq f & \\ f \sim g & :\Longleftrightarrow & f - g \prec g & \text{“asymptotic equivalence”} \end{array}$$

The derivation of K is **small** if $\partial\mathcal{o} \subseteq \mathcal{o}$. (This implies the continuity of ∂.) Then $\partial\mathcal{O} \subseteq \mathcal{O}$, so we get a derivation on $\mathrm{res}(K)$.

Examples

1. For $K = \mathbb{T}$:

$$(\Gamma, +, \leqslant) \cong \big(\{\text{transmonomials}\}, \cdot, \succcurlyeq\big).$$

2. For $K = \mathbb{R}(\ell_0, \ell_1, \dots) \subseteq \mathbb{T}$:

$$\Gamma = \bigoplus_n \mathbb{Z} e_n, \qquad e_n = v\ell_n, \qquad \ell_n = \underbrace{\log\log\cdots\log}_{n \text{ times}} x,$$

$$e_n < m\, e_{n+1} < 0 \qquad \text{for all } m > 0 \text{ and all } n.$$

In both cases $\mathcal{O} = \mathbb{R} + o$, so $\operatorname{res}(K) \cong \mathbb{R}$.

Valued differential fields

An **ordered differential field** is a differential field K equipped with an ordering making it an ordered field. We can then turn K into a valued field with dominance relation

$$f \preccurlyeq g \quad :\Longleftrightarrow \quad |f| \leqslant c|g| \text{ for some } c \in C.$$

Example

Let K be a HARDY field. Then K becomes an ordered field via

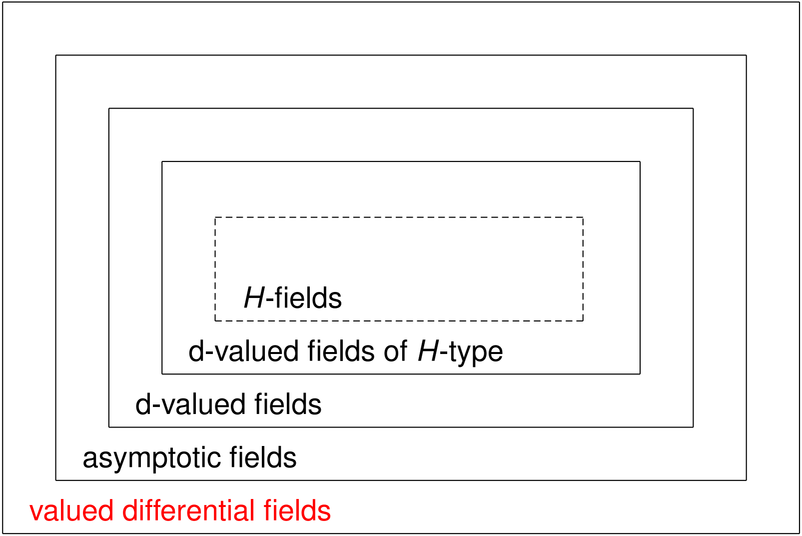
$$f > 0 \quad :\Longleftrightarrow \quad f(t) > 0 \text{, eventually.}$$

For $g \neq 0$, we have:

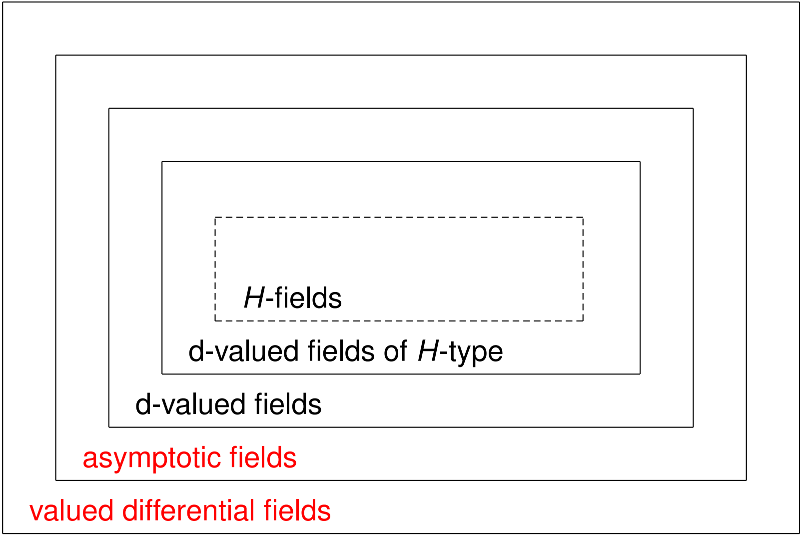
$$f \preccurlyeq g \iff \lim_{t \to +\infty} \frac{f(t)}{g(t)} \in \mathbb{R}, \qquad f \prec g \iff \lim_{t \to +\infty} \frac{f(t)}{g(t)} = 0,$$

$$f \asymp g \iff \lim_{t \to +\infty} \frac{f(t)}{g(t)} \in \mathbb{R}^{\times}, \qquad f \sim g \iff \lim_{t \to +\infty} \frac{f(t)}{g(t)} = 1.$$

Valued differential fields



Valued differential fields



Asymptotic fields

A valued differential field K is an **asymptotic field** if

$$\textit{for all nonzero } f, g \not\asymp 1, \qquad f \preccurlyeq g \quad \Longleftrightarrow \quad f' \preccurlyeq g'.$$

We say that K is of H-**type** (or H-**asymptotic**) if in addition

$$\textit{for all nonzero } f, g \prec 1, \qquad f \preccurlyeq g \quad \Longrightarrow \quad f^\dagger \succcurlyeq g^\dagger.$$

Examples

- Let K be a HARDY field. Then for nonzero $f, g \not\asymp 1$:

$$f \preccurlyeq g \iff \lim_{t\to+\infty} \frac{f(t)}{g(t)} \in \mathbb{R} \underset{\text{L'HÔPITAL}}{\iff} \lim_{t\to+\infty} \frac{f'(t)}{g'(t)} \in \mathbb{R} \iff f' \preccurlyeq g'.$$

  So K is asymptotic. One can check that K is of H-type.
- Every valued differential subfield of $\mathbb{T}$ is H-asymptotic.

Let K be an asymptotic field. We can define functions

$$\Gamma^{\neq} := \Gamma \setminus \{0\} \to \Gamma$$

by

$$\gamma = vg \mapsto \gamma' := vg', \qquad \gamma = vg \mapsto \psi(\gamma) := \gamma^{\dagger} := \gamma' - \gamma = vg^{\dagger}.$$

The pair (Γ, ψ), with $\psi(0) := \infty$, is an **asymptotic couple**, i.e.,

(AC1) $\psi(\alpha + \beta) \geqslant \min\big(\psi(\alpha), \psi(\beta)\big)$;

(AC2) $\psi(k\alpha) = \psi(\alpha)$ for all $k \in \mathbb{Z}^{\neq}$;

(AC3) $0 < \alpha < \beta \implies \alpha' < \beta'$.

