DISTALITY IN VALUED FIELDS AND RELATED STRUCTURES

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ABSTRACT. We investigate distality and existence of distal expansions in valued fields and related structures. In particular, we characterize distality in a large class of ordered abelian groups, provide an AKE-style characterization for henselian valued fields, and demonstrate that certain expansions of fields, e.g., the differential field of logarithmic-exponential transseries, are distal. As a new tool for analyzing valued fields we employ a relative quantifier elimination for pure short exact sequences of abelian groups.

CONTENTS

INTRODUCTION

Distal theories were introduced in [62] as a way to distinguish those NIP theories in which no stable behavior of any kind occurs. Examples include all (weakly) o-minimal theories (e.g., the theory of the exponential ordered field of reals) and all P-minimal theories (such as the theory of the field of p-adic numbers and its analytic expansion from [24]); see the introduction of [18] for a detailed discussion. Distality has been investigated both from the point of view of pure model theory [6, 7, 14, 49] and in connection to the extremal combinatorics of restricted families of graphs. Indeed, as demonstrated in [18], distality of a theory is equivalent to a definable version of the *strong Erdős-Hajnal Property*. Further results in [11, 17] show that many of the combinatorial consequences of distality, including the strong Erdős-Hajnal Property, improved regularity lemmas and various generalized incidence bounds, continue to hold for structures which are merely *interpretable* in distal structures. Curiously, finding a distal expansion also appears to be the easiest way of establishing these combinatorial results in a given structure. This motivates the question: *which NIP structures admit distal expansions?* Currently, the only known reason for *not* having a distal expansion comes from interpreting an infinite field of positive characteristic; see Section 2 below, where we also point out that more generally, every infinite distal unital ring without zero-divisors has characteristic zero.

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The aim of this paper is to investigate both issues—distality and existence of distal expansions—in the setting of valued fields and various related structures: ordered abelian groups, short exact sequences of abelian group, valued fields with operators. This provides new examples in which the aforementioned combinatorial results hold, and along the way yields some general tools to address these problems in similar settings. The question of classifying NIP (valued) fields is currently an active area of research motivated by various versions of Shelah's Conjecture. (See [27, 37, 42, 48] and references therein for some recent results.) In particular, good understanding has been achieved in the dp-minimal case [45, 46]; see Section 6.6 for more details. (We recall the definition of dp-minimality in Section 1.1.) Our results demonstrate that some of the issues in this program simplify in the distal case, where infinite fields of positive characteristic are ruled out, while new complications arise due to the fact that distality is not preserved under taking reducts.

As a practical matter, we will not in general set out to prove from scratch that the structures we are interested in are distal (or not distal). Instead, whenever possible we will view structures as mild expansions of certain distal reducts, and then study how distality passes from the reduct up to the original structure. For instance, in Section 7 we show that certain expansions of valued fields by unary operators are distal by reducing the problem to the reduct of said valued field without the additional operators. For this reason, we will often rely on abstract criteria which (under certain circumstances) show how the distality of a structure can be deduced from the distality of a suitably chosen reduct.

In Section 1 we recall basic results and notions around distality, as well as prove some auxiliary lemmas for verifying that certain expansions in an abstract model-theoretic setting are distal. In Section 2 we briefly discuss distal fields and rings. Using Hahn products we give an example of an infinite unital ring of prime characteristic which has a distal expansion.

In Section 3 we then study distality in the class of ordered abelian groups. While every ordered abelian group G is NIP by [35], distality may fail due to the presence of infinite stable quotients of the form G/nG. Theorem 3.13 makes this precise by characterizing distality in a large class of ordered abelian groups. To properly state this result requires the many-sorted language $\mathcal{L}_{\mathrm{qe}}$ of Cluckers and Halupczok [19], so we only mention here a consequence and save the discussion of $\mathcal{L}_{\mathrm{qe}}$ and the full statement of Theorem 3.13 for Section 3.

Corollary. *Let G be a strongly dependent ordered abelian group; then*

$$G \text{ is distal} \iff G \text{ is dp-minimal} \iff G \text{ is non-singular (i.e., } G/pG \text{ finite for every prime } p).$$

In Section 4 we consider distality in short exact sequences of abelian groups with extra structure. That is, we consider short exact sequences of abelian groups $0 \to A \to B \to C \to 0$ viewed in a natural way as three-sorted structures with the corresponding morphisms named as primitives, and with arbitrary additional structure allowed on the sorts A and C. In Section 4.1 we give a general quantifier elimination result for *pure* short exact sequences, i.e., where the image of A is assumed to be a pure subgroup of B. (This applies when C is torsion-free.) In this case only sorts for the quotients A/nA and certain induced maps $B \to A/nA$ have to be added in order to eliminate quantification over B; see Corollary 4.3 for the precise statement. This generalizes a result in [15], where all of the quotients A/nA ($n \geq 1$) were assumed to be finite. Using this quantifier elimination, we show in Section 4.2 that such a pure short exact sequence is distal (has a distal expansion) if and only if both A and C are distal (have distal expansions, respectively). Note that the theory of a pure short exact sequence is interpretable in the theory of the direct product $A \times C$, as explained at the beginning of Section 4.1; however in general, distality is not preserved under passing to reducts, thus a precise description of the definable sets is necessary for our purpose. In Sections 4.3, 4.4, and 4.5 we consider variants and extensions of our quantifier elimination theorem. We expect these elimination theorems for short exact sequences to have many uses. As an illustration, we employ some of these variants in Section 5 to prove some quantifier elimination theorems for henselian valued fields of characteristic zero.

In Section 6 we consider distality in henselian valued fields. Relying on the results of the previous sections, in Sections 6.1 and 6.2 we prove the following Ax-Kochen-Eršov (AKE) type characterization. Recall that a valued field K with valuation $v\colon K^\times = K \setminus \{0\} \to \Gamma = v(K^\times)$ is said to be *finitely ramified* if for each $n \geq 1$ there are only finitely many $\gamma \in \Gamma$ such that $0 \leq \gamma \leq v(n)$. If $\Gamma \neq \{0\}$, then this clearly implies that the field K has characteristic zero; if K has equicharacteristic zero, then K is always finitely ramified.

Main Theorem. *Let K be a henselian valued field, viewed as a structure in the language of rings augmented by a predicate for the valuation ring, with value group Γ and residue field $\boldsymbol{k}$. Then K is distal* (*has a distal expansion*) *if and only if*

1. *K is finitely ramified, and*
2. *both Γ and $\boldsymbol{k}$ are distal* (*respectively, have distal expansions*).

In this case $\boldsymbol{k}$ is either finite or of characteristic zero.

For example, this theorem implies that a finitely ramified henselian valued field K with regular non-singular value group is distal if and only if the residue field of K is distal; this generalizes the well-known facts that each p-adically closed field is distal, and that a real closed valued field is distal iff its residue field is real closed.

In Section 6.3 we consider Jahnke's results [42] on naming a henselian valuation in the distal case. In Section 6.5 we formulate a conjectural classification of fields admitting a distal expansion: a (pure) NIP field does not have a distal expansion if and only if it interprets an infinite field of positive characteristic. We show that this statements holds modulo Shelah's conjecture on NIP fields and a conjecture on distal expansions of ordered abelian groups from Section 3. For this, we rely on definability theorems of Koenigsmann-Jahnke [43], in a similar way as Johnson [47, Chapter 9]. In Section 6.6 we concentrate on the dp-minimal case; based on Johnson's results [46], we observe that our conjecture does hold unconditionally for dp-minimal fields.

Finally, in Section 7 we show that a certain "forgetful functor" argument preserves distality. Utilizing this, we exhibit expansions of (valued) fields with additional operators (e.g., derivations) which are distal. Examples include the differential field of transseries [2] and certain topological fields with a generic derivation in the sense of [36, 67]. This also implies that the theory of differentially closed fields of characteristic zero admits a distal expansion (Corollary 7.7). These techniques also yield that analytic expansions of distal valued fields of characteristic zero are distal (Corollary 7.10).

Conventions and notations. Throughout, m and n (possibly with decorations) range over the set $\mathbb{N} = \{0, 1, 2, \dots\}$. In general we adopt the model theoretic conventions of Appendix B of [2]. In particular, $\mathcal{L}$ can be a many-sorted language. Given a complete $\mathcal{L}$-theory T, we will sometimes consider a model $\mathbb{M} \models T$ and a cardinal $\kappa(\mathbb{M}) > |\mathcal{L}|$ such that $\mathbb{M}$ is $\kappa(\mathbb{M})$-saturated and every reduct of $\mathbb{M}$ is strongly $\kappa(\mathbb{M})$-homogeneous. Such a model is called a *monster model* of T. Then every model of T of size $\leq \kappa(\mathbb{M})$ can be elementarily embedded into $\mathbb{M}$. "Small" will mean "of size $< \kappa(\mathbb{M})$". We use x, y, z (sometimes with decorations) to denote multivariables. Unless otherwise specified, all multivariables are assumed to have *finite* size, and the size of such a multivariable x is denoted by $|x|$. We shall write "$\models \theta$" to indicate that θ is an $\mathcal{L}_{\mathbb{M}}$-formula and $\mathbb{M} \models \theta$. Likewise, "$\Phi(x) \models \Theta(x)$" will mean that $\Phi(x)$ and $\Theta(x)$ are small sets of $\mathcal{L}_{\mathbb{M}}$-formulas such that every $a \in \mathbb{M}_x$ realizing $\Phi(x)$ also realizes $\Theta(x)$. We write "$\varphi(x) \models \Theta(x)$" to abbreviate $\{\varphi(x)\} \models \Theta(x)$, etc.

Given linearly ordered sets I and J we denote by $I^\frown J$ the concatenation of I and J, that is, the set $K := I \cup J$ (disjoint union) equipped with the linear ordering extending both the orderings of I and J such that $I < J$. If, say, $I = \{i\}$ is a singleton, we also write $I^\frown J = i^\frown J$. Similarly, given sequences $a = (a_i)_{i \in I}$ and $b = (b_j)_{j \in J}$ in $\mathbb{M}_x$, where I, J are linearly ordered sets, we let $a^\frown b$ denote the sequence $(c_k)_{k \in K}$ where $K = I^\frown J$ and $c_i = a_i$ for $i \in I$, $c_j = b_j$ for $j \in J$. We extend this notation to the concatenation of several (finitely many) linearly ordered sets respectively sequences in the natural way. If $a = (a_i)_{i \in I}$ is a sequence and $J \subseteq I$, we let $a_J := (a_j)_{j \in J}$. By convention "indiscernible sequence" means "$\emptyset$-indiscernible sequence".

1. Preliminaries on Distality

Throughout this section $\mathcal{L}$ is a language and T is a complete $\mathcal{L}$-theory. We also fix a monster model $\mathbb{M}$ of T. The definitions below do not depend on the choice of this monster model.

1.1. Two ways of defining distality.

Distality has many facets, and can be introduced in a number of equivalent ways. In this subsection we present two of them: by means of *indiscernible sequences,* and via *honest definitions.*

Definition 1.1. We say that T is **distal** if for every small parameter set $B \subseteq \mathbb{M}$, every indiscernible sequence $a = (a_i)_{i \in I}$ in $\mathbb{M}_x$, and every $i \in I$, the following holds: if

(1) $I^< = I^{<i} := \{j \in I : j < i\}$ and $I^> = I^{>i} := \{j \in I : i < j\}$ are infinite, and
(2) $a_{I \setminus \{i\}}$ is B-indiscernible,

then a is B-indiscernible. We say that an $\mathcal{L}$-structure is **distal** if its theory is distal.

While the definition of distality given above involves checking a certain condition for all infinite linearly ordered sets $I^<$ and $I^>$, standard arguments show that this definition is equivalent to the variant where $I^<$ and $I^>$ are fixed infinite linearly ordered sets. More precisely, fix a linearly ordered set $I = I^{<\frown} i^\frown I^>$ where $I^<, I^>$ are infinite; then the theory T is distal if for every small parameter set $B \subseteq \mathbb{M}$, an indiscernible sequence $(a_i)_{i \in I}$ in $\mathbb{M}_x$ is B-indiscernible provided $(a_i)_{i \in I \setminus \{i\}}$ is B-indiscernible. For this reason, in practice we can (and often will) assume that $I^<$ and $I^>$ are "nice" infinite linearly ordered sets such as $\mathbb{Q}$ or $[0, 1]$.

Definition 1.1 can be localized to a particular indiscernible sequence:

Definition 1.2 ([62, Definition 2.1]). Let $a = (a_i)_{i \in I}$ be an indiscernible sequence in $\mathbb{M}_x$. Then a is **distal** if for every indiscernible sequence $a' = (a'_i)_{i \in I'}$ in $\mathbb{M}_x$ with the same EM-type as a and $I' = I_1{}^\frown I_2{}^\frown I_3$ where I_1, I_2, I_3 are dense without endpoints, and all $c, d \in \mathbb{M}_x$, the following holds: if the sequences

$$a'_{I_1}{}^\frown c^\frown a'_{I_2}{}^\frown a'_{I_3} \quad \text{and} \quad a'_{I_1}{}^\frown a'_{I_2}{}^\frown d^\frown a'_{I_3}$$

are indiscernible, then so is $a'_{I_1}{}^\frown c^\frown a'_{I_2}{}^\frown d^\frown a'_{I_3}$.

Definitions 1.1 and 1.2 are connected by the following fact.

Fact 1.3 ([62, Lemma 2.7]). *Suppose T is NIP, and let $a = (a_i)_{i \in I}$ be an indiscernible sequence in $\mathbb{M}_x$; then the following are equivalent:*

(1) *a is distal;*
(2) *for every small parameter set $B \subseteq \mathbb{M}$, $b \in \mathbb{M}_x$, and B-indiscernible sequence $a' = (a'_i)_{i \in I'}$ in $\mathbb{M}_x$ with $I' = I_1{}^\frown I_2$, I_1 and I_2 without endpoints, having the same EM-type as a, if $a'_{I_1}{}^\frown b^\frown a'_{I_2}$ is indiscernible, then it is also B-indiscernible.*

In particular, T is distal if and only if every infinite indiscernible sequence is distal.

It is well-known that if T is distal, then T is NIP; for instance, see [34, Proposition 2.8]. *Distality* can be thought of as a notion of *pure instability* among NIP theories. The following fact (which follows from [62, Corollary 2.15]) is evidence for this point of view.

Fact 1.4. *If T is distal then no infinite non-constant indiscernible sequence is totally indiscernible.*

In the dp-minimal case we also have a converse. We first recall the definition of dp-minimality. Recall that a *cut* in a linearly ordered set I is a downward closed subset of I; such a cut $\mathfrak{c}$ is *trivial* if $\mathfrak{c} = \emptyset$ or $\mathfrak{c} = I$. We let $\overline{I}$ be the set of nontrivial cuts in I, totally ordered by inclusion; if I does not have a largest element, then the map which sends $i \in I$ to the cut $\{j \in I : j \leq i\}$ is an embedding $I \to \overline{I}$ of ordered sets, and we then identify I with its image under this embedding. Now the theory T is called dp-**minimal** if for each indiscernible sequence $a = (a_i)_{i \in I}$ in $\mathbb{M}_x$ indexed by a dense linearly ordered set I and each $c \in \mathbb{M}_y$ there is a cut $\mathfrak{i} \in \overline{I}$ such that the sequences $(a_i)_{i<\mathfrak{i}}$ and $(a_i)_{i>\mathfrak{i}}$ are c-indiscernible. (This is not the original definition from [54], but equivalent to it thanks to [61, Lemma 1.4].)