We say that (Γ, ψ) is of ***H*-type**, or ***H*-asymptotic**, if in addition

(HC) $0 < \alpha \leqslant \beta \Longrightarrow \psi(\alpha) \geqslant \psi(\beta)$.

Example

Suppose $K = \mathbb{R}(\ell_0, \ell_1, \ell_2, \dots)$, so

$$\Gamma = \bigoplus_n \mathbb{Z} e_n \quad \text{where } e_n = v\ell_n < 0.$$

We have

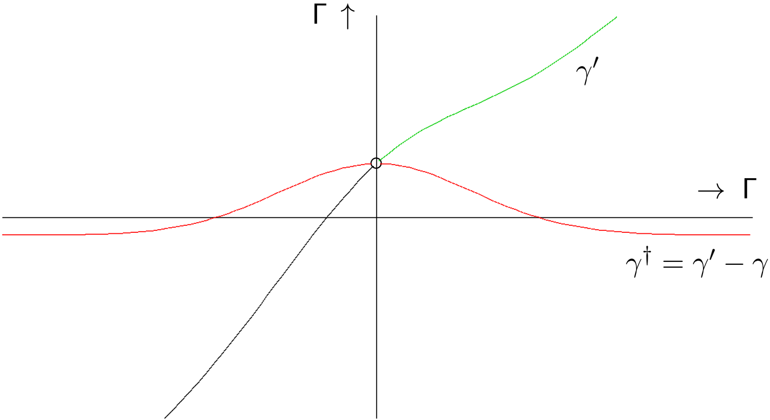
$$\ell_n' = \left(\log \ell_{n-1}\right)' = \frac{\ell_{n-1}'}{\ell_{n-1}} \quad \Longrightarrow \quad \ell_n' = \frac{1}{\ell_0 \cdots \ell_{n-1}}$$

$$\Longrightarrow \quad \ell_n^\dagger = \frac{1}{\ell_0 \cdots \ell_n}$$

Thus

$$(\Gamma^>)^\dagger = \left\{ v\left(\frac{1}{\ell_0}\right),\ v\left(\frac{1}{\ell_0 \ell_1}\right),\ \dots,\ v\left(\frac{1}{\ell_0 \cdots \ell_n}\right), \dots \right\}$$

Here is a picture of a typical *H*-asymptotic couple.



We always have $(\Gamma^>)^\dagger < (\Gamma^>)'$. (Even if (Γ, ψ) is not of *H*-type.)

What happens near the little circle is important.

Let (Γ, ψ) be an *H*-asymptotic couple. Exactly one of the following statements holds:

1. $(\Gamma^{>})^{\dagger} < \gamma < (\Gamma^{>})'$ *for a (necessarily unique)* γ.
   We call such γ a **gap** in (Γ, ψ).
2. $(\Gamma^{>})^{\dagger}$ *has a largest element.*
   We say that (Γ, ψ) is **grounded**.
3. $(\Gamma^{>})^{\dagger}$ *has no supremum; equivalently:* $\Gamma = (\Gamma^{\neq})'$.
   We say that (Γ, ψ) has **asymptotic integration.**

We use similar terminology for *H*-asymptotic fields.

Examples

1. $K = \mathbb{R}$ (but there are also more interesting examples);
2. $K = \mathbb{R}(\ell_0, \dots, \ell_n)$: then $\max(\Gamma^{>})^{\dagger} = v\left(\frac{1}{\ell_0 \cdots \ell_n}\right)$;
3. $K = \mathbb{R}(\ell_0, \ell_1, \ell_2, \dots)$, or $K = \mathbb{T}$.

The class of asymptotic fields is very robust, e.g., closed under

- taking substructures, compositional conjugation;
- algebraic extensions;
- coarsening and specialization.

Definition

Let Δ be a convex subgroup of Γ, with ordered quotient group $\dot{\Gamma} := \Gamma/\Delta$. Then K with its valuation replaced by

$$K^{\times} \xrightarrow{v} \Gamma \xrightarrow{\gamma \mapsto \gamma+\Delta} \dot{\Gamma}$$

is an asymptotic field, called the **coarsening** of K by Δ.

The class of asymptotic fields is very robust, e.g., closed under

- taking substructures, compositional conjugation;
- algebraic extensions;
- coarsening and specialization.

Important special case

Suppose K is H-asymptotic. Then

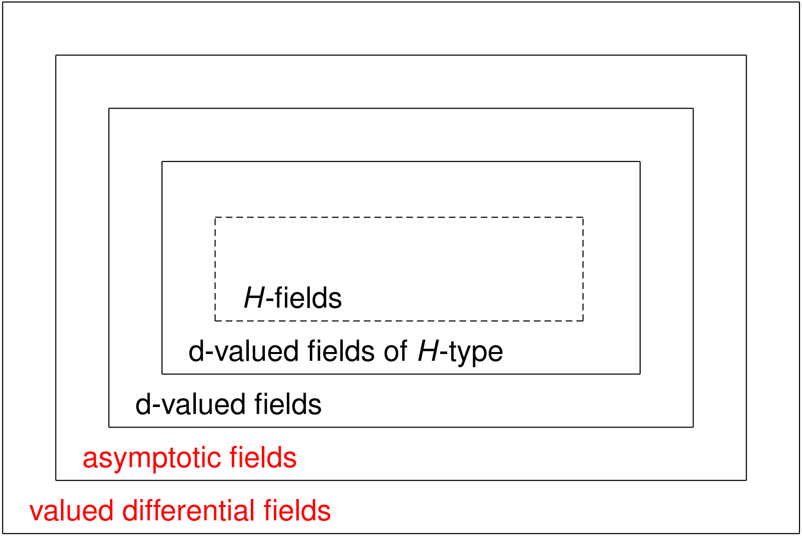
$$\Gamma^\flat := \{\gamma : \gamma^\dagger > 0\}$$