Fact 1.5 ([62, Lemma 2.10]). *If T is* dp-*minimal and every non-constant indiscernible sequence of singletons is not totally indiscernible, then T is distal. In particular, if T is* dp-*minimal and every sort of $\mathbb{M}$ expands a linearly ordered set, then T is distal.*

Linear orders in distal theories also occur on indiscernible sequences:

Corollary 1.6. *Suppose T is distal, and let $a = (a_i)_{i \in I}$ be a non-constant indiscernible sequence in $\mathbb{M}_x$. Then there is an $\mathcal{L}$-formula $\theta(u, x, y, w)$ and some n such that for all $I_0, I_1 \subseteq I$ of size n and all $i, j \in I$ such that $I_0 < i, j < I_1$ we have*

$$i < j \iff \models \theta(a_{I_0}, a_i, a_j, a_{I_1}).$$

Proof. By 1.4, a is not totally indiscernible, and for every indiscernible sequence which is not totally indiscernible there are such θ and n; see, e.g., the explanation after [13, Fact 3.1]. □

In the following we sometimes employ $\mathcal{L}$-formulas whose free variables have been separated into multivariables x, y thought of as *object* and *parameter* variables, respectively. We use the notation $\varphi(x; y)$ to indicate that the free variables of the $\mathcal{L}$-formula φ are contained among the components of the multivariables x, y (which we also assume to be disjoint). We refer to $\varphi(x; y)$ as a *partitioned $\mathcal{L}$-formula.* Given $a \in \mathbb{M}_x$ and $B \subseteq \mathbb{M}_y$ we let

$$\mathrm{tp}_\varphi(a|B) := \big\{\varphi(x; b) : b \in B,\ \models \varphi(a; b)\big\} \cup \big\{\neg\varphi(x; b) : b \in B,\ \models \neg\varphi(a; b)\big\}$$

be the φ-type of a over B.

Definition 1.7. Let $\varphi(x; y)$ be a partitioned $\mathcal{L}$-formula, and let $y_1, y_2, \dots$ be disjoint multivariables of the same sort as y. A partitioned $\mathcal{L}$-formula $\psi(x; y_1, \dots, y_n)$ is a (uniform) **strong honest definition** for $\varphi(x; y)$ (in T) if for every $a \in \mathbb{M}_x$ and finite $B \subseteq \mathbb{M}_y$ with $|B| \geq 2$, there are $b_1, \dots, b_n \in B$ such that

$$\models \psi(a; b_1, \dots, b_n) \quad \text{and} \quad \psi(x; b_1, \dots, b_n) \models \mathrm{tp}_\varphi(a|B).$$

Remark. A strong honest definition for $\varphi(x; y)$ remains a strong honest definition for $\neg\varphi(x; y)$. Moreover, if $\psi(x; y_1, \dots, y_m)$, $\psi'(x; y'_1, \dots, y'_n)$ are strong honest definitions for the partitioned $\mathcal{L}$-formulas $\varphi(x; y)$, $\varphi'(x; y)$, respectively, with all multivariables y_i, y'_j disjoint, then $\psi \wedge \psi'$ is a strong honest definitions for $\varphi \wedge \varphi'$.

By [14, Theorem 21] we have:

Fact 1.8. *The following are equivalent:*

(1) *T is distal;*
(2) *every partitioned $\mathcal{L}$-formula $\varphi(x; y)$ has a strong honest definition in T.*

When proving distality of a particular structure, Definition 1.1 is typically easier to verify. On the other hand, occasionally 1.8(2) is more useful since it ultimately gives more information about definable sets, and obtaining bounds on the complexity of strong honest definitions is important for combinatorial applications.

1.2. **Reduction to singletons.** In order to verify that a theory is distal, it is enough to check distality for "singletons". There are two ways to interpret this claim. First, we observe that existence of strong honest definitions for all formulas reduces to formulas in a single free variable.

Proposition 1.9. *Suppose every partitioned $\mathcal{L}$-formula $\varphi(x;y)$ with $|x| = 1$ has a strong honest definition in T. Then every partitioned $\mathcal{L}$-formula $\varphi(x;y)$ with $|x|$ arbitrary has a strong honest definition in T, so T is distal.*

Proof. We argue by induction on the size $|x|$ of x, with the base case $|x| = 1$ given by the assumption. Assume that $x = (x_0, x_1)$, and let a partitioned $\mathcal{L}$-formula $\varphi(x_0, x_1; y)$ be given. By the inductive assumption, take a strong honest definition $\psi(x_0; z_1, \dots, z_n)$ for the partitioned $\mathcal{L}$-formula $\varphi(x_0; x_1, y)$, where $z_i = (x_{1i}, y_i)$ for $i = 1, \dots, n$. Set

$$\chi(x_0; x_1, \vec{y}\,) := \psi\big(x_0; (x_1, y_1), \dots, (x_1, y_n)\big) \qquad \text{where } \vec{y} := (y_1, \dots, y_n),$$

let

$$\begin{aligned}\chi^+(x_1; y, \vec{y}\,) &:= \forall x_0 \big(\chi(x_0; x_1, \vec{y}\,) \to \varphi(x_0; x_1, y)\big),\\ \chi^-(x_1; y, \vec{y}\,) &:= \forall x_0 \big(\chi(x_0; x_1, \vec{y}\,) \to \neg\varphi(x_0; x_1, y)\big),\end{aligned}$$

and by inductive assumption, let $\rho^+(x_1; \vec{y}^{\,+})$ and $\rho^-(x_1; \vec{y}^{\,-})$ be strong honest definitions for χ^+ and χ^-, respectively; here $\vec{y}^{\,+} = (\vec{y}^{\,+}_1, \dots, \vec{y}^{\,+}_{n^+})$ for some n^+, and similarly with $-$ in place of $+$. We claim that

$$\gamma(x_0, x_1; \vec{y}, \vec{y}^{\,+}, \vec{y}^{\,-}) := \chi(x_0; x_1, \vec{y}\,) \wedge \rho^+(x_1; \vec{y}^{\,+}) \wedge \rho^-(x_1; \vec{y}^{\,-})$$

is a strong honest definition for $\varphi(x; y)$. To see this let $a_i \in \mathbb{M}_{x_i}$ $(i = 0, 1)$ and a finite $B \subseteq \mathbb{M}_y$ with $|B| \geq 2$ be given. Applying ψ to a_0 and the set of parameters $\{a_1\} \times B$, we obtain some $\vec{b} \in B^n$ such that

$$\models \chi(a_0; a_1, \vec{b}) \quad \text{and} \quad \chi(x_0; a_1, \vec{b}) \models \operatorname{tp}_\varphi\big(a_0 \big| \{a_1\} \times B\big).$$

Next choose $\vec{b}^{\,+} \in \big(B \times \{\vec{b}\}\big)^{n^+}$ such that

$$\models \rho^+(a_1; \vec{b}^{\,+}) \quad \text{and} \quad \rho^+(x_1; \vec{b}^{\,+}) \models \operatorname{tp}_{\chi^+}\big(a_1 \big| B \times \{\vec{b}\,\}\big).$$

Then for any $a_1' \models \rho^+(x_1, \vec{b}^{\,+})$ and $b \in B$ we have

$$\begin{aligned}\models \chi(x_0, a_1', \vec{b}\,) \to \varphi(x_0, a_1', b) \quad &\iff \quad \models \chi(x_0, a_1, \vec{b}\,) \to \varphi(x_0, a_1, b)\\ &\iff \quad \models \varphi(a_0, a_1, b).\end{aligned}$$

Similarly, we find $\vec{b}^{\,-} \in \big(B \times \{\vec{b}\,\}\big)^{n^-}$ such that for any $a_1' \models \rho^-(x_1, \vec{b}^{\,-})$ and $b \in B$ we have

$$\models \chi(x_0, a_1', \vec{b}\,) \to \neg\varphi(x_0, a_1', b) \quad \iff \quad \models \neg\varphi\,(a_0, a_1, b)\,.$$

Combining, we see that for all $a_1' \models \rho^+(x_1, \vec{b}^{\,+}) \wedge \rho^-(x_1, \vec{b}^{\,-})$ and $a_0' \models \chi(x_0, a_1', \vec{b})$ and each $b \in B$ we have $\models \varphi(a_0', a_1', b) \leftrightarrow \varphi(a_0, a_1, b)$. Thus

$$\gamma(x_0, x_1; \vec{b}, \vec{b}^{\,+}, \vec{b}^{\,-}) \models \operatorname{tp}_\varphi(a_0 a_1 | B) \quad \text{and} \quad \models \gamma(a_0, a_1; \vec{b}, \vec{b}^{\,+}, \vec{b}^{\,-})$$

hold, as wanted. □

Remark. Let $f(m)$ be the smallest possible number of parameters n in a strong honest definition $\psi(x; y_1, \dots, y_n)$ for partitioned $\mathcal{L}$-formulas $\varphi(x; y)$ with $|x| \leq m$. It follows from the proof that if $f(1)$ is finite, then $f(m) \leq 2f(1) + f(m-1)$ for $m \geq 1$; so $f(m) \leq (2m-1)f(1)$ for all $m \geq 1$. This gives a naive upper bound on the growth of the size of distal cell decompositions, an important parameter in combinatorial applications of distality isolated in [11, Section 2]. It is an interesting (and challenging) problem to determine optimal bounds in various theories of interest, e.g., in o-minimal or P-minimal theories.

Secondly, in terms of indiscernible sequences we have the following equivalence.

Proposition 1.10. *The following are equivalent:*

(1) *T is distal;*
(2) *for every indiscernible sequence $a = (a_i)_{i\in I}$ in $\mathbb{M}_x$, $i \in I$ such that $I^{<i}$ and $I^{>i}$ are infinite, and $b \in \mathbb{M}_y$ with $|y| = 1$, if $a_{I\setminus\{i\}}$ is b-indiscernible, then so is a;*
(3) *for every indiscernible sequence $a = (a_i)_{i\in I}$ in $\mathbb{M}_x$ where $|x| = 1$, $i \in I$ such that $I^{<i}$ and $I^{>i}$ are infinite, and $b \in \mathbb{M}_y$, if $a_{I\setminus\{i\}}$ is b-indiscernible, then so is a.*

Proof. It is not hard to see that the condition in (2) can be iterated to obtain the same conclusion with y an arbitrary multivariable, which is sufficient to satisfy Definition 1.1. (Alternatively, Proposition 1.9 provides a more explicit version of this argument.) The equivalence of (1) and (3) is established in [62, Theorem 2.28]. (See also Proposition 1.17 below for a discussion.) □

Corollary 1.11. *The following are equivalent:*

(1) *T is not distal;*
(2) *there is an indiscernible sequence $a = (a_i)_{i\in\mathbb{Q}}$ in $\mathbb{M}_x$ and some $b \in \mathbb{M}_y$ such that $a_{\mathbb{Q}\setminus\{0\}}$ is b-indiscernible, and some partitioned $\mathcal{L}$-formula $\varphi(x; y)$ such that*

$$\models \varphi(a_i; b) \iff i \neq 0;$$

(3) *the same statement as in* (2) with $|x| = 1$.

Proof. To show (1) ⇒ (3), assume that the condition in Proposition 1.10(3) fails. Then we can take some indiscernible sequence $a = (a_i)_{i\in\mathbb{Q}}$ in $\mathbb{M}_x$ where $|x| = 1$ and some $b \in \mathbb{M}_y$ such that $a_{\mathbb{Q}\setminus\{0\}}$ is b-indiscernible, but a is not. Thus we can take an $\mathcal{L}$-formula $\psi(x_1, \dots, x_n; y)$, where $x_1, \dots, x_n$ are single variables of the same sort as x, as well as finite subsets I_1, I_2 of $\mathbb{Q}$ with $|I_1| + |I_2| = n - 1$ and $I_1 < 0 < I_2$, such that

(1) $\models \neg\psi(a_{I_1}, a_0, a_{I_2}; b)$;
(2) $\models \psi(a_{J_1}, a_j, a_{J_2}; b)$ for all $J_1, J_2 \subseteq \mathbb{Q} \setminus \{0\}$ and $j \in \mathbb{Q} \setminus \{0\}$ with $|J_1| + |J_2| = n - 1$ and $J_1 < j < J_2$.

Let $y' := (y, y_1, y_2)$ where $y_1 = (x_1, \dots, x_m)$, $y_2 = (x_{m+2}, \dots, x_n)$, $m = |I_1|$, and set

$$\varphi(x; y') := \psi(y_1, x, y_2, y), \quad b' := (b, a_{I_1}, a_{I_2}) \in \mathbb{M}_{y'}.$$

Choose $\varepsilon \in \mathbb{Q}$ with $I_1 < -\varepsilon < 0 < \varepsilon < I_2$ and set $I' := \{i \in \mathbb{Q} : -\varepsilon < i < \varepsilon\}$. Then the sequence $a_{I'}$ is indiscernible and $a_{I'\setminus\{0\}}$ is b'-indiscernible; moreover, for $i \in I'$ we have

$$\models \varphi(a_i; b') \iff i \neq 0.$$

It follows that (3) holds. Finally, (3) ⇒ (2) and (2) ⇒ (1) are obvious. □

Remark 1.12. Let $a = (a_i)_{i\in\mathbb{Q}}$ be an indiscernible sequence in $\mathbb{M}_x$ and $b \in \mathbb{M}_y$ such that $a_{\mathbb{Q}\setminus\{0\}}$ is b-indiscernible. It is easy to see that the set of $\mathcal{L}$-formulas $\varphi(x; y)$ violating the conclusion of (2) in Corollary 1.11 (that is, such that $\models \varphi(a_0; b)$ or $\models \neg\varphi(a_i; b)$ for some, or equivalently, all $i \neq 0$) is closed under *positive* boolean combinations.

Remark 1.13. Let $\mathbb{Q}_\infty = \mathbb{Q} \cup \{\infty\}$ where $\infty \notin \mathbb{Q}$ is a new symbol and the usual ordering of $\mathbb{Q}$ is extended to a total ordering of $\mathbb{Q}_\infty$ with $\mathbb{Q} < \infty$. Then Corollary 1.11 and Remark 1.12 remain true with the linearly ordered set $\mathbb{Q}$ replaced by $\mathbb{Q}_\infty$. (This is used in the proof of Theorem 4.6 below.)

1.3. **Induced structure and mild expansions.** From [62] we record the following. (For part (2) use [62, Corollary 2.9] along with Fact 1.3.)

Fact 1.14.

(1) *If T is distal, then so is every complete theory bi-interpretable with T.*
(2) *Naming a small set of constants does not affect distality: if $\mathbb{M}$ is distal, then for each small $A \subseteq \mathbb{M}$, the $\mathcal{L}_A$-structure $\mathbb{M}_A$ is also distal.*

In what follows, we will often be in a situation when T is NIP and we have a definable set $D \subseteq \mathbb{M}_x$ (often, a sort) such that the induced structure on D is distal. More precisely, denote the full induced structure on D by D_{ind}; that is, we introduce the one-sorted language $\mathcal{L}_{\text{ind}}$ which contains, for each $\mathcal{L}$-formula $\varphi(y_1, \dots, y_n)$ where each y_i is a multivariable of the same sort as x, an n-ary relation symbol R_φ; then D_{ind} is the $\mathcal{L}_{\text{ind}}$-structure with underlying set D where each relation symbol R_φ is interpreted by $\varphi^{\mathbb{M}} \cap D^n$. The following is then straightforward by Definition 1.1.

Lemma 1.15. *If T is distal, then D_{ind} is also distal.*

We have the following lemmas in the converse direction. *In the rest of this subsection we assume that T is NIP, and we let D be an $\emptyset$-definable set such that D_{ind} is distal.* Our goal is to conclude that under suitable circumstances, T itself is distal.

Lemma 1.16. *Let $B \subseteq \mathbb{M}$ be small and $b \in \mathbb{M}_y$, and let $(a_i)_{i\in\mathbb{Q}}$ be a B-indiscernible sequence of elements from D. If $(a_i)_{i\in\mathbb{Q}\setminus\{0\}}$ is Bb-indiscernible, then so is $(a_i)_{i\in\mathbb{Q}}$.*

Proof. If a fails the conclusion of the lemma, then using distality of a (in the sense of Definition 1.2), following the proof of [62, Lemma 2.7] gives a contradiction to T being NIP. □

We also have a dual fact, where the sequence may be anywhere in $\mathbb{M}$, but the new parameters are coming from our distal set D. (A similar observation is stated in [29, Remark 4.26].)