is a convex subgroup of Γ. More generally, so is

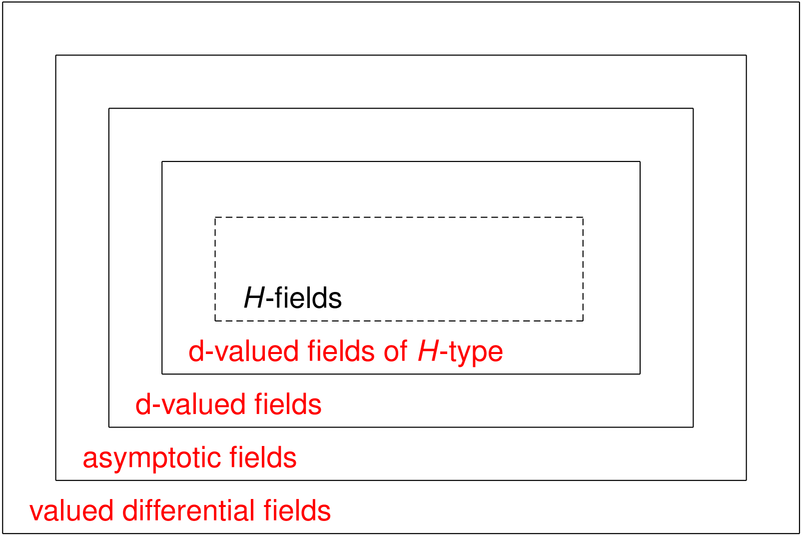
$$\Gamma^\flat_\phi := \{\gamma : \gamma^\dagger > v\phi\} \qquad \text{for } \phi \in K^\times.$$

If $K = \mathbb{T}$, then $\Gamma^\flat = \{vf : f \in \mathbb{T}^\times, f^n \prec \mathrm{e}^x \text{ for all } n\}$.

Differential-valued fields



Differential-valued fields



Let K be an asymptotic field. Then $C \subseteq \mathcal{O}$ and

$$c \mapsto c + \scriptstyle{\mathcal{O}}\displaystyle: \ C \to \operatorname{res}(K) = \mathcal{O}/\scriptstyle{\mathcal{O}} \qquad \text{is injective.}$$

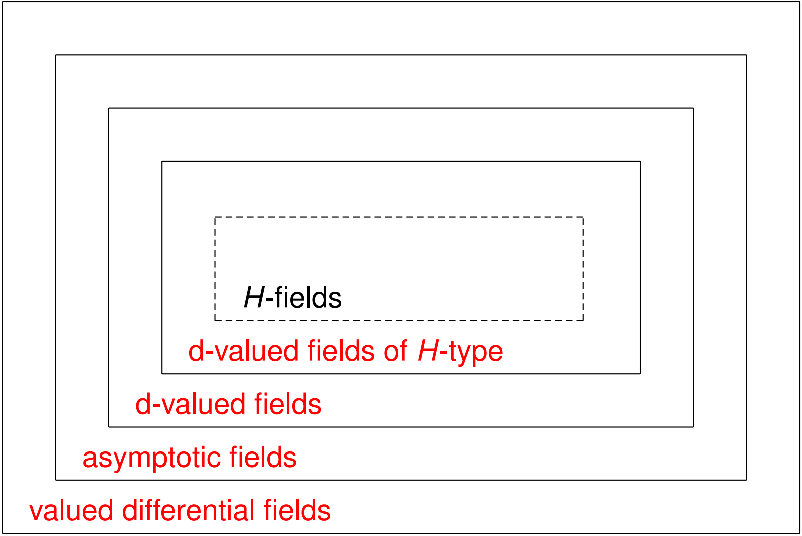
Definition

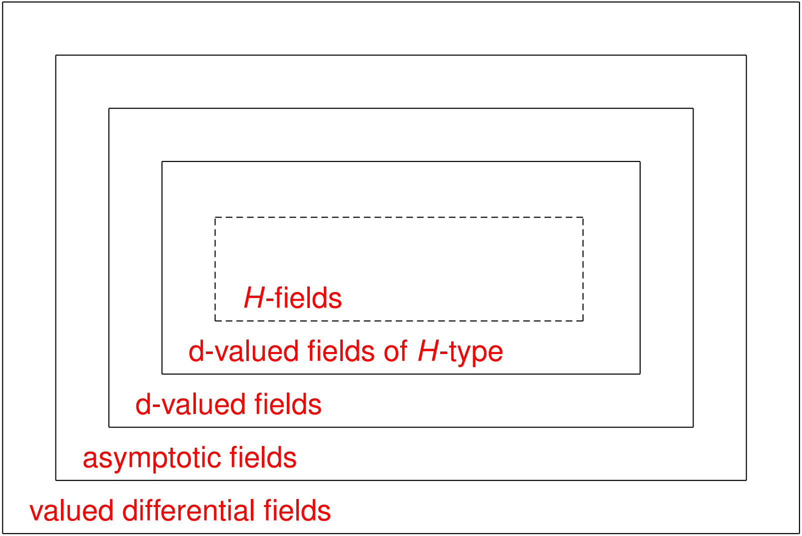
We say that K is d-**valued** if $\mathcal{O} = C + \scriptstyle{\mathcal{O}}$; equivalently, for each $f \asymp 1$ there is some $c \in C$ with $f \sim c$.

These were defined and first studied by ROSENLICHT (1980s), who in the process also introduced asymptotic couples.

The class of d-valued fields is not as robust as that of asymptotic fields, for example, not closed under taking substructures: consider

$$\mathbb{Q}\left(\sqrt{2 + x^{-1}}\right) \subseteq \mathbb{T}.$$


H-fields
d-valued fields of *H*-type
d-valued fields
asymptotic fields
valued differential fields


H-fields
d-valued fields of *H*-type
d-valued fields
asymptotic fields
valued differential fields

Definition

Let K be an *ordered* differential field. Then K is an ***H*-field** if

(H1) $f \succ 1 \;\Rightarrow\; f^\dagger > 0$; and

(H2) for each $f \asymp 1$ there is some $c \in C$ with $f \sim c$.

Every *H*-field, as valued differential field, is *H*-asymptotic and d-valued (by (H2)).

Each compositional conjugate K^ϕ of an *H*-field K with $\phi \in K$, $\phi > 0$, is an *H*-field.

Examples

- every ordered differential subfield $K \supseteq \mathbb{R}$ of $\mathbb{T}$;
- every HARDY field $K \supseteq \mathbb{R}$.

Recall that $\mathbb{T}$ is real closed, as well as closed under exponentiation and integration. This motivates the following:

Definition

Let K be an H-field. We say that K is LIOUVILLE **closed** if

1. K is real closed;
2. for each $f \in K$ there is some $y \in K$ with $y \neq 0$, $y^{\dagger} = f$; and
3. for each $g \in K$ there is some $z \in K$ with $z' = g$.

A LIOUVILLE **closure** of an H-field K is a minimal LIOUVILLE closed H-field extension of K.