Proposition 1.17. *Let $a = (a_i)_{i\in\mathbb{Q}}$ be an indiscernible sequence in $\mathbb{M}_x$ and $b \in D^N$, where $N \in \mathbb{N}$. If $(a_i)_{i\in\mathbb{Q}\setminus\{0\}}$ is b-indiscernible, then so is $(a_i)_{i\in\mathbb{Q}}$.*

This proposition can be shown along the same lines as the proof of [62, Theorem 2.28]; we provide the details for the sake of completeness and correcting some inaccuracies there. First we recall some terminology and facts from [62].

A nontrivial cut $\mathfrak{c}$ in a linearly ordered set I is *dedekind* if $\mathfrak{c}$ does not have a largest and $I \setminus \mathfrak{c}$ does not have a smallest element. Let $a = (a_i)_{i\in I}$ be an ($\emptyset$-) indiscernible sequence in $\mathbb{M}_x$ where I is endless, and $B \subseteq \mathbb{M}$ is an arbitrary parameter set. Recall that since T is NIP, the $\mathcal{L}_B$-formulas $\varphi(x)$ with the property that the set of $i \in I$ with $\models \varphi(a_i)$ is cofinal in I form a complete x-type $\lim(a|B)$ over B. (See, e.g., [63, Proposition 2.8].) Given a dedekind cut $\mathfrak{c}$ in I, letting $\mathfrak{c}^+$ denote the complement $I \setminus \mathfrak{c}$ of $\mathfrak{c}$ ordered by the reverse ordering, we set

$$\lim_-(\mathfrak{c}|B) := \lim(a_{\mathfrak{c}}|B), \qquad \lim_+(\mathfrak{c}|B) := \lim(a_{\mathfrak{c}^+}|B).$$

(Here a is understood from the context.) We say that $b \in \mathbb{M}_x$ *fills* $\mathfrak{c}$ in a if the sequence $a_{\mathfrak{c}}{}^\frown b^\frown a_{I\setminus\mathfrak{c}}$ is indiscernible.

Fact 1.18 (Strong base change, [62, Lemma 2.8])**.** *Let $a = (a_i)_{i\in I}$ be an indiscernible sequence in $\mathbb{M}_x$ and $A \subseteq \mathbb{M}_x$ be a small parameter set containing all a_i. Let also $(\mathfrak{c}_\lambda)_{\lambda\in\Lambda}$ be a family of pairwise distinct dedekind cuts in I, and for each $\lambda \in \Lambda$, let a_λ fill the cut $\mathfrak{c}_\lambda$ in a. Then there exists a family $(a'_\lambda)_{\lambda\in\Lambda}$ in $\mathbb{M}_x$ such that $(a'_\lambda) \equiv_a (a_\lambda)$ and $\text{tp}(a'_\lambda|A) = \lim_+(\mathfrak{c}_\lambda|A)$ for all $\lambda \in \Lambda$.*

Let $a = (a_i)_{i\in I}$ and $b = (b_j)_{j\in J}$ be sequences in $\mathbb{M}_x$ and $\mathbb{M}_y$, respectively, indexed by linearly ordered sets I, J. We say that a is b-indiscernible if a is B-indiscernible where $B := \{b_j : j \in J\}$. If a is b-indiscernible and b is a-indiscernible, then a, b are said to be **mutually indiscernible**.

Definition 1.19. ([62, Definition 2.12]) Indiscernible sequences $a = (a_i)_{i\in I}$ and $b = (b_i)_{i\in I}$ are **weakly linked** if for all *disjoint* subsets $I_1, I_2 \subseteq I$, the sequences a_{I_1} and b_{I_2} are mutually indiscernible.

The following is [62, Lemma 2.14(1)]. It is stated there with the additional assumption that the sequence of pairs $(a_i, b_i)_{i\in I}$ is indiscernible; however, this assumption is not needed, and this point is important in the proof of Proposition 1.17 given below.

Lemma 1.20. *Let $a = (a_i)_{i\in I}$ and $b = (b_i)_{i\in I}$ be weakly linked indiscernible sequences, where a is distal; then a and b are mutually indiscernible.*

Proof. We may arrange that I is dense. To show that a is indiscernible over b, let $I' \subseteq I$ be an arbitrary finite set; it is enough to show that a is $b_{I'}$-indiscernible. Now $a_{I\setminus I'}$ is $b_{I'}$-indiscernible as a, b are weakly linked. Since a is distal, repeatedly applying Fact 1.3 we conclude that a is $b_{I'}$-indiscernible.

Towards a contradiction assume that b is not a-indiscernible. This yields finite subsets I_1, I_2 of I such that b_{I_2} is not a_{I_1}-indiscernible. But then by indiscernibility of a over b_{I_2}, there exists some set I_1' disjoint from I_2 such that $a_{I_1'} \equiv_{b_{I_2}} a_{I_1}$; in particular, b_{I_2} is not $a_{I_1'}$-indiscernible, contradicting that a, b are weakly linked. □

Proof of Proposition 1.17. Toward a contradiction assume that the sequence $(a_i)_{i\in\mathbb{Q}\setminus\{0\}}$ is b-indiscernible but $(a_i)_{i\in\mathbb{Q}}$ is not. We will show that then there is an indiscernible sequence (b_n) with $b_n \equiv b$ which is not distal (in the sense of Definition 1.2); since $b_n \in D^N$, this will contradict distality of D_{ind}. We proceed by establishing a sequence of claims. In Claims 1.21–1.23 below we let I be a dense linearly ordered set without endpoints and $\mathfrak{c}$ be a dedekind cut in I.

Claim 1.21. *There is a b-indiscernible sequence $(a_i')_{i\in I}$ and some a' filling the cut $\mathfrak{c}$ in (a_i') such that $\operatorname{tp}(a', b) \neq \operatorname{tp}(a_i', b)$ for all $i \in I$.*

Proof. By assumption $a := (a_i)_{i\in\mathbb{Q}}$ is not b-indiscernible, so we find finite subsets J_1, J_2 of $\mathbb{Q}$ and a nonzero rational number j such that $J_1 < 0, j < J_2$ and

$$a_{J_1}{}^\frown a_0{}^\frown a_{J_2} \not\equiv_b a_{J_1}{}^\frown a_j{}^\frown a_{J_2}. \tag{1.1}$$

We may assume $J_1, J_2 \neq \emptyset$; let $j_1 := \max J_1$, $j_2 := \min J_2$, and set

$$a_j' := a_{J_1}{}^\frown a_j{}^\frown a_{J_2} \qquad \text{for } j \in J := (j_1, j_2) \subseteq \mathbb{Q}.$$

Then (1.1) holds for all $j \in J \setminus \{0\}$, the sequence $(a_j')_{j\in J}$ is still indiscernible, $(a_j')_{j\in J\setminus\{0\}}$ is b-indiscernible, and $\operatorname{tp}(a_0', b) \neq \operatorname{tp}(a_j', b)$ for $j \in J \setminus \{0\}$. Using compactness, this yields the claim. □

Let now (a_i') and a' be as in Claim 1.21; to simplify notation (and since we have no use of our original sequence $(a_i)_{i\in\mathbb{Q}}$ anymore), we now rename $(a_i')_{i\in I}$, a' as $(a_i)_{i\in I}$, a, respectively. Thus

- $(a_i)_{i\in I}$ is b-indiscernible, and
- a fills the cut $\mathfrak{c}$ in (a_i) and satisfies $\operatorname{tp}(a, b) \neq \operatorname{tp}(a_i, b)$ for all $i \in I$.

We also fix an $\mathcal{L}$-formula $\theta(x, y)$ such that $\models \neg\theta(a, b) \wedge \theta(a_i, b)$ for all $i \in I$.

Claim 1.22. *Let $\mathfrak{c}'$ be a dedekind cut in I with $\mathfrak{c} \subseteq \mathfrak{c}'$. Then there exists an $a' \in \mathbb{M}_x$ such that*

1. *a' fills the cut $\mathfrak{c}'$ in (a_i),*
2. *$\operatorname{tp}(a, b) = \operatorname{tp}(a', b)$, so in particular $\models \neg\theta(a', b)$.*

Proof. As $(a_i)_{i\in I}$ is b-indiscernible, we can choose a' satisfying (1) and (2) by compactness: given finite subsets $I_1 \subseteq \mathfrak{c}_\alpha$ and $I_2 \subseteq I \setminus \mathfrak{c}_\alpha$ there is a b-automorphism of $\mathbb{M}$ which sends a_{I_1}, a_{I_2} to a_{J_1}, a_{J_2}, respectively, where $J_1 \subseteq \mathfrak{c}$, $J_2 \subseteq I \setminus \mathfrak{c}$. □

In the next claim we let α, β be ordinals, and let r, s (also with decorations) range over α respectively β. We also assume that we have a strictly increasing sequence $(\mathfrak{c}_r)$ of dedekind cuts in I with $\mathfrak{c}_0 = \mathfrak{c}$.

Claim 1.23. *There exists an array $(a_{r,s})$ and a sequence (b_s) such that:*

(1) *if $s < s'$, then $\models \theta(a_{r,s'}, b_s)$;*
(2) *$\models \neg\theta(a_{r,s}, b_s)$;*
(3) *for all $r_0 < \cdots < r_n$ and pairwise distinct $s_0, \dots, s_n$, we have*
$$(a_{r_0,s_0}, \dots, a_{r_n,s_n}) \equiv (a_{i_0}, \dots, a_{i_n})$$
for some (equivalently, all) $i_0 < \cdots < i_n$ in I;
(4) *$b_s \equiv b$.*

Proof. By Claim 1.22 we obtain a sequence $a' = (a'_r)$ such that for all r,

- a'_r fills the cut $\mathfrak{c}_r$ in (a_i), and
- $\models \neg\theta(a'_r, b)$.

Let $a := (a_i)$. By induction on β we choose sequences (a_s) and tuples (b_s), with $a_s = (a_{r,s})$, such that

(a) $a_{r,s} \models \lim_+(\mathfrak{c}_r|aa_{<s}b_{<s})$, where $a_{<s} := (a_{s'})_{s'<s}$ and $b_{<s} := (b_{s'})_{s'<s}$; and
(b) $b_s a_s \equiv_a ba'$.

We start with $a_0 := a'$ and $b_0 := b$. Then (a) holds since a'_r fills the cut $\mathfrak{c}_r$ in $a = (a_i)$, and (b) holds trivially. Assume that (a_s) and tuples (b_s) have been chosen, for some given value of β. Applying Fact 1.18 to the family $(\mathfrak{c}_r)$ of dedekind cuts in I and the family $a' = (a'_r)$, where each a'_r fills $\mathfrak{c}_r$ in a, and a set of parameters A containing all components of a, $a_{<\beta}$, and $b_{<\beta}$, we find a sequence $a_\beta = (a_{r,\beta})$ such that $a_{r,\beta} \models \lim_+(\mathfrak{c}_r|aa_{<\beta}b_{<\beta})$ for each r (so (a) is satisfied for β in place of s) and $a_\beta \equiv_a a'$. Using this, we can move a' to a_β by an automorphism over a, and let b_β be the corresponding image of b; then (b) holds for β in place of s.

Now let $(a_{r,s})$ and (b_s) be sequences as just constructed, satisfying (a), (b). We check that (1)–(4) are satisfied.

(1) Let r and $s < s'$ with $\models \neg\theta(a_{r,s'}, b_s)$. By (a) we have $a_{r,s'} \models \lim_+(\mathfrak{c}_r|b_s)$, hence we can take some $i \in I$ such that $\models \neg\theta(a_i, b_s)$. But by (b) we have $b_s \equiv_{a_i} b$, hence $\models \neg\theta(a_i, b)$, contradicting our choice of θ.
(2) By (b) and choice of a'.
(3) Indeed, let $r_0 < \cdots < r_n$ and pairwise distinct $s_0, \dots, s_n$ be given, and let $\varphi(x_0, \dots, x_n)$ be an $\mathcal{L}$-formula with $\models \varphi(a_{r_0,s_0}, \dots, a_{r_n,s_n})$. Take the unique $k \in \{0, \dots, n\}$ such that $s_k = \max\{s_0, \dots, s_n\}$. Then by (a), for sufficiently large $i_k \in \mathfrak{c}_{r_k}$ we have
$$\models \varphi(a_{r_0,s_0}, \dots, a_{r_{k-1},s_{k-1}}, a_{i_k}, a_{r_{k+1},s_{k+1}}, \dots, a_{r_n,s_n}).$$
Repeating this procedure for the maximum of $\{s_0, \dots, s_{k-1}, s_{k+1}, \dots, s_n\}$, etc., we can thus successively choose $i_0 < \cdots < i_n$ in I (as $\mathfrak{c}_{r_0} \subset \cdots \subset \mathfrak{c}_{r_n}$) such that $\models \varphi(a_{i_0}, \dots, a_{i_n})$, which is sufficient to conclude the claim.
(4) is immediate by (b). □

For the following claim, recall our standing convention that m, n range over $\mathbb{N}$.

Claim 1.24. *There exists an array $(a_{m,n})$ and a sequence (b_n) satisfying (1)–(4) of Claim 1.23 for $\alpha = \beta = \omega$ such that additionally*

(5) *(a_n, b_n) is indiscernible, where $a_n = (a_{m,n})$, and*
(6) *$\big((a_{m,n})_n\big)$ is B-indiscernible where $B = \{b_0, b_1, \dots\}$.*

Proof. We take an ordinal α sufficiently large compared to $|T|$ (how large will become clear during the course of the rest of the proof), and then an ordinal $\beta \geq \alpha$ and sufficiently large compared to α (also to be determined). Next, we take a linearly ordered set I which has more than $|\alpha|$ many dedekind cuts, so that we can choose a strictly increasing sequence $(\mathfrak{c}_r)$ of dedekind cuts in I. Then Claim 1.23 applies and yields $(a_{r,s})$ and (b_s) having properties (1)–(4) in that claim. Set $a_s = (a_{r,s})$.

Assuming that β is large enough compared to α, Erdős-Rado and compactness (see, e.g., [63, Proposition 1.1]) give us an indiscernible sequence (a'_n, b'_n) such that for every $l \in \omega$ there exist some $s_0 < \cdots < s_l$ such that $(a'_k, b'_k)_{k \le l} \equiv (a_{s_k}, b_{s_k})_{k \le l}$. In particular, $(a'_{r,n})$ and (b'_n) satisfy (1)–(4) for $\beta = \omega$, and (5) holds as well.

Assuming α is large enough compared to $|T|$, we can similarly find a B'-indiscernible sequence $\big((a''_{m,n})_n\big)$, where $B' = \{b'_0, b'_1, \dots\}$, such that for every $l \in \omega$ there exist some $r_0 < \cdots < r_l$ such that

$$\big((a''_{k,n})\big)_{k \le l} \equiv_B \big((a'_{r_k,n})\big)_{k \le l}.$$

In particular, $(a''_{m,n})$, (b'_n) still satisfy (1)–(5), and (6) holds as well. □

Let now $(a_{m,n})$ and (b_n) be as in Claim 1.24; so (1)–(6) in Claims 1.23 and 1.24 hold.

Claim 1.25. *The sequences $(a_{n,n})$ and (b_n) are weakly linked, but not mutually indiscernible.*

Proof. First note that $(a_{n,n})$ is indiscernible by (3) applied with η given by $\eta(n) = n$ for each n, and (b_n) is indiscernible by (5). Clearly, the sequences are not mutually indiscernible because we have $\models \theta(a_{n,n}, b_m)$ for all $m < n$ by (1), but $\models \neg\theta(a_{n,n}, b_n)$ for all n by (2).