Theorem

Every H-field K has exactly one or exactly two LIOUVILLE *closures, up to isomorphism over K.*

What can go wrong when forming LIOUVILLE closures may be seen from the asymptotic couple (Γ, ψ) of K. Recall that exactly one of the following holds:

1. K has a gap γ: $(\Gamma^{>})^{\dagger} < \gamma < (\Gamma^{>})'$
2. K is grounded: $(\Gamma^{>})^{\dagger}$ *has a largest element.*
3. K has asymptotic integration: $(\Gamma^{>})^{\dagger}$ *has no supremum.*

In 1 we have *two* LIOUVILLE closures: if $\gamma = vg$, then we have a choice when adjoining $\int g$: make it $\succ 1$ or $\prec 1$.

In 2 we have *one* LIOUVILLE closure: if $vg = \max\,(\Gamma^{>})^{\dagger}$, then $\int g \succ 1$ in each LIOUVILLE closure of K.

In 3 we may have *one or two* LIOUVILLE closures.

Order 2 linear differential equations in transseries

Since $\mathbb{T}$ is LIOUVILLE CLOSED, each linear differential equation

$$y' + fy = g \qquad (f, g \in \mathbb{T})$$

has a nonzero solution $y \in \mathbb{T}$. *What other kinds of algebraic differential equations have solutions in* $\mathbb{T}$*?*

Examples (2nd order linear)

- $y'' = -y$ has *no* solution $y \in \mathbb{T}^{\times}$;
- $y'' = xy$ has *two* $\mathbb{R}$-linearly independent solutions in $\mathbb{T}$:

$$\mathrm{Ai} = \frac{e^{-\xi}}{2\pi^{1/2}x^{1/4}} \sum_n (-1)^n \frac{a_n}{\xi^n}$$

$$\mathrm{Bi} = \frac{e^{\xi}}{\pi^{1/2}x^{1/4}} \sum_n (-1)^n \frac{a_n}{\xi^n} \qquad (\xi = \tfrac{2}{3}x^{3/2},\ a_n \in \mathbb{R}).$$

Order 2 linear differential equations in transseries

Let K be a LIOUVILLE closed H-field. For $f \in K$ and $y \in K^\times$,

$$4y'' + fy = 0 \iff \omega(2y^\dagger) = f$$

where $\omega(z) := -(2z' + z^2)$.

Hence

$$\omega(K) = \{f \in K : 4y'' + fy = 0 \text{ for some } y \in K^\times\}.$$

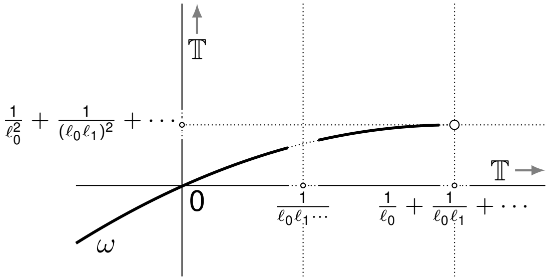
Example ($K = \mathbb{T}$)

$$\gamma_n := \ell_n^\dagger = \frac{1}{\ell_0 \cdots \ell_n}$$

$$\lambda_n := -\gamma_n^\dagger = \frac{1}{\ell_0} + \frac{1}{\ell_0\ell_1} + \cdots + \frac{1}{\ell_0\ell_1\cdots\ell_n}$$

$$\omega_n := \omega(\lambda_n) = \frac{1}{\ell_0^2} + \frac{1}{(\ell_0\ell_1)^2} + \cdots + \frac{1}{(\ell_0\ell_1\cdots\ell_n)^2}$$

One can show that the sequence (ω_n) is cofinal in $\omega(\mathbb{T})$, and that $\omega(\mathbb{T})$ is downward closed in $\mathbb{T}$ (as a consequence of the newtonianitynewtoniannity of $\mathbb{T}$).



Definition

Call an H-field K with asymptotic integration **ω-free** if $\omega(K)$ has no supremum in K. (This is not quite the definition of ω-free used in our book, but equivalent to it for LIOUVILLE closed K.)

Newtonianity

Newtonian is a version of "d-henselian" satisfied by $\mathbb{T}$, which says that certain kinds of d-polynomials in one variable over K have a zero $y \preccurlyeq 1$ in K. The definition involves compositional conjugation.

It guarantees, for example, that the PAINLEVÉ II equation

$$y'' = 2y^3 + xy + \alpha \qquad (\alpha \in C,\ x' = 1)$$

has a solution in $y \preccurlyeq 1$ in K.

We chose the adjective "newtonian" since it is this property that allows us to develop a NEWTON diagram method for differential polynomials.

ω-freeness and newtonianity will be discussed in more detail in JORIS' next talk.

II. The Main Results

The main results

From now on, we view each H-field K as a (model-theoretic) structure where we single out the primitives

$$0,\ 1,\ +,\ \cdot,\ \partial \text{ (derivation)},\ \leqslant \text{ (ordering)},\ \preccurlyeq \text{ (dominance)}.$$

Theorem A

The following statements about K axiomatize a model complete theory T^{nl}: K is

1. *a* LIOUVILLE *closed H-field;*
2. *ω-free;*
3. *newtonian.*

Moreover, $\mathbb{T}$ is a model of these axioms.

(The inclusion of $\preccurlyeq$ is necessary.)

The theory T^{nl} is not complete. It has exactly two completions:

- $T^{\text{nl}}_{\text{small}}$: small derivation;
- $T^{\text{nl}}_{\text{large}}$: large derivation.

Thus $T^{\text{nl}}_{\text{small}} = \text{Th}(\mathbb{T})$.

Corollary

$\mathbb{T}$ *is decidable; in particular:* there is an algorithm which, given d-polynomials $P_1, \ldots, P_m \in \mathbb{Q}(x)\{Y_1, \ldots, Y_n\}$, decides whether $P_1(y) = \cdots = P_m(y) = 0$ for some $y \in \mathbb{T}^n$.

There is no such algorithm if $\mathbb{T}$ is replaced by its *H*-subfield of exponential transseries.

Theorem A is the main step towards a quantifier elimination for $\mathbb{T}$, in a slightly extended language.

Let $\mathcal{L}^{\iota}_{\Lambda,\Omega}$ be our language $\mathcal{L} = \{0, 1, +, \cdot, \partial, \leqslant, \preccurlyeq\}$ augmented by a unary function symbol ι and unary predicates Λ, Ω.

Extend T^{nl} to the $\mathcal{L}^{\iota}_{\Lambda,\Omega}$-theory $T^{\mathrm{nl},\iota}_{\Lambda,\Omega}$ by adding as defining axioms for these new symbols the universal closures of

$$\big[a \neq 0 \longrightarrow a \cdot \iota(a) = 1\big] \ \& \ \big[a = 0 \longrightarrow \iota(a) = 0\big],$$
$$\Lambda(a) \ \longleftrightarrow \ \exists y\big[y \succ 1 \ \& \ a = -y^{\dagger\dagger}\big],$$
$$\Omega(a) \ \longleftrightarrow \ \exists y\big[y \neq 0 \ \& \ 4y'' + ay = 0\big].$$

For a model K of T^{nl} this makes both $\Lambda(K)$ and $\Omega(K) = \omega(K)$ downward closed.

Example ($K = \mathbb{T}$)

$$f \in \Lambda(\mathbb{T}) \iff f < \lambda_n = \frac{1}{\ell_0} + \frac{1}{\ell_0\ell_1} + \cdots + \frac{1}{\ell_0\ell_1\cdots\ell_n} \text{ for some } n,$$

$$f \in \Omega(\mathbb{T}) \iff f < \omega_n = \frac{1}{\ell_0^2} + \frac{1}{\ell_0^2\ell_1^2} + \cdots + \frac{1}{\ell_0^2\ell_1^2\cdots\ell_n^2} \text{ for some } n.$$

Theorem B

$T^{\mathrm{nl},\iota}_{\Lambda,\Omega}$ admits quantifier elimination.

The predicates Λ and Ω act as switchmen when constructing extensions of K: If an element γ in an H-field extension of K solving $\gamma^\dagger = -\lambda \in K$ is a gap, then $\Lambda(\lambda)$ tells us to choose $\int \gamma \succ 1$, while $\neg\Lambda(\lambda)$ forces $\int \gamma \prec 1$. Likewise, Ω controls what happens when we adjoin λ with $\omega(\lambda) = \omega \in K$.

III. The Next Lectures

Lecture 3 JORIS will discuss the main “machine” behind the proof of our theorems: the NEWTON diagram method.

Lecture 4 I will sketch the main steps in the proofs of Theorems A and B, and give some applications.

Lecture 5 LOU will speak about further developments.

Tutorial: Model Theory of Transseries

Lecture 4

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Overview

I. Newtonization

II. Strategy for the Proof of the Main Results

III. Applications

I. Newtonization

Reminders from the last lecture

Let K be a d-valued field of H-type with asymptotic integration. Recall that $C \cong \operatorname{res}(K)$. Suppose for simplicity that Γ is divisible and K is equipped with a "monomial group" $\mathfrak{M}$.

From JORIS' last lecture recall the definition of the NEWTON *polynomial* $N_P \in C\{Y\}^{\neq}$ of $P \in K\{Y\}^{\neq}$: eventually

$$P^{\phi} = \mathfrak{d} \cdot N_P + R_P \qquad \text{where } \mathfrak{d} = \mathfrak{d}_{\phi} \in \mathfrak{M} \text{ and } R_P \prec^{\flat}_{\phi} \mathfrak{d}.$$

If K is ω-free, then $N_P \in C[Y](Y')^{\mathbb{N}}$, and N_P doesn't change if we pass from K to an extension (of d-valued fields of H-type).

We put $\operatorname{ndeg} P := \deg N_P$ (the NEWTON degree of P).

We say that K is *newtonian* if every $P \in K\{Y\}$ with $\operatorname{ndeg} P = 1$ has a zero in $\mathcal{O}$. (Mostly useful in combination with ω-freeness.)

Constructing immediate extensions

Some reminders from general valuation theory

Let (a_ρ) be an ordinal-indexed sequence in K. Then

1. (a_ρ) **pseudoconverges to** $a \in K$ if $v(a - a_\rho)$ is eventually strictly increasing; notation: $a_\rho \rightsquigarrow a$;
2. (a_ρ) is **divergent** if it has no pseudolimit in K;
3. (a_ρ) is a **pseudocauchy sequence** in K if eventually

$$\tau > \sigma > \rho \quad \Longrightarrow \quad a_\tau - a_\sigma \prec a_\sigma - a_\rho;$$

   equivalently: (a_ρ) has a pseudolimit in an extension of K.

Declare $(a_\rho) \sim (b_\sigma)$ if (a_ρ), (b_σ) have the same pseudolimits in all extensions of K, and set

$$\boldsymbol{a} := c_K(a_\rho) = \text{equivalence class of } (a_\rho).$$

Constructing immediate extensions

Let L be an extension of K. Then $C \subseteq C_L$, and naturally $\Gamma \hookrightarrow \Gamma_L$.

If $C_L = C$ and $\Gamma_L = \Gamma$, then L is an **immediate** extension of K. In this case, every $a \in L \setminus K$ is a pseudolimit of a divergent pc-sequence in K.

Conversely, we can always adjoin pseudolimits in immediate extensions, as we now explain.

We introduce a classification of pc-sequences (a_ρ) in K:

1. d-*algebraic type* over K: $P(b_\lambda) \rightsquigarrow 0$ for some $P \in K\{Y\}$ and pc-sequence $(b_\lambda) \sim (a_\rho)$ in K;
2. d-*transcendental type* over K: not of d-algebraic type.

Any P as in 1, chosen so that $Q(b_\lambda) \not\rightsquigarrow 0$ whenever $Q \in K\{Y\}$ has lower complexity than P and $(b_\lambda) \sim (a_\rho)$, is a **minimal** d-**polynomial** of (a_ρ) over K.

Constructing immediate extensions

Theorem (d-analogues of KAPLANSKY's theorems)

Let (a_ρ) be a divergent pc-sequence in K.

1. *Suppose (a_ρ) is of* d*-algebraic type over K with minimal* d*-polynomial P over K.*

   There is some a in an immediate extension of K with $a_\rho \rightsquigarrow a$ and $P(a) = 0$, and for each b in an extension of K with $a_\rho \rightsquigarrow b$ and $P(b) = 0$ there is a K-isomorphism $K\langle a\rangle \to K\langle b\rangle$ with $a \mapsto b$.

2. *Suppose (a_ρ) is of* d*-transcendental type over K.*

   There is some a in an immediate extension of K with $a_\rho \rightsquigarrow a$, and for each b in an extension of K with $a_\rho \rightsquigarrow b$ there is a K-isomorphism $K\langle a\rangle \to K\langle b\rangle$ with $a \mapsto b$.

A consequence: if K is ω-free and has no proper immediate d-algebraic extension, then K is newtonian.

Reducing to NEWTON degree 1

The proof of the following important fact uses the full machinery of NEWTON diagrams, including its most complicated part ("unraveling": differential TSCHIRNHAUS transformations) for dealing with "almost multiple zeros" (only hinted at by JORIS in his last lecture):

Theorem

Suppose K is ω-free. Let (a_ρ) be a divergent pc-sequence in K with minimal d*-polynomial P over K. Then* $\operatorname{ndeg}_{\boldsymbol{a}} P = 1$*, i.e.,*

$$\operatorname{ndeg} P_{+a_\rho,\times(a_{\rho+1}-a_\rho)} = 1 \qquad \textit{for sufficiently large } \rho.$$

We now discuss how these facts can be used to embed K into a newtonian d-valued field in a "minimal" way.