Given a finite tuple $\boldsymbol{i} = (i_0, \dots, i_{n-1}) \in \mathbb{N}^n$, we write $a_{\boldsymbol{i}} := (a_{i_0,i_0}, \dots, a_{i_{n-1},i_{n-1}})$ and $b_{\boldsymbol{i}} := (b_{i_0}, \dots, b_{i_{n-1}})$. We say that such a tuple is *strictly increasing* if $i_0 < \cdots < i_{n-1}$. To show that $(a_{n,n})$ and (b_n) are weakly linked, it is enough to show that for all strictly increasing $\boldsymbol{i}, \boldsymbol{i}', \boldsymbol{j}, \boldsymbol{j}' \in \mathbb{N}^n$ we have:

$$(*_1) \qquad (\boldsymbol{i} \cup \boldsymbol{i}') \cap (\boldsymbol{j} \cup \boldsymbol{j}') = \emptyset \implies a_{\boldsymbol{i}} b_{\boldsymbol{j}} \equiv a_{\boldsymbol{i}'} b_{\boldsymbol{j}'}.$$

(Here in the antecedent we identify the tuples $\boldsymbol{i}$, $\boldsymbol{i}'$, $\boldsymbol{j}$, $\boldsymbol{j}'$ with the corresponding subsets of $\mathbb{N}$.) First note that by (5) and (6), for strictly increasing $\boldsymbol{i}, \boldsymbol{i}', \boldsymbol{j}, \boldsymbol{j}' \in \mathbb{N}^n$ we easily have

$$(*_2) \qquad \boldsymbol{i}\boldsymbol{j} \equiv^{\mathrm{qf}}_{<} \boldsymbol{i}'\boldsymbol{j}' \implies a_{\boldsymbol{i}} b_{\boldsymbol{j}} \equiv a_{\boldsymbol{i}'} b_{\boldsymbol{j}'},$$

where $\equiv^{\mathrm{qf}}_{<}$ indicates the equality of quantifier-free types in the language of ordered sets. Hence in order to prove $(*_1)$, it is enough to show that for any finite tuples $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{i}'$, $\boldsymbol{j}'$ of natural numbers with $\boldsymbol{i} \cap \boldsymbol{j} = \emptyset$ and $\boldsymbol{i}' \cap \boldsymbol{j}' = \emptyset$ and $i_1, i_2, j \in \mathbb{N}$ we have

$$(*_3) \qquad \boldsymbol{i}, \boldsymbol{j} < i_1 < j < i_2 < \boldsymbol{i}', \boldsymbol{j}' \implies a_{(i_1)} \equiv_{a_{\boldsymbol{i}} a_{\boldsymbol{i}'} b_{\boldsymbol{j}} b_j b_{\boldsymbol{j}'}} a_{(i_2)}.$$

Indeed, suppose $\boldsymbol{i}, \boldsymbol{i}', \boldsymbol{j}, \boldsymbol{j}' \in \mathbb{N}^n$ are strictly increasing with $(\boldsymbol{i} \cup \boldsymbol{i}') \cap (\boldsymbol{j} \cup \boldsymbol{j}') = \emptyset$ as in $(*_1)$. We claim that we can use $(*_2)$ and $(*_3)$ to arrange that $\boldsymbol{i}\boldsymbol{j} \equiv^{\mathrm{qf}}_{<} \boldsymbol{i}'\boldsymbol{j}'$. To see this let $\boldsymbol{i} = (i_0, \dots, i_{n-1})$ and $\boldsymbol{j} = (j_0, \dots, j_{n-1})$, and suppose we have $k, l \in \{0, \dots, n-1\}$ with $i_k < j_l$ whereas $i'_k > j'_l$. If $k = n-1$, then we take any integer $\widetilde{i}_k > j_l$; otherwise, using $(*_2)$ we first arrange that $i_{k+1} - j_l$ is as large as necessary so that we may take an integer $\widetilde{i}_k \notin \boldsymbol{j} \cup \boldsymbol{j}'$ with $j_l < \widetilde{i}_k < i_{k+1}$. In both cases set $\widetilde{i}_m := i_m$ for $m \neq k$ and consider the strictly increasing tuple $\widetilde{\boldsymbol{i}} := (\widetilde{i}_0, \dots, \widetilde{i}_{n-1}) \in \mathbb{N}^n$; then by $(*_3)$ we have $a_{\boldsymbol{i}} b_{\boldsymbol{j}} \equiv a_{\widetilde{\boldsymbol{i}}} b_{\boldsymbol{j}}$. Thus by induction on the number of pairs (k, l) with $i_k < j_l$ and $i'_k > j'_l$, we arrive at the case $\boldsymbol{i}\boldsymbol{j} \equiv^{\mathrm{qf}}_{<} \boldsymbol{i}'\boldsymbol{j}'$, and then $a_{\boldsymbol{i}} b_{\boldsymbol{j}} \equiv a_{\boldsymbol{i}'} b_{\boldsymbol{j}'}$ follows from $(*_2)$.

To show $(*_3)$, let now $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{i}'$, $\boldsymbol{j}'$ be finite tuples of natural numbers with $\boldsymbol{i}, \boldsymbol{j} < i_1 < j < i_2 < \boldsymbol{i}', \boldsymbol{j}'$, and towards a contradiction assume that we have an $\mathcal{L}$-formula $\psi(x, y, z)$ (for suitable disjoint multivariables x, y, z), such that with $\varphi(x, y) := \psi(x, y, a_{\boldsymbol{i}} a_{\boldsymbol{i}'} b_{\boldsymbol{j}} b_{\boldsymbol{j}'})$, we have $\models \varphi(a_{(i_1)}, b_j)$, but $\models \neg\varphi(a_{(i_2)}, b_j)$. Recall that T is NIP, so we may let m be the alternation number of the partitioned $\mathcal{L}$-formula $\varphi(x; y, z)$. (See [63, Section 2.1].) In view of $(*_2)$, we can arrange:

$$(*_4) \qquad i_1 < j - m < j < j + m < i_2,$$

$$(*_5) \qquad \models \varphi(a_{i,n}, b_j) \text{ for all } i, n \text{ with } j - m < n < j \text{ and } j - m < i < j + m,$$

$$(*_6) \qquad \models \neg\varphi(a_{i,n}, b_j) \text{ for all } i, n \text{ with } j < n < j + m \text{ and } j - m < i < j + m.$$

To see this first replace the tuple $(\boldsymbol{i}, \boldsymbol{j}, i_1, j, i_2, \boldsymbol{i}', \boldsymbol{j}')$ by a tuple with the same order type so that $i_1 + m < j < i_2 - m$; modifying φ accordingly, we then still have $\models \varphi(a_{(i_1)}, b_j) \wedge \neg\varphi(a_{(i_2)}, b_j)$ by $(*_2)$, and $(*_4)$ holds. Next, note that if $i_1 < n < j$, then the tuple $(\boldsymbol{i}, \boldsymbol{j}, i_1, j, i_2, \boldsymbol{i}', \boldsymbol{j}')$ has the same order type as the tuple $(\boldsymbol{i}, \boldsymbol{j}, n, j, i_2, \boldsymbol{i}', \boldsymbol{j}')$, so $\models \varphi(a_{(n)}, b_j)$ by $(*_2)$. Similarly we see that $\models \neg\varphi(a_{(n)}, b_j)$ for $j < n < i_2$. Property (6) then implies $(*_5)$ and $(*_6)$.

Now let $\eta \colon \omega \to \omega$ be an injective function such that

- $\eta(n) = n$ for $|n - j| \geq m$,
- $\eta(n) < j$ for even n with $|n - j| < m$, and
- $\eta(n) > j$ for odd n with $|n - j| < m$.

Then the sequence $(a_{n,\eta(n)})$ is indiscernible by (4), and the truth value of the formula $\varphi(x; b_j)$ alternates $> m$ times on it by the choice of η and $(*_5)$ and $(*_6)$, a contradiction. □

By Lemma 1.20 and Claim 1.25, we conclude that the indiscernible sequence (b_n) is not distal, and $b_n \equiv b$ for all n by (4), as promised. □

Corollary 1.26. *Suppose* $\mathbb{M} \subseteq \operatorname{acl}(D)$. *Then* T *is distal.*

Proof. We verify that T satisfies Definition 1.1. Let $a = (a_i)_{i \in \mathbb{Q}}$ be an indiscernible sequence, and let some tuple b such that $a_{\mathbb{Q}\setminus\{0\}}$ is b-indiscernible be given. By assumption, there is some $d \in D^n$ such that $b \subseteq \operatorname{acl}(d)$. By Ramsey and compactness, moving d by an automorphism over b, we may assume that $a_{\mathbb{Q}\setminus\{0\}}$ is d-indiscernible. By Proposition 1.17, a is d-indiscernible, hence it is also b-indiscernible as desired. □

Corollary 1.27. *T is distal if and only if T^{eq} is distal.*

Proof. If T^{eq} is distal then so is T, by Lemma 1.15. For the converse note that since T is NIP, so is T^{eq}, and $\mathbb{M}^{\mathrm{eq}} \subseteq \operatorname{acl}(\mathbb{M})$, where acl is taken in the structure $\mathbb{M}^{\mathrm{eq}}$. Hence the previous corollary applies to T^{eq} in place of T. □

1.4. **Distal expansions.** We say that T **has a distal expansion** if there is an expansion $\mathcal{L}^*$ of $\mathcal{L}$ and a complete distal $\mathcal{L}^*$-theory T^* which contains T. We also say that an $\mathcal{L}$-structure **has a distal expansion** if it can be expanded to a distal structure (in some language expanding $\mathcal{L}$). Clearly, if an $\mathcal{L}$-structure $\boldsymbol{M}$ has a distal expansion, then so does its complete theory; the converse holds if $\boldsymbol{M}$ is sufficiently saturated.

Lemma 1.28. *Suppose T is interpretable in a complete distal $\mathcal{L}^*$-theory T^* (for some language $\mathcal{L}^*$). Then T has a distal expansion.*

Proof. The theory T is definable in $(T^*)^{\mathrm{eq}}$, which is distal by Corollary 1.27. Hence we may replace T^* by $(T^*)^{\mathrm{eq}}$ and assume that T is definable in T^*. Now Lemma 1.15 yields a distal expansion of T. □

So for example, the theory ACF_0 of algebraically closed fields of characteristic zero has a distal expansion, since it is interpretable (in fact, definable) in the theory RCF of real closed ordered fields: if K is a real closed ordered field then its algebraic closure is $K[\mathrm{i}]$ (where $\mathrm{i}^2 = -1$), and the field $K[\mathrm{i}]$ is 0-definable in K.

1.5. **Distality and the Shelah expansion.** Let $\boldsymbol{M}$ be an $\mathcal{L}$-structure. Recall that the *Shelah expansion* of $\boldsymbol{M}$ is the structure $\boldsymbol{M}^{\mathrm{Sh}}$ in the language $\mathcal{L}^{\mathrm{Sh}}$ obtained from $\boldsymbol{M}$ by naming all *externally definable* subsets of $\boldsymbol{M}$, i.e., sets of the form

$$\phi(x,b)^{\boldsymbol{N}} \cap M_x = \big\{a \in M_x : \boldsymbol{N} \models \phi(a,b)\big\}$$

with $\phi(x,y)$ an $\mathcal{L}$-formula and $b \in N_y$ for some elementary extension $\boldsymbol{N} \succeq \boldsymbol{M}$. (Here we can replace $\boldsymbol{N}$ by an elementary extension if necessary and thus always assume $\boldsymbol{N}$ is sufficiently saturated.)

Fact 1.29.
(1) $\boldsymbol{M}$ *is NIP if and only if* $\boldsymbol{M}^{\mathrm{Sh}}$ *is NIP* (*Shelah* [60], *see also* [13]);
(2) $\boldsymbol{M}$ *is distal if and only if* $\boldsymbol{M}^{\mathrm{Sh}}$ *is distal* (*Boxall-Kestner* [7]).

This implies the following remark on how the operations of taking Shelah expansions and reducts interact with distality:

Lemma 1.30. *Let $\mathcal{L}'$ be an expansion of the language $\mathcal{L}$ and let $\boldsymbol{M}'$ be an $\mathcal{L}'$-structure whose $\mathcal{L}$-reduct is $\boldsymbol{M}$. If $\boldsymbol{M}'$ is distal, then $\boldsymbol{M}^{\mathrm{Sh}}$ has a distal expansion, namely $(\boldsymbol{M}')^{\mathrm{Sh}}$.*

Proof. We first note that $(\boldsymbol{M}')^{\mathrm{Sh}}$ is indeed an expansion of $\boldsymbol{M}^{\mathrm{Sh}}$, since every sufficiently saturated $\boldsymbol{N} \succeq \boldsymbol{M}$ can be expanded to an $\mathcal{L}'$-structure $\boldsymbol{N}'$ such that $\boldsymbol{N}' \succeq \boldsymbol{M}'$. Hence $\boldsymbol{M}^{\mathrm{Sh}}$ is a reduct of $(\boldsymbol{M}')^{\mathrm{Sh}}$, and the latter is distal by Fact 1.29(2). ☐

2. Distal Fields and Rings

We emphasize the following important fact:

Fact 2.1 ([18, Corollary 6.3])**.** *No distal structure interprets an infinite field of positive characteristic.*

We first observe that this generalizes from fields to rings without zero-divisors. *In the rest of this section we let R be a ring; here and in the rest of this paper, all rings are assumed to be unital.*

Fact 2.2 (Jacobson, see e.g., [52, Theorem 12.10])**.** *Assume that for every $r \in R$ there is some $n \geq 2$ such that $r^n = r$. Then R is commutative.*

Recall that the *characteristic* $\operatorname{char}(R)$ of R is the smallest $n \geq 1$ such that $n \cdot 1 = 0$, if such an n exists, and $\operatorname{char}(R) := 0$ otherwise. For $a \in R$ we let $C(a) := \{b \in R : ab = ba\}$, a subring of R. We also let $Z(R) := \bigcap_{a \in R} C(a)$, a commutative subring of R, the *center* of R.

Proposition 2.3. *Suppose R is infinite without zero-divisors and interpretable in a distal structure. Then R has characteristic zero.*

Proof. Note that R having no zero-divisors implies that the only nilpotent element of R is 0. First assume that R is commutative. Then R is an integral domain, and interprets its fraction field F. But F is of characteristic 0 by Fact 2.1, and hence so is R. Now suppose R is not commutative. In this case, Fact 2.2 yields some $r \in R$ such that $r^n \neq r$ for all $n \geq 2$. Then the powers r^n of r are pairwise distinct, so the definable commutative subring $R' = Z(C(r))$ of R is infinite. By what we just showed, $\operatorname{char}(R') = 0$, hence $\operatorname{char}(R) = 0$. ☐

Here is a slight strengthening of this proposition. An idempotent e of R is said to be *central* if $e \in Z(R)$, and *centrally primitive* if e is central, $e \neq 0$, and e cannot be written as a sum $e = a + b$ of two nonzero central idempotents $a, b \in R$ with $ab = 0$. For every central idempotent e of R, the ideal Re of R is a ring with multiplicative identity e; we have a surjective ring morphism $r \mapsto re \colon R \to Re$, and if R has no zero-divisors, then so does Re.

Corollary 2.4. *Suppose R is infinite and interpretable in a distal structure, and that for every centrally primitive idempotent e of R, the ring Re is finite or has no zero-divisors. Then R has characteristic zero.*

Proof. Let $B(R)$ be the set of central idempotents of R forms a boolean subring of R. Since R has NIP, $B(R)$ is finite. Thus there are some $n \geq 1$ and centrally primitive idempotents $e_1, \dots, e_n$ of R such that $R = Re_1 \oplus \cdots \oplus Re_n$ (internal direct sum of ideals of R); see [52, §22]. For some $i \in \{1, \dots, n\}$, the ring Re_i is infinite, and hence has no zero-divisors; by Proposition 2.3 we have $\operatorname{char}(Re_i) = 0$ and thus $\operatorname{char}(R) = 0$. ☐

In the next three subsections we show that the hypothesis of not having zero-divisors cannot be dropped in Proposition 2.3. To produce an example, we employ a certain valued $\mathbb{F}_p$-vector space; here and below, we fix a prime p.

2.1. **Hahn spaces over $\mathbb{F}_p$.** We first define a language $\mathcal{L}$ and an $\mathcal{L}$-theory T whose intended model is the Hahn product $H = H(\mathbb{Q}, \mathbb{F}_p)$, that is, the abelian group of all sequences $h = (h_q)_{q\in\mathbb{Q}}$ in $\mathbb{F}_p$ with well-ordered support

$$\operatorname{supp} h := \big\{q \in \mathbb{Q} : h_q \neq 0\big\} \subseteq \mathbb{Q},$$

equipped with the valuation $v\colon H \to \mathbb{Q}_\infty$ satisfying

$$v(h) = \min(\operatorname{supp} h) \quad \text{for } 0 \neq h \in H,$$

which makes H into a valued abelian group. (See, e.g., [2, p. 74].) Let $\mathcal{L}$ be the two-sorted language with sorts s_{g} (for the underlying abelian group) and s_{v} (for the value set), and the following primitives: a copy $\{0, -, +\}$ of the language of abelian groups on the sort s_{g}; a copy $\{\leq, \infty\}$ of the language of ordered sets with an additional constant symbol ∞ on the sort s_{v}, as well as a a function symbol v of sort $s_{\mathrm{g}}s_{\mathrm{v}}$. Next we define T^- to be the (universal) $\mathcal{L}$-theory whose models $(G, S; \dots)$ satisfy:

(4) $(S; \leq)$ is a linearly ordered set with largest element ∞,
(5) $(G; 0, -, +)$ is an abelian group with $pG = \{0\}$ (and hence is an $\mathbb{F}_p$-vector space in a natural way),
(6) $v\colon G \to S$ is a (not necessarily surjective) $\mathbb{F}_p$-vector space valuation: for every $g, h \in G$,
  (a) $v(g) = \infty$ iff $h = 0$,
  (b) $v(g + h) \geq \min\big(v(g), v(h)\big)$,
  (c) $v(kg) = v(g)$ for every $k \in \mathbb{Z} \setminus p\mathbb{Z}$.
(7) for all $g, h \in G$ with $vg = vh \neq \infty$ there is $k \in \{1, \dots, p-1\}$ such that $v(g - kh) > vg$ (the Hahn space property [2, p. 94]).

Finally, we define T to be the $\mathcal{L}$-theory containing T^- whose models $(G, S; \dots)$ satisfy in addition:

(8) the ordered set $(S; \leq)$ is dense without smallest element, and
(9) the map $v\colon G \to S$ is surjective.

Note that if $(G, S; \dots)$ is a model of T^- which satisfies (9), then (G, S, v) is a Hahn space over $\mathbb{F}_p$ in the sense of [2, Section 2.3]. All structures in the following two subsections will be models of T^-; we will denote them by (G, S), (G', S'), (G^*, S^*), and their valuation indiscriminately by v.

2.2. **Quantifier elimination.** There are three relevant extension lemmas for models of T^-:

Lemma 2.5. *Let $s \in S \setminus v(G)$. Then there are an extension (G', S') of (G, S) and $g' \in G'$ such that*

(1) *$v(g') = s$, and*
(2) *given any embedding $i\colon (G, S) \to (G^*, S^*)$ and an element $g^* \in G^*$ such that $v(g^*) = i(s)$, there is an embedding $i'\colon (G', S') \to (G^*, S^*)$ which extends i such that $i'(g') = g^*$.*

Furthermore, given any (G', S') and $g' \in G'$ which satisfy (1) *and* (2)*, we have $G' = G \oplus \mathbb{F}_p g'$ (internal direct sum of $\mathbb{F}_p$-vector spaces), $S' = S$, $v(G') = v(G) \cup \{s\}$, and the embedding i' in* (2) *is unique.*

Proof. Let g' be an element of an $\mathbb{F}_p$-vector space extension of G with $g' \notin G$, and set $G' := G \oplus \mathbb{F}_p g'$, and extend $v\colon G \to S$ to a map $G' \to S$, also denoted by v, such that $v(g + kg') = \min(vg, s)$ for $g \in G$, $k \in \mathbb{F}_p^\times$. One verifies easily that then (G', S) is a model of T^- and (1), (2) hold. □

Lemma 2.6. *Let P be a cut in S with $P \neq S$. Then there is an extension (G', S') of (G, S) and some $s' \in S'$ such that*

(1) *s' realizes P, that is, $P < s' < S \setminus P$,*
(2) *given any embedding $i\colon (G, S) \to (G^*, S^*)$ and an element $s^* \in S^*$ such that $i(P) < s^* < i(S \setminus P)$, there is an embedding $i'\colon (G', S') \to (G^*, S^*)$ which extends i such that $i'(s') = s^*$.*

Furthermore, given any (G', S') and $s' \in S'$ which satisfy (1) and (2), we have $G = G'$, $S' = P^\frown s'^\frown (S \setminus P)$, and the embedding i' in (2) is unique.

The easy proof of this lemma is left to the reader. Iterating the previous two lemmas routinely implies:

Corollary 2.7. *Every model (G, S) of T^- has a T-closure, that is, an extension (G', S') to a model of T such that every embedding $(G, S) \to (G^*, S^*)$ into a model of T extends to an embedding $(G', S') \to (G^*, S^*)$.*

We recall some basic definitions about pseudoconvergence in valued abelian groups; our reference for this material is [2, Section 2.2]. Let (g_ρ) be a sequence in G indexed by elements of an infinite well-ordered set without largest element. Then (g_ρ) is said to be a *pseudocauchy* sequence (abbreviated: a *pc-sequence*) if there is some index ρ_0 such that for all indices $\tau > \sigma > \rho > \rho_0$ we have $v(g_\tau - g_\sigma) > v(g_\sigma - g_\rho)$. Given $g \in G$, we write $g_\rho \rightsquigarrow g$ if the sequence $\big(v(g - g_\rho)\big)$ in S is eventually strictly increasing. We say that a pc-sequence (g_ρ) in G is *divergent* if there is no $g \in G$ with $g_\rho \rightsquigarrow g$. The next lemma is immediate from [2, Lemma 2.3.1].

Lemma 2.8. *Let (g_ρ) be a divergent pc-sequence in G. Then there is an extension (G', S') of (G, S) and some $g' \in G'$ such that:*

1. *$g_\rho \rightsquigarrow g'$, and*
2. *given any embedding $i\colon (G, S) \to (G^*, S^*)$ and an element $g^* \in G^*$ such that $i(g_\rho) \rightsquigarrow g^*$, there is an embedding $i'\colon (G', S') \to (G^*, S^*)$ which extends i such that $i'(g') = g^*$.*

Furthermore, given any (G', S') and $g' \in G'$ which satisfy (1) and (2), we have $G' = G \oplus \mathbb{F}_p g'$ (internal direct sum of $\mathbb{F}_p$-vector spaces), $S' = S$, and the embedding i' in (2) is unique.

We now combine the embedding lemmas above to show:

Proposition 2.9. *The $\mathcal{L}$-theory T has QE.*

Proof. By Corollary 2.7 and one of the standard QE tests (see, e.g., [2, Corollary B.11.11]), it suffices to show: Let $(G, S) \subsetneq (G_1, S_1)$ be a proper extension of models of T and (G^*, S^*) be an $|G|^+$-saturated elementary extension of (G, S); then the natural inclusion $(G, S) \to (G^*, S^*)$ extends to an embedding $(G', S') \to (G^*, S^*)$ of a substructure (G', S') of (G_1, S_1) properly extending (G, S).

If $S \neq S_1$, pick an arbitrary $g_1 \in G_1$ with $s_1 := v(g_1) \in S_1 \setminus S$. Then $|G|^+$-saturation of (G^*, S^*) yields an element s^* of S^* such that for each $s \in S$ we have $s < s^*$ iff $s < s_1$, and by Lemma 2.6, setting $G' := G \oplus \mathbb{F}_p g_1$ and $S' := S \cup \{s_1\}$ gives rise to a substructure (G', S') of (G_1, S_1) with the required property.

Now suppose $S = S_1$. Then $G \neq G_1$; pick an arbitrary $g_1 \in G_1 \setminus G$. Then [2, Lemma 2.2.18] yields a divergent pc-sequence (g_ρ) in G with $g_\rho \rightsquigarrow g_1$, and $|G|^+$-saturation of (G^*, S^*) yields an element g^* of G^* with $g_\rho \rightsquigarrow g^*$ (see the proof of [2, Lemma 2.2.5]). In this case, setting $G' := G \oplus \mathbb{F}_p g_1$ and $S' := S$ we obtain a substructure (G', S') of (G_1, S_1) with the required property. □

Corollary 2.10. *The $\mathcal{L}$-theory T is complete; it is the model completion of T^-.*

Hence if $(G, S) \models T$ and G_0 is a subgroup of G with $v(G_0) = S$, then (G_0, S) is an elementary substructure of (G, S). In particular, we have $(H_0, \mathbb{Q}) \preceq (H, \mathbb{Q})$ where $H_0 := \big\{h \in H : \operatorname{supp}(h) \text{ finite}\big\}$.

Remark. The previous proposition and its corollary can also be deduced (in a one-sorted setting) from more general results in [51].

2.3. **Indiscernible sequences.** Let $(G, S) \models T$. In the following two lemmas we prove some properties of nonconstant indiscernible sequences in G. For this let $(g_i)_{i \in I}$ be a sequence in G where I is a nonempty linearly ordered set without a largest or smallest element. We let I^* be the set I equipped with the reversed ordering $\geq$.

Lemma 2.11. *Suppose (g_i) is nonconstant and indiscernible. Then exactly one of the following holds:*

(1) $v(g_i - g_j) < v(g_j - g_k)$ *for all* $i < j < k$ *in* I *(we say that* (g_i) *is* pseudocauchy*); or*
(2) $v(g_i - g_j) > v(g_j - g_k)$ *for all* $i < j < k$ *in* I *(so the sequence* $(g_i)_{i \in I^*}$ *is pseudocauchy).*

Proof. Choose elements $0 < 1 < \cdots < p+1$ of I and consider the $p+1$ elements $h_0 := g_0 - g_{p+1}, \dots, h_p := g_p - g_{p+1}$ of G. Let m, n range over $\{0, \dots, p\}$. We have three cases to consider:

Case 1: *$v(h_m) = v(h_n)$ for all m, n.* Then by the Hahn axiom, for $m \geq 1$ we get $k_m \in \{1, \dots, p-1\}$ such that $v(h_0 - k_m b_m) > v(h_0)$. By the pigeonhole principle, there are $1 \leq m < n$ such that $k_m = k_n$. Now note that

$$v(h_0) \;<\; v\big((h_0 - k_m h_m) - (h_0 - k_n h_n)\big) \;=\; v\big(k_m(h_n - h_m)\big) \;=\; v(h_n - h_m) \;=\; v(g_n - g_m)$$

and thus

$$v(g_n - g_{p+1}) \;=\; v(h_n) \;=\; v(h_0) \;<\; v(g_n - g_m)$$

and so we are in case (2), by indiscernibility.

Case 2: *There are $m < n$ such that $v(h_m) < v(h_n)$.* Then by indiscernibility we are in case (1).

Case 3: *There are $m < n$ such that $v(h_m) > v(h_n)$.* We will actually show that this case cannot happen. Note that in this case

$$v(g_m - g_n) \;=\; v\big(h_m - h_n\big) \;=\; v(h_n) \;=\; v(g_n - g_{p+1}).$$

Thus by indiscernibility, for all $i < j < k < l$ in I we have

$$v(g_i - g_j) \;=\; v(g_j - g_k) \;=\; v(g_k - g_l)$$

and thus taking an element $i < m$ in I we have

$$v(h_m) \;=\; v(g_m - g_{p+1}) \;=\; v(g_i - g_m) \;=\; v(g_m - g_n) \;=\; v(g_n - g_{p+1}) \;=\; v(h_n),$$

a contradiction. □

In the rest of this subsection we let $A \subseteq G$ and $B \subseteq S$.

Lemma 2.12. *Suppose (g_i) is nonconstant and AB-indiscernible, and let $s \in v(A) \cup B$. Then either*

(1) $v(g_i - g_j) > s$ *for all* $i \neq j$, *or*
(2) $v(g_i - g_j) < s$ *for all* $i \neq j$.

Proof. By Lemma 2.11 we have $v(g_i - g_j) \neq v(g_k - g_l)$ for all $i < j < k < l$, and with $\square \in \{<, =, >\}$, by s-indiscernibility of (g_i): if $v(g_i - g_j) \,\square\, s$ for some pair $i < j$, then $v(g_i - g_j) \,\square\, s$ for all $i < j$. □

The two lemmas above motivate the following definition:

Definition 2.13. We say that (g_i) is **pre-AB-indiscernible** if

(1) exactly one of the following is true:
   (a) $(g_i)_{i \in I}$ is pseudocauchy, or
   (b) $(g_i)_{i \in I^*}$ is pseudocauchy;
(2) for each $s \in v(A) \cup B$, either
   (a) $v(g_i - g_j) > s$ for all $i \neq j$, or
   (b) $v(g_i - g_j) < s$ for all $i \neq j$;
(3) for every $a \in A$, exactly one of the following is true:
   (a) $\big(v(g_i - a)\big)$ is constant,
   (b) $\big(v(g_i - a)\big)$ is strictly increasing,
   (c) $\big(v(g_i - a)\big)$ is strictly decreasing.

If (g_i) is nonconstant and AB-indiscernible, then it is pre-AB-indiscernible, by Lemmas 2.11 and 2.12 and A-indiscernibility of (g_i). To show a converse, we first record some properties of pre-AB-indiscernible sequences. We say that (g_i) is a "pc-sequence" if it is pseudocauchy.

Lemma 2.14. *Suppose (g_i) is a pre-AB-indiscernible pc-sequence; then for each i the value $s_i := v(g_i - g_j)$, where $j > i$, does not depend on j, and*

$(2')$ *for each $s \in v(A) \cup B$, either*
  - (a) *$s_i > s$ for all i, or*
  - (b) *$s_i < s$ for all i,*

$(3')$ *for each $a \in A$, either*
  - (a) *$\big(v(g_i - a)\big)$ is constant, and $s_i > v(g_j - a)$ for each i, j, or*
  - (b) *$s_i = v(g_i - a)$ for all i.*

Proof. The first statement is clear since (g_i) is a pc-sequence, and implies $(2')$ by property (2) in Definition 2.13. To show $(3')$, let $a \in A$. Suppose (3)(a) in Definition 2.13 holds, and let s be the common value of the $v(g_i - a)$; then $v(g_i - g_j) \geq s$ for all $i < j$, and since (s_i) is strictly increasing and I does not have a smallest element, we obtain $s_i = v(g_i - g_j) > s$ for $i < j$. If (3)(b) holds, then $s_i = v(g_i - a)$ for each i. Case (3)(c) does not occur: otherwise, for $i < j < k$ we have

$$s_i = v(g_i - g_j) = v\big((g_i - a) + (a - g_j)\big) = v(g_j - a)$$

and similarly $s_i = v(g_k - a)$, which is impossible. This yields $(3')$. □

We now arrive at our classification of nonconstant indiscernible sequences from G:

Proposition 2.15. *Suppose A is a subgroup of G and (g_i) is nonconstant. Then (g_i) is AB-indiscernible iff (g_i) is pre-AB-indiscernible.*

Proof. Suppose (g_i) is pre-AB-indiscernible. To show that (g_i) is AB-indiscernible we can assume that (g_i) is a pc-sequence; so for each i the value $s_i := v(g_i - g_j)$, where $j > i$, does not depend on j. For $a \in A$ such that $\big(v(g_i - a)\big)$ is constant, denote by s_a the common value of the $v(g_i - a)$. Let now

$$t(x_1, \dots, x_n) = k_1 x_1 + \cdots + k_n x_n + a \qquad (k_1, \dots, k_n \in \mathbb{Z},\ a \in A)$$

be an $\mathcal{L}_A$-term of sort s_{g}. By quantifier elimination (Proposition 2.9) and Lemma 2.14 it is enough to show that

- $v\big(t(g_{i_1}, \dots, g_{i_n})\big)$ is constant and contained in $v(A)$ for $i_1 < \cdots < i_n$, or
- there is $g \in A$ such that $v(g_i - a)$ is constant and $v\big(t(g_{i_1}, \dots, g_{i_n})\big) = s_a$ for $i_1 < \cdots < i_n$, or
- there is an $m \in \{1, \dots, n\}$ such that $v\big(t(g_{i_1}, \dots, g_{i_n})\big) = s_{i_m}$ for $i_1 < \cdots < i_n$.