Definition (an analogue of henselization of valued fields)

A **newtonization** of K is a newtonian extension of K which K-embeds into each newtonian extension of K.

Theorem

Suppose K is ω-free. Then K has a newtonization. Moreover, if L is a newtonization of K, then

- *L is an immediate extension of K;*
- *no proper differential subfield of L containing K is newtonian.*

We note the following consequence, which is a key ingredient for the proof of our main results.

Newton-Liouville closure

Corollary

Suppose K is an ω-free H-field. There is a newtonian Liouville closed H-field extension K^{nl} of K which embeds over K into each newtonian Liouville closed H-field extension of K. Any such K^{nl} is d-algebraic over K. Its constant field is a real closure of C.

We call K^{nl} the Newton-Liouville **closure** of K.

If K is ω-free, then each d-algebraic H-field extension of K is ω-free, and hence K has a unique Liouville closure up to isomorphism over K.

Thus one can obtain K^{nl} by alternating newtonization with taking Liouville closures.

Main ingredients for obtaining a newtonization

These are the results on constructing immediate extensions, the theorem on “reduction to ndeg 1”, and the following:

Lemma

Suppose K is newtonian. Let (a_ρ) be a pc-sequence in K and $P \in K\{Y\}$ with $\operatorname{ndeg}_{\boldsymbol{a}} P = 1$:

$$\operatorname{ndeg} P_{+a_\rho, \times(a_{\rho+1}-a_\rho)} = 1 \qquad \text{for sufficiently large } \rho.$$

Then there is some $a \in K$ with $P(a) = 0$ and $a_\rho \rightsquigarrow a$.

By our assumptions, for sufficiently large ρ we get $z_\rho \in K$ with

$$P(z_\rho) = 0 \quad \text{and} \quad z_\rho - a_\rho \preccurlyeq a_{\rho+1} - a_\rho.$$

We claim that for large enough ρ we can upgrade this to “$\asymp$” (and so take $a := z_\rho$ for large enough ρ). For this one shows that the zeros of P can’t “accumulate.”

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Then there is some $a \in K$ with $P(a) = 0$ and $a_\rho \rightsquigarrow a$.

With not too much extra work, this lemma also yields:

Corollary (assuming K ω-free)

K is newtonian $\iff$ K has no proper immediate d-algebraic extension.

II. Strategy for the Proof of the Main Results

Let $\mathcal{L} = \{0, 1, +, \cdot, \partial, \leqslant, \preccurlyeq\}$ and let

$$T^{\text{nl}} = \text{the theory of newtonian LIOUVILLE closed } H\text{-fields},$$

that is, the $\mathcal{L}$-theory axiomatized by

- the axioms for LIOUVILLE closed H-fields;
- the ω-freeness axiom; and
- the axiom scheme of newtonianity.

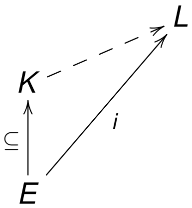
Every H-field extends to a model of T^{nl}, and in JORIS' lectures we heard that $\mathbb{T} \models T^{\text{nl}}$.

Theorem A

T^{nl} is model complete. (Hence T^{nl} is the model companion of the theory of H-fields.)

Strategy for the proof of Theorem A

By the familiar model completeness test of A. ROBINSON, it suffices to solve the following embedding problem:



Let E be an ω-free H-subfield of some $K \models T^{\mathrm{nl}}$, and let i be an embedding of E into a very saturated $L \models T^{\mathrm{nl}}$. Then i extends to an embedding $K \hookrightarrow L$.

We first make some preliminary reductions. First,

$$\begin{aligned} C_L \text{ is real closed, very saturated } &\Rightarrow\ i|C_E \text{ extends to } j\colon C \hookrightarrow C_L \\ &\Rightarrow\ j \text{ to } E(C) \hookrightarrow L. \end{aligned}$$

Since $E(C)$ is d-algebraic over E, it remains ω-free.

Strategy for the proof of Theorem A

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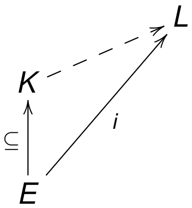


Let E be an ω-free H-subfield of some $K \models T^{\mathrm{nl}}$ such that $C_E = C$, and let i be an embedding of E into a very saturated $L \models T^{\mathrm{nl}}$. Then i extends to an embedding $K \hookrightarrow L$.

Next, suppose $\Gamma_E^{<}$ is not cofinal in $\Gamma^{<}$. Take $y \in K^{>}$, $y^* \in L^{>}$ with

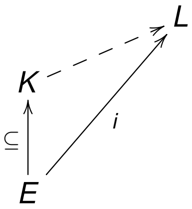
$$\Gamma_E^{<} < vy < 0, \qquad \Gamma_{iE}^{<} < vy^* < 0.$$

Now $E\langle y\rangle$ is grounded, but it extends to an ω-free H-field "$E\langle y\rangle_\omega = E\langle y, \log y, \log\log y, \ldots\rangle$" in a canonical way.

So i extends to an embedding $E\langle y\rangle_\omega \hookrightarrow L$ with $y \mapsto y^*$.

Strategy for the proof of Theorem A

By the familiar model completeness test of A. ROBINSON, it suffices to solve the following embedding problem:



Let E be an ω-free H-subfield of some $K \models T^{\mathrm{nl}}$ such that $C_E = C$ and $\Gamma_E^{<}$ is cofinal in $\Gamma^{<}$, and let i be an embedding of E into a very saturated $L \models T^{\mathrm{nl}}$. Then i extends to an embedding $K \hookrightarrow L$.

This has the nice consequence that now we don't need to worry about preserving ω-freeness anymore: *every* differential subfield of K containing E is an ω-free H-subfield of K.

Strategy for the proof of Theorem A

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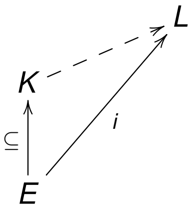
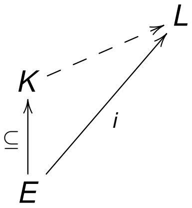


Let E be an ω-free H-subfield of some $K \models T^{\mathrm{nl}}$ such that $C_E = C$ and $\Gamma_E^{<}$ is cofinal in $\Gamma^{<}$, and let i be an embedding of E into a very saturated $L \models T^{\mathrm{nl}}$. Then i extends to an embedding $K \hookrightarrow L$.

Now we have the following three cases:

1. E is not newtonian and LIOUVILLE closed;
2. E is newtonian and LIOUVILLE closed, and there is some $y \in K \setminus E$ such that $E\langle y\rangle | E$ is immediate;
3. E is newtonian and LIOUVILLE closed, but there is no $y \in K \setminus E$ such that $E\langle y\rangle | E$ is immediate.

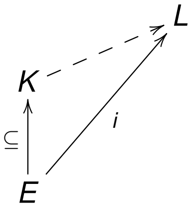
Strategy for the proof of Theorem A



Case 1

E is not newtonian and LIOUVILLE closed.

Then we can extend i to an embedding $E^{\mathrm{nl}} \hookrightarrow L$ of the NEWTON-LIOUVILLE closure E^{nl} of E inside K.



Case 2

E is newtonian and LIOUVILLE closed, and we have $y \in K \setminus E$ with $E\langle y\rangle|E$ immediate.

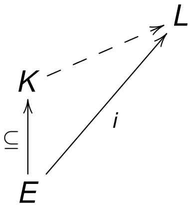
Take a divergent pc-sequence (a_ρ) in E such that $a_\rho \rightsquigarrow y$.

By saturation, take $z \in L$ with $i(a_\rho) \rightsquigarrow z$.

Since E is has no proper immediate d-algebraic extension, (a_ρ) is of d-transcendental type over E.

Thus i extends to $E\langle y\rangle \hookrightarrow L$ with $y \mapsto z$.

Strategy for the proof of Theorem A



Case ❸

E is newtonian and LIOUVILLE closed, and for no $y \in K \setminus E$ is $E\langle y\rangle|E$ immediate.

In this case it turns out that for each $f \in K \setminus E$, the *cut of f in the ordered set E* uniquely determines the isomorphism type of $E\langle f\rangle$ over E (and so we can again appeal to saturation).

Let's look at this case in some more detail.

Here we are approximating f by iterated exponential integrals.

Strategy for the proof of Theorem A

Setting

Let $E \subseteq K$ be an extension of ω-free newtonian LIOUVILLE closed H-fields with $C_E = C$, and suppose E is maximal in K: for no $y \in K \setminus E$ is $E\langle y\rangle|E$ immediate.

Then no divergent pc-sequence in E has a pseudolimit in K.

Definition

Let $f \in K \setminus E$. Then $v(f - E) \subseteq \Gamma$ has a largest element, and we call $b \in E$ a **best approximation** to f if

$$v(f - b) = \max v(f - E).$$

Note that then $v(f - b) \notin \Gamma_E$ since $C = C_E$.

Strategy for the proof of Theorem A

Setting

Let $E \subseteq K$ be an extension of ω-free newtonian LIOUVILLE closed H-fields with $C_E = C$, and suppose E is maximal in K: for no $y \in K \setminus E$ is $E\langle y\rangle|E$ immediate.

Let $f \in K \setminus E$. Pick a best approximation $b_0 \in E$ to $f_0 := f$. Then $f_1 := (f_0 - b_0)^\dagger \notin E$ since E is LIOUVILLE closed and $C_E = C$. So we can take a best approximation b_1 to f_1, etc.

We get sequences (f_n) in $K \setminus E$ and (a_n), (b_n) in E such that

- $a_n^\dagger = b_n$ is a best approximation to f_n, and
- $f_{n+1} = (f_n - b_n)^\dagger$.

$$\text{“}\, f \;=\; b_0 + \mathrm{e}^{\int f_1} \;=\; b_0 + \mathrm{e}^{\int b_1 + \mathrm{e}^{\int f_2}} \;=\; \cdots \text{”}.$$

Strategy for the proof of Theorem A

Setting

Let $E \subseteq K$ be an extension of ω-free newtonian LIOUVILLE closed H-fields with $C_E = C$, and suppose E is maximal in K: for no $y \in K \setminus E$ is $E\langle y\rangle|E$ immediate.

Then for each $P \in E\{Y\}$ one can expand $P(f) \in K$ as a polynomial in the "monomials"

$$\mathfrak{m}_n := (f_n - b_n)/a_{n+1} \in E\langle f\rangle.$$

Using this one gets detailed information about the asymptotic couple of $E\langle f\rangle$: with $\mu_n := v\mathfrak{m}_n \in \Gamma_{E\langle f\rangle}$,

- $\Gamma_{E\langle f\rangle} = \Gamma_E \oplus \bigoplus_n \mathbb{Z}\mu_n$, and $\Gamma_E^<$ is cofinal in $\Gamma_{E\langle f\rangle}^<$;
- $\psi\left(\Gamma_{E\langle f\rangle}^>\right) = \psi(\Gamma_E^>) \cup \left\{\mu_0^\dagger < \mu_1^\dagger < \cdots\right\}$ with $\mu_n^\dagger \notin \Gamma_E$.

All this turns out to only depend on the cut of f in E!

Before we move on to Theorem B, we record:

Corollary

The completions of T^{nl} are $T^{\mathrm{nl}}_{\mathrm{small}} = \mathrm{Th}(\mathbb{T})$ and $T^{\mathrm{nl}}_{\mathrm{large}}$.

To see this, we note that the ω-free H-field $E := \mathbb{Q}(\ell_0, \ell_1, \dots)$ embeds into each LIOUVILLE closed H-field with small derivation, in particular into $\mathbb{T}$.

So the NEWTON-LIOUVILLE closure

$$\mathbb{T}^{\mathrm{da}} := \{f \in \mathbb{T} : f \text{ is d-algebraic}\}$$

of E inside $\mathbb{T}$ is a prime model of $T^{\mathrm{nl}}_{\mathrm{small}}$.

Similarly, the NEWTON-LIOUVILLE closure of E^{ϕ} ($\phi = x^{-2}$), is a prime model of $T^{\mathrm{nl}}_{\mathrm{large}}$.

Let $\mathcal{L}^{\iota}_{\Lambda,\Omega} = \mathcal{L} \cup \{\iota, \Lambda, \Omega\}$ and let

$$T^{\mathrm{nl},\iota}_{\Lambda,\Omega} = T^{\mathrm{nl}} + \text{the universal closures of}$$

$$\big[a \neq 0 \longrightarrow a \cdot \iota(a) = 1\big] \ \& \ \big[a = 0 \longrightarrow \iota(a) = 0\big],$$

$$\Lambda(a) \ \longleftrightarrow \ \exists y \big[y \succ 1 \ \& \ a = -y^{\dagger\dagger}\big],$$

$$\Omega(a) \ \longleftrightarrow \ \exists y \big[y \neq 0 \ \& \ 4y'' + ay = 0\big].$$

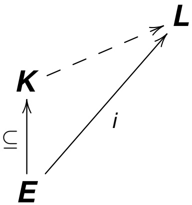
We denote $\mathcal{L}^{\iota}_{\Lambda,\Omega}$-structures by boldface letters: $\boldsymbol{K} = (K, \Lambda, \Omega)$.