For this we can assume $k_m \notin p\mathbb{Z}$ for some m, since otherwise $t(g) = a$ for all $g \in G$, and we are done; take m minimal such that $k_m \notin p\mathbb{Z}$. Set $k := k_1 + \cdots + k_n$. We distinguish two cases:

Case 1: $k \in p\mathbb{Z}$. Then

$$t(h_1, \dots, h_n) = k_1(h_1 - h_n) + \cdots + k_{n-1}(h_{n-1} - h_n) + a \qquad \text{for all } h_1, \dots, h_n \in G.$$

Let $s := va$. If $(2')$(a) in Lemma 2.14 holds, then $v\big(t(g_{i_1}, \dots, g_{i_n})\big) = s$ for $i_1 < \cdots < i_n$ in I; if $(2')$(b) holds, then $m < n$, and $v\big(t(g_{i_1}, \dots, g_{i_n})\big) = s_{i_m}$ for $i_1 < \cdots < i_n$ in I.

Case 2: $k \notin p\mathbb{Z}$. Then we can take $g \in A$ such that

$$t(h_1, \dots, h_n) = k_1(h_1 - h_n) + \cdots + k_{n-1}(h_{n-1} - h_n) + k(h_n - h) \qquad \text{for all } h_1, \dots, h_n \in G.$$

If $(3')$(a) holds, then $v\big(t(g_{i_1}, \dots, g_{i_n})\big) = s_a$ for $i_1 < \cdots < i_n$ in I; whereas if $(3')$(b) holds, then $v\big(t(g_{i_1}, \dots, g_{i_n})\big) = s_{i_m}$ for $i_1 < \cdots < i_n$ in I. □

Corollary 2.16. *T is distal.*

Proof. By Corollary 1.26 it suffices to prove that the structure induced on the group sort $s_{\rm g}$ of models of T is distal. For this, suppose $(g_i)_{i\in I}$ as above is indiscernible, the linearly ordered set I is dense, and 0 is an element of I such that $(g_i)_{i\in I^{\neq}}$ is AB-indiscernible, where $I^{\neq} := I \setminus \{0\}$; by Proposition 1.10, it is enough to show that then $(g_i)_{i\in I}$ is AB-indiscernible. This is clear if $(g_i)_{i\in I}$ is constant; thus we may assume that $(g_i)_{i\in I}$ is nonconstant. Replacing A by the subgroup of G generated by A we can also arrange that A is a subgroup of G, and by Lemma 2.11, that $(g_i)_{i\in I}$ is a pc-sequence. Let $s_i := v(g_i - g_j)$ where $j > i$ is arbitrary. Let $s \in v(A) \cup B$; if $s_i > s$ for all $i \in I^{\neq}$, then also $s_0 > s$, and similarly with "$<$" in place of "$>$". Together with Lemma 2.12 applied to $(g_i)_{i\in I^{\neq}}$, this implies that (2) in Definition 2.13 holds. Similarly, using Lemma 2.14(3′) for $(g_i)_{i\in I^{\neq}}$ we see that statement (3) in Definition 2.13 holds: Let $a \in A$. Suppose $\big(v(g_i - a)\big)_{i\in I^{\neq}}$ is constant and $s_i > v(g_j - a)$ for all $i, j \in I^{\neq}$; then $s_i > v(g_j - a)$ for all $i \in I$, $j \in I^{\neq}$ and thus $v(g_0 - a) = v\big((g_0 - g_j) + (g_j - a)\big) = v(g_j - a)$ for $j \neq 0$, hence (3)(a) holds. If $s_i = v(g_i - a)$ for $i \neq 0$, then $v(g_0 - a) = v\big((g_0 - g_j) + (g_j - a)\big) = s_0$ for $j > 0$, hence (3)(b) holds. This shows that $(g_i)_{i\in I}$ is pre-AB-indiscernible, and hence AB-indiscernible by Proposition 2.15. □

We now use the above to give our promised example of an infinite ring of positive characteristic interpretable a distal structure.

Example. Suppose $R = \mathbb{F}_p \times H$, where $H = H(\mathbb{Q}, \mathbb{F}_p)$ is as in the beginning of Section 2.1, equipped with the componentwise addition and multiplication given by

$$(k, g) \cdot (l, h) := (kl, kg + lh) \qquad \text{for } k, l \in \mathbb{F}_p,\ g, h \in H.$$

Then R is a commutative ring of characteristic p, with multiplicative identity $1 = (1, 0)$. Moreover, R is interpretable in the $\mathcal{L}$-structure $(H, \mathbb{Q}) \models T$, which is distal by Corollary 2.16.

Remark. Distality for a more general class of valued abelian groups and certain related structures is established in [16], and is used there to demonstrate that in fact every abelian group (in the pure group language) admits a distal expansion.

In the remainder of this section we point out a consequence of Fact 2.1 for henselian valued fields with a distal expansion.

2.4. **NIP in henselian valued fields.** *In this subsection K is a henselian valued field with value group Γ and residue field $\boldsymbol{k}$.* We view K as a model-theoretic structure $(K, \mathcal{O})$, where $\mathcal{O}$ is the valuation ring of K. We recall the following facts; the proofs below are courtesy of Franziska Jahnke.

Fact 2.17. *Suppose K is finitely ramified and $\boldsymbol{k}$ is NIP and perfect; then $(K, \mathcal{O})$ is NIP.*

Proof. In the case $\operatorname{char} \boldsymbol{k} = 0$ this follows from Delon [22] (using also [35]), and for $\operatorname{char} \boldsymbol{k} > 0$ and unramified K this was shown by Bélair [5]. We reduce the finitely ramified case with $\operatorname{char} \boldsymbol{k} = p > 0$ to these cases. We use the notation and terminology of [2, Section 3.4]. First, after passing to an elementary extension we can assume that $(K, \mathcal{O})$ is $\aleph_1$-saturated. Let $\Delta := \Delta_0$ be the smallest convex subgroup of Γ containing vp, and let $\dot{K}$ be the corresponding specialization of K. Then $\dot{K}$ has characteristic zero, cyclic value group Δ_0, and residue field isomorphic to $\boldsymbol{k}$; saturation implies that $\dot{K}$ is complete. It is well-known (see, e.g. [69, Theorem 22.7]) that therefore $\dot{K}$ is a finite extension of a complete unramified discretely valued subfield L with the same residue field $\boldsymbol{k}$ as $\dot{K}$. By [5], $(L, \mathcal{O}_L)$ is NIP, hence so is $(\dot{K}, \mathcal{O}_{\dot{K}})$. Now the Δ-coarsening $(K, \dot{\mathcal{O}})$ of K has residue field $\dot{K}$, and hence is NIP by [22]. The valuation ring of $\dot{K}$ is definable in the pure field $\dot{K}$ [50, Lemma 3.6]. Hence $\mathcal{O}$ is definable in $(K, \dot{\mathcal{O}})$, and thus $(K, \mathcal{O})$ is NIP. □

See Corollaries 5.18 and 5.23 below for versions of the preceding fact where $\boldsymbol{k}$ and Γ are permitted to have additional structure. Here is a partial converse of Fact 2.17:

Fact 2.18. *Suppose $(K, \mathcal{O})$ is NIP and $\boldsymbol{k}$ is finite; then K is finitely ramified.*

Proof. We may assume that $(K, \mathcal{O})$ is $\aleph_0$-saturated. This time, we let Δ be the biggest convex subgroup of Γ *not* containing vp, and let $\dot{K}$ be the corresponding specialization of K. Then $\dot{K}$ has characteristic p, value group Δ, and residue field isomorphic to $\boldsymbol{k}$. The Shelah expansion of $(K, \mathcal{O})$ interprets every convex subgroup of Γ, and hence also the valued field $(\dot{K}, \mathcal{O}_{\dot{K}})$; in particular, $(\dot{K}, \mathcal{O}_{\dot{K}})$ is NIP, by Fact 1.29(1). Now [48, Proposition 5.3] implies that $\Delta = \{0\}$, since $\boldsymbol{k}$ is finite. Hence for every $\gamma > 0$ in Γ there is some n such that $n\gamma \geq vp$. Saturation yields some n such that for every $\gamma > 0$ in Γ we have $n\gamma \geq vp$; hence K is finitely ramified. □

Combining 2.1 and 1.15 with 2.18 implies:

Corollary 2.19. *If $(K, \mathcal{O})$ has a distal expansion, then K is finitely ramified and $\boldsymbol{k}$ has characteristic zero or is finite.*

Remark 2.20. If K is finitely ramified and $\boldsymbol{k}$ is finite, $p = \operatorname{char} \boldsymbol{k}$, then K has a specialization which is p-adically closed of finite p-rank. (Let $\Delta = \Delta_0$ be as in the proof of Fact 2.17 and let $\dot{K}$ be the Δ-specialization of K; then $\dot{K}$ is henselian of mixed characteristic $(0, p)$ with cyclic value group and finite residue field $\boldsymbol{k}$, hence is p-adically closed of finite p-rank [55, Theorem 3.1].)

See [1, Section 5.1] for a conjectural characterization of all NIP henselian valued fields.

3. Distality in Ordered Abelian Groups

In 1984, Gurevich and Schmitt [35] showed that every ordered abelian group is NIP. In this section, we investigate distality for ordered abelian groups; the main result is Theorem 3.13 below. As a warmup, in Section 3.1 we characterize distality for those ordered abelian groups which have quantifier elimination in the Presburger language (see Theorem 3.2). This already applies to a variety of familiar ordered abelian groups since it includes every ordered abelian group which is elementarily equivalent to an archimedean one. *In the rest of this section we assume $m, n \geq 1$, and we let p, q range over the set of prime numbers.*

An ordered abelian group G is said to be *non-singular* if G/pG is finite for every p. The following fact from [45, Proposition 5.1] will be used several times:

Fact 3.1. *An ordered abelian group is* dp*-minimal if and only if it is non-singular.*

3.1. The case of QE in $\mathcal{L}_{\text{Pres}}$.

In this subsection we consider ordered abelian groups in the *Presburger language*

$$\mathcal{L}_{\text{Pres}} = \big\{0,\, 1,\, +,\, -,\, <,\, (\equiv_m)\big\}.$$

We naturally construe a given ordered abelian group G as an $\mathcal{L}_{\text{Pres}}$-structure: the symbols $0, +, -, <$ have their usual interpretations; the constant symbol 1 is interpreted by the least positive element of G, provided G has one, and by 0 otherwise; and for each m, the binary relation symbol $\equiv_m$ is interpreted as equivalence modulo m, i.e., for $g, h \in G$,

$$g \equiv_m h \quad :\Longleftrightarrow \quad g - h \in mG.$$

In the rest of this subsection G is an ordered abelian group, and all ordered abelian groups will be construed as $\mathcal{L}_{\text{Pres}}$-structures. Recall that an ordered abelian group is *regular* if it is elementarily equivalent to an archimedean ordered abelian group; moreover, G is regular if either $|G/nG| = n$ for each $n \geq 1$, or nG is dense in G for each $n \geq 1$. In the first case, G is elementarily equivalent to $(\mathbb{Z}; +, <)$, whereas any two dense regular ordered abelian groups G, H are elementarily equivalent iff for each p either G/pG and H/pH are infinite or $|G/pG| = |H/pH|$. (See [59, 73].) In this subsection we show the following.

Theorem 3.2. *Suppose G is regular; then the following are equivalent:*

(1) *G is distal;*
(2) *G is* dp*-minimal;*
(3) *G is non-singular.*

Theorem 3.2 applies to archimedean G, so the ordered abelian groups $(\mathbb{Z};+,<)$, $(\mathbb{Q};+,<)$, and $(\mathbb{Z}_{(2)};+,<)$ are distal, whereas $(\mathbb{Q}^{>0};\cdot,<)$ is not.

The rest of this subsection is devoted to proving Theorem 3.2. We rely on the following:

Fact 3.3 (Weispfenning, [71])**.** *An ordered abelian group is regular if and only if it has QE in $\mathcal{L}_{\text{Pres}}$.*

We first note that the direction (2) $\Rightarrow$ (1) in Theorem 3.2 holds by Fact 1.5. Furthermore, the equivalence (2) $\Leftrightarrow$ (3) is Fact 3.1. Thus it suffices to establish (1) $\Rightarrow$ (3). We will actually prove the contrapositive. For the rest of the subsection we thus fix some p and assume:

(1) G is regular;
(2) G/pG is infinite;
(3) G is sufficiently saturated.

We shall prove that under these assumptions, G is not distal. By QE in $\mathcal{L}_{\text{Pres}}$, we can easily describe indiscernible sequences in a single variable:

Lemma 3.4. *A sequence $(a_i)_{i\in I}$ in G is indiscernible if and only if for all $i_1<\cdots<i_n$ and $j_1<\cdots<j_n$ from I, $k,k_1,\dots,k_n\in\mathbb{Z}$, and $m\geq 2$ we have*

(1) $k\cdot 1+\sum_{l=1}^n k_l a_{i_l}>0 \iff k\cdot 1+\sum_{l=1}^n k_l a_{j_l}>0$;
(2) $k\cdot 1+\sum_{l=1}^n k_l a_{i_l}=0 \iff k\cdot 1+\sum_{l=1}^n k_l a_{j_l}=0$; *and*
(3) $k\cdot 1+\sum_{l=1}^n k_l a_{i_l}\equiv_m 0 \iff k\cdot 1+\sum_{l=1}^n k_l a_{j_l}\equiv_m 0$.

We think of (1) and (2) in Lemma 3.4 as *geometric conditions* and of (3) as *algebraic conditions*. It is easy to prescribe a certain choice of geometric conditions in a rapidly increasing sequence; here we say that a sequence $(a_i)_{i\in I}$ in G is *rapidly increasing* if for all $i<j$ from I and m, n,

$$0\leq m1<na_i<a_j.$$

(That is, $a_i>1$ for all i, and the a_i and 1 lie in distinct archimedean classes.)

Lemma 3.5. *Suppose $(a_i)_{i\in I}$ is a rapidly increasing sequence in G. Then for all $i_1<\cdots<i_n$ and $j_1<\cdots<j_n$ from I and all $k,k_1,\dots,k_n\in\mathbb{Z}$, we have*

(1) $k\cdot 1+\sum_{l=1}^n k_l a_{i_l}>0 \iff k\cdot 1+\sum_{l=1}^n k_l a_{j_l}>0 \iff (k_n,\dots,k_1,k)>_{\text{lex}}(0,\dots,0)$, *and*
(2) $k\cdot 1+\sum_{l=1}^n k_l a_{i_l}=0 \iff k\cdot 1+\sum_{l=1}^n k_l a_{j_l}=0 \iff k=k_1=\cdots=k_n=0$.

In general, it is more difficult to prescribe all of the algebraic conditions which hold in an indiscernible sequence, but once we have an indiscernible sequence in G we can use the following:

Lemma 3.6. *Suppose $(a_i)_{i\in I}$ is an indiscernible sequence in G. Then for all distinct $i_1,\dots,i_n$ and distinct $j_1,\dots,j_n$ from I, all $k,k_1,\dots,k_n\in\mathbb{Z}$ and $m\geq 2$, we have*

(1) $k\cdot 1+\sum_{l=1}^n k_l a_{i_l}=0 \iff k\cdot 1+\sum_{l=1}^n k_l a_{j_l}=0$, *and*
(2) $k\cdot 1+\sum_{l=1}^n k_l a_{i_l}\equiv_m 0 \iff k\cdot 1+\sum_{l=1}^n k_l a_{j_l}\equiv_m 0$.