Theorem B

$T^{\mathrm{nl},\iota}_{\Lambda,\Omega}$ *admits quantifier elimination.*

Strategy for the proof of Theorem B

Again, we need to solve an embedding problem:



*Let **E** be a substructure of some **K** $\models T^{\mathrm{nl},\iota}_{\Lambda,\Omega}$ and let i be an embedding of **E** into a very saturated **L** $\models T^{\mathrm{nl},\iota}_{\Lambda,\Omega}$. Then i extends to an embedding **K** $\hookrightarrow$ **L**.*

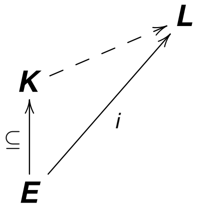
In order to tackle this, we need to first investigate the substructures of models of $T^{\mathrm{nl},\iota}_{\Lambda,\Omega}$.

Since we included ι in $\mathcal{L}^{\iota}_{\Lambda,\Omega}$, such substructures are valued ordered differential *fields.*

However, they are not automatically H-fields $\rightsquigarrow$ **pre-*H*-fields.**

Strategy for the proof of Theorem B

Again, we need to solve an embedding problem:



Let $\boldsymbol{E}$ *be a substructure of some* $\boldsymbol{K} \models T^{\mathrm{nl},\iota}_{\Lambda,\Omega}$ *and let* i *be an embedding of* $\boldsymbol{E}$ *into a very saturated* $\boldsymbol{L} \models T^{\mathrm{nl},\iota}_{\Lambda,\Omega}$*. Then* i *extends to an embedding* $\boldsymbol{K} \hookrightarrow \boldsymbol{L}$*.*

The pairs (Λ, Ω) of subsets of a pre-H-field E such that $\boldsymbol{E} = (E, \Lambda, \Omega)$ embeds into a model of $T^{\mathrm{nl},\iota}_{\Lambda,\Omega}$ are characterized by the axioms for $\Lambda\Omega$**-cuts** in E. We show:

- every ω-free pre-H-field has just one $\Lambda\Omega$-cut;
- $\boldsymbol{E}$ has an extension $\boldsymbol{E}^* = (E^*, \dots)$, where E^* is an ω-free H-field, which embeds over E into any model of $T^{\mathrm{nl},\iota}_{\Lambda,\Omega}$ extending $\boldsymbol{E}$.

Strategy for the proof of Theorem B

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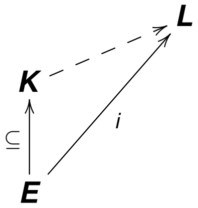

Let $\boldsymbol{E}$ *be a substructure of some* $\boldsymbol{K} \models T^{\mathrm{nl},\iota}_{\Lambda,\Omega}$ *and let i be an embedding of* $\boldsymbol{E}$ *into a very saturated* $\boldsymbol{L} \models T^{\mathrm{nl},\iota}_{\Lambda,\Omega}$*. Then i extends to an embedding* $\boldsymbol{K} \hookrightarrow \boldsymbol{L}$*.*

These two facts allow us to focus henceforth, for embedding purposes, on ω-free *H*-fields, and we can forget about $\Lambda\Omega$-cuts.

Theorem B now follows from the embedding theorem that we used in proving Theorem A. (The embedding theorem is somewhat stronger than model completeness of T^{nl}, since E there is only assumed to be ω-free.)

III. Applications

Corollary

1. $\mathbb{T}$ *is* o-minimal at $+\infty$*: if $X \subseteq \mathbb{T}$ is definable, then there is some $f \in \mathbb{T}$ with $(f, +\infty) \subseteq X$ or $(f, +\infty) \cap X = \emptyset$.*
2. *All definable subsets of $\mathbb{R}^n \subseteq \mathbb{T}^n$ are* semialgebraic.
3. $\mathbb{T}$ *has* NIP*.*

An instance of 1: if P is a one-variable d-polynomial over $\mathbb{T}$, then there is some $f \in \mathbb{T}$ and $\sigma \in \{\pm 1\}$ with sign $P(y) = \sigma$ for all $y > f$. (Related to old theorems of BOREL, HARDY, ...)

An illustration of 2: the set of $(c_0, \ldots, c_n) \in \mathbb{R}^{n+1}$ such that

$$c_0 y + c_1 y' + \cdots + c_n y^{(n)} = 0, \qquad 0 \neq y \prec 1$$

has a solution in $\mathbb{T}$ is a semialgebraic subset of $\mathbb{R}^{n+1}$.

One can strengthen 3 to "$\mathbb{T}$ is distal" (of infinite dp-rank).

Eliminate the primitives $\preccurlyeq$, Λ, Ω, ι using “ideal” elements, thus reducing quantifier-free formulas to a very simple form:

Let $K \models T^{\mathrm{nl}}$. In an immediate H-field extension L of K we find some element λ with

$$\begin{array}{rrcll} & \Lambda(K) & < & \lambda & < \ K \setminus \Lambda(K), \\ \text{so} & \Omega(K) & < & \omega := \omega(\lambda) & < \ K \setminus \Omega(K). \end{array}$$

Next take some c^* in an H-field extension L^* of L with

$$C < c^* < K^{>C}.$$

Then for each 0-definable $X \subseteq K^n$ there is a quantifier-free formula φ in the language $\mathcal{L}_{\mathrm{OR}}$ of ordered rings such that

$$X \ = \ \{a \in K^n : \ L^* \models \varphi(a, a', \dots, a^{(r)}, \lambda, \omega, c^*)\}.$$

We illustrate this by establishing ❸ through reduction to NIP for real closed fields: Suppose $R \subseteq K^m \times K^n$ is 0-definable and independent. We just do the case $m = n = 1$. Thus for every $N \geqslant 1$ there are $a_1, \dots, a_N \in K$ and $b_I \in K$ ($I \subseteq \{1, \dots, N\}$) with

$$R(a_i, b_I) \iff i \in I.$$

Take a quantifier-free $\mathcal{L}_{\mathrm{OR}}$-formula φ such that for all $a, b \in K$:

$$R(a, b) \iff L^* \models \varphi(a, a', \dots, a^{(r)}, b, b', \dots, b^{(r)}, \lambda, \omega, c^*).$$

Thus the relation $R^* \subseteq (L^*)^{r+1} \times (L^*)^{r+4}$ given by

$$R^*(a_0, \dots, a_r, b_0, \dots, b_{r+3}) \iff L^* \models \varphi(a_0, \dots, a_r, b_0, \dots, b_{r+3})$$

is independent and (q.f.-) definable in the $\mathcal{L}_{\mathrm{OR}}$-structure L^*. ↯