Proof. The sequence (a_i) is indiscernible in the $\{0,1,+,-,(\equiv_m)\}$-reduct of G. However, this reduct is just (an expansion by definitions and constants of) the underlying abelian group of G, which is stable. Thus the sequence (a_i) in this reduct is totally indiscernible, which implies the conclusion of the lemma. □

Proposition 3.7. *G is not distal.*

Proof. First, using Ramsey we obtain a rapidly increasing indiscernible sequence $(b_i)_{i\in(-1,1)}$ in G such that $b_i \not\equiv_p b_j$ for all $i<j$ from $(-1,1)$. The argument uses that G/pG is infinite and that each coset of pG is cofinal in G. We will use (b_i) to obtain our counterexample to distality. For this, consider the collection $\Phi(x)$ of $\mathcal{L}_{\text{Pres}}$-formulas, with $x=(x_i)_{i\in(-1,1]}$, consisting exactly of the following formulas:

(Φ1) for every $i<j$ from $(-1,1]$ and every m, n, the formula

$$0 \le m1 < nx_i < x_j,$$

(Φ2) for every $i_1<\cdots<i_n$ from $(-1,1)$ and $k,k_1\dots,k_n\in\mathbb{Z}$, if $G\models k\cdot 1+\sum_{l=1}^n k_l b_{i_l}\equiv_m 0$, the formulas

$$k\cdot 1+\textstyle\sum_{l=1}^n k_l x_{i_l}\equiv_m 0 \quad\text{and}\quad \big(k\cdot 1+\sum_{l=1}^n k_l x_{i_l}\equiv_m 0\big)[x_1/x_0],$$

and otherwise the formulas

$$k\cdot 1+\textstyle\sum_{l=1}^n k_l x_{i_l}\not\equiv_m 0 \quad\text{and}\quad \big(k\cdot 1+\sum_{l=1}^n k_l x_{i_l}\not\equiv_m 0\big)[x_1/x_0],$$

where $[x_1/x_0]$ denotes replacing each occurrence of x_0 in the preceding expression by x_1, and

(Φ3) the formula $x_0\equiv_p x_1$.

Thus $\Phi(x)$ expresses that $(x_i)_{i\in(-1,1]}$ is rapidly increasing and satisfies the same algebraic conditions as (b_i), x_0 and x_1 have the same algebraic relations with $(x_i)_{i\in(-1,1)\setminus\{0\}}$, however x_1 and x_0 are congruent modulo p.

Claim 3.8. *$\Phi(x)$ is finitely satisfiable in G.*

Proof of Claim. Let $\Phi_0\subseteq\Phi$ be finite. Set $b_i^*:=b_i$ for $i\in(-1,1)$; then clearly $(b_i^*)_{i\in(-1,1)}$ satisfies all formulas from (Φ1), (Φ2), and (Φ3) which do not involve x_1. We claim that we can choose $b_1^*\in G$ so that $(b_i^*)_{i\in(-1,1]}$ satisfies Φ_0. To see this let N be the product of all moduli occurring in Φ_0, and pick b_1^* to be a sufficiently large member of the coset b_0+pNG. The "sufficiently large" ensures that all formulas in Φ_0 coming from (Φ1) are satisfied, the choice of N ensures that $b_0\equiv_m b_1^*$ for all relevant m, and thus all formulas from (Φ2) are satisfied, and clearly $b_0\equiv_p b_1^*$. □

By the claim and after replacing our original sequence $(b_i)_{i\in(-1,1)}$, we can assume that we have some $b_1\in G$ such that $(b_i)_{i\in(-1,1]}$ realizes $\Phi(x)$. It is clear that $(b_i)_{i\in(-1,1)}$ is indiscernible, and that $(b_i)_{i\in(-1,1)}$ is *not* b_1-indiscernible. It remains to establish:

Claim 3.9. *$(b_i)_{i\in(-1,1)\setminus\{0\}}$ is b_1-indiscernible.*

Proof of Claim. It is sufficient to show that $(b_i)_{i\in(-1,1]\setminus\{0\}}$ is indiscernible. By (Φ1) this sequence is rapidly increasing, thus by Lemma 3.5 the geometric conditions (1) and (2) of Lemma 3.4 hold. It suffices to check (3) from Lemma 3.4. Let $i_1<\cdots<i_{n-1}<i_n=1$ from $(-1,0)\cup(0,1]$ and $j_1<\cdots<j_n$ from $(-1,1)$, and let $k,k_1,\dots,k_n\in\mathbb{Z}$; it is sufficient to show that then

$$k\cdot 1+\textstyle\sum_{l=1}^n k_l b_{i_l}\equiv_m 0 \iff k\cdot 1+\sum_{l=1}^n k_l b_{j_l}\equiv_m 0.$$

Now

$$k\cdot 1+\textstyle\sum_{l=1}^n k_l b_{i_l}\equiv_m 0 \iff \big(k\cdot 1+\sum_{l=1}^n k_l b_{i_l}\equiv_m 0\big)[b_0/b_1]$$

by (Φ2), and

$$\big(k\cdot 1+\textstyle\sum_{l=1}^n k_l b_{i_l}\equiv_m 0\big)[b_0/b_1] \iff k\cdot 1+\sum_{l=1}^n k_l b_{j_l}\equiv_m 0,$$

by Lemma 3.6 and the fact that $(b_i)_{i\in(-1,1)}$ is indiscernible. □

This concludes the proof of the proposition. □

3.2. **A review of the Cluckers-Halupczok language.** In the rest of the section, we consider ordered abelian groups which do not in general have QE in $\mathcal{L}_{\mathrm{Pres}}$. We use the language $\mathcal{L}_{\mathrm{qe}}$ introduced by Cluckers and Halupczok [19] (see also [39]) for their (relative) quantifier elimination result for ordered abelian groups. This language is similar in spirit to one introduced by Gurevich and Schmitt [35], however it is more in line with our modern paradigm of many-sorted languages and perhaps a little more intuitive.

The rest of the subsection is taken essentially from [19]. In what follows G is an ordered abelian group and we use the notation $H \Subset G$ to denote that H is a convex subgroup of G. We introduce $\mathcal{L}_{\mathrm{qe}}$ and at the same time describe how G is viewed as an $\mathcal{L}_{\mathrm{qe}}$-structure $\boldsymbol{G}$. We begin by listing the sorts of $\mathcal{L}_{\mathrm{qe}}$: besides the main sort $\mathcal{G}$ whose underlying set is that of the ordered abelian group G, these are the **auxiliary sorts** $\mathcal{S}_p$, $\mathcal{T}_p$, $\mathcal{T}_p^+$ (one for each p) associated with G. Here is how they are interpreted in $\boldsymbol{G}$:

Definition 3.10.

(1) For $a \in G \setminus pG$, let $G_p(a)$ be the largest convex subgroup of G such that $a \notin G_p(a) + pG$, and for $a \in pG$ let $G_p(a) := \{0\}$; then the underlying set of sort $\mathcal{S}_p$ is $\big\{G_p(a) : a \in G\big\}$;
(2) for $b \in G$, set $G_p^-(b) := \bigcup \big\{G_p(a) : a \in G,\ b \notin G_a\big\}$, where the union over the empty set is declared to be $\{0\}$; then the underlying set of sort $\mathcal{T}_p$ is $\big\{G_p^-(b) : b \in G\big\}$;
(3) For $b \in G$, define $G_p^+(b) := \bigcap \big\{G_p(a) : a \in G,\ b \in G_p(a)\big\}$, where the intersection over the empty set is G; then the underlying set of sort $\mathcal{T}_p^+$ is $\big\{G_p^+(b) : b \in G\big\}$.

Below we don't distinguish notationally between the sort $\mathcal{S}_p$ and its underlying set (so we can write $\mathcal{S}_p = \big\{G_p(a) : a \in G\big\}$), and similar for the other auxiliary sorts. We let α range over (the underlying sets of) the auxiliary sorts. In each case, α is a convex subgroup of G; if we want to stress this role of α as a convex subgroup of G (rather than as an abstract element of the underlying set of a certain sort of the structure $\boldsymbol{G}$), we denote it by G_α, and we let $\pi_\alpha \colon G \twoheadrightarrow G/G_\alpha$ be the natural surjection. We let 1_α denote the minimal positive element of G/G_α if the ordered abelian group G/G_α is discrete, and set $1_\alpha := 0 \in G/G_\alpha$ otherwise; for $k \in \mathbb{Z}$ we let $k_\alpha := k \cdot 1_\alpha$. For $a, b \in G$ and $\diamond$ denoting one of the relation symbols $=$, $<$, or $\equiv_m$ we also write $a \diamond_\alpha b + k_\alpha$ if $\pi_\alpha(a) \diamond \pi_\alpha(b) + k_\alpha$ holds in the ordered abelian group G/G_α. We also set

$$G_\alpha^{[m]} := \bigcap_{G_\alpha \subsetneq H \Subset G} (H + mG) \qquad \text{and} \qquad a \equiv_{n,\alpha}^{[m]} b \ :\Longleftrightarrow\ a - b \in G_\alpha^{[m]} + nG \quad (a, b \in G).$$

We now describe the primitives of the $\mathcal{L}_{\mathrm{qe}}$-structure $\boldsymbol{G}$; these are:

(G1) on the main sort $\mathcal{G}$, the usual primitives $0, +, -, \leq$ of the language of ordered abelian groups;
(G2) binary relations "$\alpha \leq \alpha'$" on $\Big(\mathcal{S}_p \,\dot\cup\, \mathcal{T}_p \,\dot\cup\, \mathcal{T}_p^+\Big) \times \Big(\mathcal{S}_q \,\dot\cup\, \mathcal{T}_q \,\dot\cup\, \mathcal{T}_q^+\Big)$, interpreted as $G_\alpha \subseteq G_{\alpha'}$ (each pair (p, q) giving rise to nine separate binary relations);
(G3) predicates for the relations $a \diamond_\alpha b + k_\alpha$, where $\diamond \in \big\{=, <, (\equiv_m)\big\}$ and $k \in \mathbb{Z}$ (each of these being ternary relations on $G \times G \times \mathcal{X}$ where $\mathcal{X} \in \{\mathcal{S}_p, \mathcal{T}_p, \mathcal{T}_p^+\}$);
(G4) for $m \geq n$, the ternary relation $x \equiv_{q^n,\alpha}^{[q^m]} y$ on $G \times G \times \mathcal{S}_p$;
(G5) a unary predicate discr of sort $\mathcal{S}_p$ which holds of α if and only if G/G_α is discrete;
(G6) for $d \in \mathbb{N}$ and n, two unary predicates of sort $\mathcal{S}_p$ defining the sets

$$\big\{\alpha \in \mathcal{S}_p : \dim_{\mathbb{F}_p} \big(G_\alpha^{[p^n]} + pG\big) \big/ \big(G_\alpha^{[p^{n+1}]} + pG\big) = d\big\} \text{ and}$$
$$\big\{\alpha \in \mathcal{S}_p : \dim_{\mathbb{F}_p} \big(G_\alpha^{[p^n]} + pG\big) \big/ (G_\alpha + pG) = d\big\}.$$

We let $\mathcal{A}$ be the set of auxiliary sorts associated to G, and let $\mathcal{L}_{\mathrm{qe}}^{\mathcal{A}}$ be the sublanguage of $\mathcal{L}_{\mathrm{qe}}$ with sorts $\mathcal{A}$ and primitives listed in (G2), (G5), (G6).

Definition 3.11. Let $\phi(x,\eta)$ be an $\mathcal{L}_{\text{qe}}$-formula, where x and η are multivariables of sort $\mathcal{G}$ and $\mathcal{A}$, respectively. We say that $\phi(x,\eta)$ is in *family union form* if

$$\phi(x,\eta) \;=\; \bigvee_{i=1}^{n} \exists\theta\big(\xi_i(\eta,\theta)\wedge\psi_i(x,\theta)\big),$$

where θ is a multivariable of sort $\mathcal{A}$, $\xi_i(\eta,\theta)$ are $\mathcal{L}_{\text{qe}}^{\mathcal{A}}$-formulas, each $\psi_i(x,\theta)$ is a conjunction of basic formulas (i.e., atomic or negated atomic formulas), and for each ordered abelian group G, viewed as an $\mathcal{L}_{\text{qe}}$-structure $\boldsymbol{G}$ as above, the formulas $\xi_i(\eta,\alpha)\wedge\psi_i(x,\alpha)$, with i ranging over $\{1,\dots,n\}$ and α over tuples of the appropriate sorts in $\boldsymbol{G}$, are pairwise inconsistent.

The following is the main result from [19]:

Fact 3.12. *In the theory of ordered abelian groups, each $\mathcal{L}_{\text{qe}}$-formula is equivalent to an $\mathcal{L}_{\text{qe}}$-formula in family union form.*

3.3. **The case where all $\mathcal{S}_p$ are finite.** The main result of this section is the following.

Theorem 3.13. *Suppose that $\mathcal{S}_p$ is finite for all p. Then G is distal if and only if G is non-singular.*

The hypothesis of the theorem holds if G is strongly dependent, by [12, 25, 30, 37]. The proof of Theorem 3.13, which we now outline, is a generalization of the proof of Theorem 3.2, using Fact 3.12.

For the rest of this section, G is an ordered abelian group such that for each p the underlying set of sort $\mathcal{S}_p$ is finite. Note that then the underlying sets of sorts $\mathcal{T}_p$ and $\mathcal{T}_p^+$ are also finite, for each p. It suffices to show that if G/pG is infinite for some p, then G is *not* distal. Here we construe G as an $\mathcal{L}_{\text{qe}}$-structure, together with constants which name all of $\mathcal{A}$; since each $\mathcal{S}_p$ is finite, the underlying sets of auxiliary sorts will not grow when we pass to an elementary extension of G. Thus we can also assume that G is sufficiently saturated. In this setting, Fact 3.12 specializes as follows:

Proposition 3.14. *In G, each $\mathcal{L}_{\text{qe}}$-formula $\phi(x)$, where x is a multivariable of sort $\mathcal{G}$, is equivalent to a finite boolean combination of atomic formulas in which the only occurring predicates are those from* (G3)*.*

Proof. In Fact 3.12, the quantifier "$\exists\theta$" can be replaced by a finite disjunction over all possible tuples of constants of the same sort as θ. Upon substitution of these constants, each "$\xi_i(\theta)$" becomes a sentence, so in the theory of G, it is equivalent to $\bot$ or $\top$. Likewise for the unary relation $\operatorname{discr}(\alpha)$, the unary "dimension" relations applied to α, and the binary relations $\alpha\le\alpha'$. Finally, as each $\mathcal{S}_p$ is finite, the ternary relations $x\equiv_{q^n,\alpha}^{[q^m]} y$ from (G4) are already taken care of by the relations $x\equiv_{q^n,\alpha'} y$: by [19, Lemma 2.4(2)] we have $G_\alpha^{[q^m]} = G_{\alpha'}+q^mG$ where α' is the successor of α in $\mathcal{S}_{q^m}$ with respect to the linear ordering $\le$ of $\mathcal{S}_{q^m}$ from (G2). □

Proposition 3.14 should be viewed as saying that G has QE in a language which is essentially a union of countably many copies of the Presburger language, one for each of the quotient groups G/G_α. With this point of view, it is fairly straightforward to generalize everything in subsection 3.1 by including "for every α" in many places. For instance, we have the following generalization of Lemma 3.4:

Lemma 3.15. *A sequence $(a_i)_{i\in I}$ in G is indiscernible iff for all $i_1<\dots<i_n$ and $j_1<\dots<j_n$ from I, all $k,k_1,\dots,k_n\in\mathbb{Z}$, all α, and $m\ge 2$, we have*

1. $\sum_{l=1}^n k_l a_{i_l} >_\alpha k_\alpha \iff \sum_{l=1}^n k_l a_{j_l} >_\alpha k_\alpha$;
2. $\sum_{l=1}^n k_l a_{i_l} =_\alpha k_\alpha \iff \sum_{l=1}^n k_l a_{j_l} =_\alpha k_\alpha$; *and*
3. $\sum_{l=1}^n k_l a_{i_l} \equiv_{m,\alpha} k_\alpha \iff \sum_{l=1}^n k_l a_{j_l} \equiv_{m,\alpha} k_\alpha$.

Next, following the proof of Theorem 3.2, the "rapidly increasing sequence" we construct here is a sequence $(a_i)_{i\in I}$ in G such that for all $i<j$ from I, all m, n, and all α,

$$0 \leq_\alpha m\cdot 1_\alpha <_\alpha n\cdot a_i <_\alpha a_j.$$

That is, the sequence (a_i) is a rapidly increasing sequence in each of the countably many quotients G/G_α. This gives rise to an appropriate generalization of Lemma 3.5. We also use the fact that the (unordered) abelian group reducts of the quotients G/G_α are all stable, to get a generalization of Lemma 3.6. Finally, the proof of Proposition 3.7 generalizes to conclude our proof of Theorem 3.13.

We conclude this section with the following conjecture.

Conjecture 3.16. *Every ordered abelian group admits a distal expansion.*

There are some partial results towards this conjecture, but the general case remains open.

4. Distality and Short Exact Sequences of Abelian Groups

In this section we prove a general quantifier elimination theorem for certain short exact sequences of abelian groups, and analyze distality in this setting. These results are used in Sections 5 and 6 below. In Section 4.1 we show our main elimination result. The remaining subsections of this section discuss an application to the preservation of distality as well as variants and refinements.

4.1. Quantifier elimination for pure short exact sequences.

Let

$$0 \to A \xrightarrow{\iota} B \xrightarrow{\nu} C \to 0$$

be a short exact sequence of morphisms of abelian groups which is *pure*, which means that $\iota(A)$ is a pure subgroup of B. (For example, this always holds if C is torsion-free.) We treat such a pure short exact sequence as a three-sorted structure (A,B,C) consisting of three abelian groups, with the two maps $\iota\colon A\to B$ and $\nu\colon B\to C$ added as primitives. If A is $\aleph_1$-saturated, then the short exact sequence splits, i.e., B is the direct sum of A and C, with ι and ν being the natural embedding respectively projection (see, e.g., [2, Corollary 3.3.38]). So the complete theory of (A,B,C) is uniquely determined by the theory of A and the theory of C. Moreover, if $(A,C,R_0,R_1,\dots)$ is an arbitrary expansion of the pair (A,C), then the theory of $(A,B,C,R_0,R_1,\dots)$ is determined by the theory of $(A,C,R_0,R_1,\dots)$. For a syntactical formulation of this observation let us fix the languages involved:

- $\mathcal{L}_{\mathrm{ac}}=\{0_{\mathrm{a}},+_{\mathrm{a}},-_{\mathrm{a}},0_{\mathrm{c}},+_{\mathrm{c}},-_{\mathrm{c}}\}$, the language of the pair (A,C) of abelian groups;
- $\mathcal{L}_{\mathrm{b}}=\{0_{\mathrm{b}},+_{\mathrm{b}},-_{\mathrm{b}}\}$, the language of abelian groups on B;
- $\mathcal{L}_{\mathrm{abc}}=\mathcal{L}_{\mathrm{ac}}\cup\mathcal{L}_{\mathrm{b}}\cup\{\iota,\nu\}$, the language of the three-sorted structure (A,B,C);
- $\mathcal{L}^*_{\mathrm{ac}}$ the language of an expansion $(A,C,R_0,R_1,\dots)$ of the $\mathcal{L}_{\mathrm{ac}}$-structure (A,C);
- $\mathcal{L}^*_{\mathrm{abc}}=\mathcal{L}_{\mathrm{abc}}\cup\mathcal{L}^*_{\mathrm{ac}}$, the language of $(A,B,C,R_0,R_1,\dots)$.

Let T_{abc} be the $\mathcal{L}_{\mathrm{abc}}$-theory of all structures arising from pure exact sequences as above. Viewing T_{abc} as a set of sentences in the expanded language $\mathcal{L}^*_{\mathrm{abc}}$, the observation above then reads as follows:

Corollary 4.1. *Every $\mathcal{L}^*_{\mathrm{abc}}$-sentence is equivalent in T_{abc} to an $\mathcal{L}^*_{\mathrm{ac}}$-sentence.*

This is also a consequence of the quantifier elimination theorem to be proved in this section. For its formulation we note that for each n, our short exact sequence fits into a commutative diagram of group morphisms

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & nA & \longrightarrow & nB & \longrightarrow & nC & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\subseteq} & & \downarrow{\scriptstyle\subseteq} & & \downarrow{\scriptstyle\subseteq} & & \\
0 & \longrightarrow & A & \xrightarrow{\iota} & B & \xrightarrow{\nu} & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\pi_n} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A/nA & \longrightarrow & B/nB & \longrightarrow & C/nC & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact rows and columns. We now expand (A, B, C) by new sorts with underlying sets A/nA together with two unary functions: the natural surjection $\pi_n \colon A \to A/nA$ and a function $\rho_n \colon B \to A/nA$, which, on $\nu^{-1}(nC)$, is the composition of the group morphisms

$$\nu^{-1}(nC) = nB + \iota(A) \to \big(nB + \iota(A)\big)/nB \xrightarrow{\sim} \iota(A)/\big(nB \cap \iota(A)\big) \xrightarrow{\sim} A/nA,$$

and zero outside $\nu^{-1}(nC)$. Note that $\rho_0 \colon B \to A$ agrees with the inverse of $\iota \colon A \xrightarrow{\sim} \iota(A)$ on $\iota(A) = \nu^{-1}(0)$ and is zero on $B \setminus \iota(A)$. (We identify A with $A/0A$ in the natural way.) Note also that $\pi_n = \rho_n \circ \iota$. Moreover, if our short exact sequence splits, and $\pi' \colon B \to A$ is a left inverse of ι, then ρ_n agrees with $\pi_n \circ \pi'$ on $\nu^{-1}(nC)$.

We denote the language of this expansion of the $\mathcal{L}_{\text{abc}}$-structure (A, B, C) by

$$\mathcal{L}_{\text{abcq}} = \mathcal{L}_{\text{abc}} \cup \{\rho_0, \rho_1, \dots, \pi_0, \pi_1, \dots\},$$

and we let T_{abcq} be the $\mathcal{L}_{\text{abcq}}$-theory of all these structures arising from a pure exact sequence as above. We also let

$$\mathcal{L}_{\text{acq}} = \mathcal{L}_{\text{ac}} \cup \{\pi_0, \pi_1, \dots\},$$

a sublanguage of $\mathcal{L}_{\text{abcq}}$. Note that the group operations on A/nA are 0-definable in the reduct of T_{abcq} to the two-sorted language $\mathcal{L}_{\text{a}} \cup \{\pi_n\}$, where $\mathcal{L}_{\text{a}} = \{0_{\text{a}}, +_{\text{a}}, -_{\text{a}}\}$ is the language of the abelian group A. Note also that π_n, ρ_n are interpretable in the $\mathcal{L}_{\text{abc}}$-reduct of T_{abcq}; in particular, if $\boldsymbol{M} = (A, B, C, \dots)$ and $\boldsymbol{M}' = (A', B', C', \dots)$ are models of T_{abcq}, then every isomorphism between the $\mathcal{L}_{\text{abc}}$-reducts of $\boldsymbol{M}$, $\boldsymbol{M}'$ extends uniquely to an $\mathcal{L}_{\text{abcq}}$-isomorphism $\boldsymbol{M} \to \boldsymbol{M}'$.

Let the multivariables x_{a}, x_{b}, x_{c} be of sort A, B and C, respectively. The $\mathcal{L}_{\text{abcq}}$-terms of the form $\rho_n\big(t(x_{\text{b}})\big)$ or $\nu\big(t(x_{\text{b}})\big)$, for an $\mathcal{L}_{\text{b}}$-term $t(x_{\text{b}})$, are called *special*.

Theorem 4.2. *In T_{abcq} every $\mathcal{L}_{\text{abc}}$-formula $\phi(x_{\text{a}}, x_{\text{b}}, x_{\text{c}})$ is equivalent to a formula*

$$\phi_{\text{acq}}\big(x_{\text{a}}, \sigma_1(x_{\text{b}}), \dots, \sigma_m(x_{\text{b}}), x_{\text{c}}\big)$$

where the σ_i are special terms and ϕ_{acq} is a suitable $\mathcal{L}_{\text{acq}}$-formula.

For example, the formula $x_{\text{b}} = 0_{\text{b}}$ is equivalent to $\rho_0(x_{\text{b}}) = 0_{\text{a}} \wedge \nu(x_{\text{b}}) = 0_{\text{c}}$. Also, x_{b} is divisible by n if and only if $\rho_n(x_{\text{b}}) = \pi_n(0_{\text{a}})$ and $\nu(x_{\text{b}})$ is divisible by n.

Proof. Let $\sigma_0, \sigma_1, \dots$ list all special terms. Given a tuple b in a model of T_{abcq} of the same sort as x_{b}, let us write $\sigma(b)$ for the tuple $\sigma_0(b), \sigma_1(b), \dots$. Assume that we have two models $\boldsymbol{M} = (A, B, C, \dots)$ and $\boldsymbol{M}' = (A', B', C', \dots)$ of T_{abcq}. We let a, b, c range over tuples in $\boldsymbol{M}$ of the same sort as x_{a}, x_{b}, x_{c}, respectively, and similarly with the tuples a', b', c' in $\boldsymbol{M}'$. Suppose we are given a, b, c in $\boldsymbol{M}$ and a', b', c' in $\boldsymbol{M}'$ such that the type of $a\sigma(b)c$ in the $\mathcal{L}_{\text{acq}}$-reduct $\boldsymbol{M}_{\text{acq}}$ of $\boldsymbol{M}$ is the same as the type of $a'\sigma(b')c'$ in the $\mathcal{L}_{\mathrm{acq}}$-reduct $\boldsymbol{M}'_{\mathrm{acq}}$ of $\boldsymbol{M}'$. It is enough to show that then abc and $a'b'c'$ have the same type in $\boldsymbol{M}$ and in $\boldsymbol{M}'$, respectively.

For this, after replacing $\boldsymbol{M}$, $\boldsymbol{M}'$ by suitably saturated elementarily equivalent structures, we may assume that there is an isomorphism $\boldsymbol{M}_{\mathrm{acq}} \xrightarrow{\cong} \boldsymbol{M}'_{\mathrm{acq}}$ with $a\sigma(b)c \mapsto a'\sigma(b')c'$. We can then also assume that the short exact sequences underlying $\boldsymbol{M}$ and $\boldsymbol{M}'$ split, thus this isomorphism extends to an isomorphism $\boldsymbol{M} \xrightarrow{\cong} \boldsymbol{M}'$. Hence we may assume that $\boldsymbol{M} = \boldsymbol{M}'$, $a = a'$, $c = c'$ and $\sigma(b) = \sigma(b')$, and it suffices to show that there is an automorphism of $\boldsymbol{M}$ which is the identity on A and C and maps b to b'.

Let B_0 denote the subgroup of B generated by b and B_0' the subgroup of B' generated by b'. Since for each $\mathcal{L}_{\mathrm{b}}$-term $t(x_{\mathrm{b}})$ we have $t(b) = 0$ if and only if $t(b') = 0$, we obtain an isomorphism $f_0 \colon B_0 \to B_0'$ such that $f_0(t(b)) = t(b')$ for all $\mathcal{L}_{\mathrm{b}}$-terms $t(x_{\mathrm{b}})$; in particular, we have $f_0(b) = b'$. Furthermore we have $\rho_n(b_0) = \rho_n(f_0(b_0))$ and $\nu(b_0) = \nu(f_0(b_0))$ for all $b_0 \in B_0$. Set

$$A_0 := B_0 \cap \iota(A) = B_0' \cap \iota(A), \qquad C_0 := \nu(B_0) = \nu(B_0').$$

The map $b_0 \mapsto \iota^{-1}\big(f_0(b_0) - b_0\big)$ is a group morphism $B_0 \to A$. Since f_0 fixes all elements of A_0, the image of $b_0 \in B_0$ under this morphism only depends on $\nu(b_0)$. So f_0 induces a group morphism $h_0 \colon C_0 \to A$ satisfying

$$f_0(b_0) = b_0 + \iota\big(h_0(\nu(b_0))\big) \qquad \text{for all } b_0 \in B_0.$$

We show now that h_0 is a *partial morphism* $C \to A$ in the sense of [74, p. 159], that is, $h_0(nC \cap C_0) \subseteq nA$ for each n: given $c \in nC \cap C_0$, choose $b_0 \in B_0$ with $\nu(b_0) = c$; since ρ_n is a group morphism on $\nu^{-1}(nC)$, we then have

$$\pi_n\big(h_0(c)\big) = \rho_n\big(\iota(h_0(c))\big) = \rho_n\big(f_0(b_0) - b_0\big) = \rho_n\big(f_0(b_0)\big) - \rho_n(b_0) = 0,$$

from which we conclude that $h_0(c) \in nA$.

Finally we may assume that A is pure injective. Then the partial morphism h_0 extends to a group morphism $h \colon C \to A$ [74, Corollary 3.3]. The formula $b \mapsto b + \iota\big(h(\nu(b))\big)$ defines an automorphism of B which together with the identity on all other sorts is an automorphism of $\boldsymbol{M}$ which maps b to b', as required. □

The following corollary generalizes Corollary 4.1; here we view T_{abcq} as a set of $\mathcal{L}^*_{\mathrm{abcq}}$-sentences.

Corollary 4.3. *In T_{abcq} every $\mathcal{L}^*_{\mathrm{abc}}$-formula $\phi^*(x_{\mathrm{a}}, x_{\mathrm{b}}, x_{\mathrm{c}})$ is equivalent to a formula*

$$\phi^*_{\mathrm{acq}}\big(x_{\mathrm{a}}, \sigma_1(x_{\mathrm{b}}), \dots, \sigma_m(x_{\mathrm{b}}), x_{\mathrm{c}}\big)$$

where the σ_i are special terms and ϕ^*_{acq} is a suitable formula in the language $\mathcal{L}^*_{\mathrm{acq}} := \mathcal{L}_{\mathrm{acq}} \cup \mathcal{L}^*_{\mathrm{ac}}$.

Proof. This has exactly the same proof as Theorem 4.2. We show instead that the corollary follows directly from the theorem itself. It is clear that the collection of all formulas equivalent in T_{abcq} to one having the form in the statement of the corollary contains all atomic formulas, is closed under boolean combinations and under quantification over A and over C. It remains to show that this collection of formulas is also closed under quantification over B. Let y_{b} be a multivariable of sort B disjoint from x_{b}, and consider the formula

$$\phi^*(x_{\mathrm{a}}, x_{\mathrm{b}}, x_{\mathrm{c}}) = \exists\, y_{\mathrm{b}}\ \psi^*\big(x_{\mathrm{a}}, \sigma_1(x_{\mathrm{b}}, y_{\mathrm{b}}), \dots, \sigma_m(x_{\mathrm{b}}, y_{\mathrm{b}}), x_{\mathrm{c}}\big)$$

with special terms σ_i and a suitable $\mathcal{L}^*_{\mathrm{acq}}$-formula ψ^*. We may assume that we have $k \in \{0, \dots, m\}$ and $n_1, \dots, n_k \in \mathbb{N}$ such that σ_i is of sort A/n_iA for $i = 1, \dots, k$ and of sort C for $i = k+1, \dots, m$. Theorem 4.2 implies that for distinct variables $z_1, \dots, z_k$ of sort A and $z_{k+1}, \dots, z_m$ of sort C, the $\mathcal{L}_{\mathrm{abcq}}$-formula

$$\exists\, y_{\mathrm{b}}\ \left(\bigwedge_{i=1}^{k} \pi_{n_i}(z_i) = \sigma_i(x_{\mathrm{b}}, y_{\mathrm{b}}) \wedge \bigwedge_{i=k+1}^{m} z_i = \sigma_i(x_{\mathrm{b}}, y_{\mathrm{b}})\right)$$

is equivalent in T_{abcq} to a formula

$$\chi\big(z_1,\dots,z_m,\tau_1(x_{\text{b}}),\dots,\tau_n(x_{\text{b}})\big)$$

where the τ_j are special terms and χ is a suitable $\mathcal{L}_{\text{acq}}$-formula. Then ϕ^* is equivalent to

$$\exists z_1\cdots\exists z_m\Big(\chi\big(z_1,\dots,z_m,\tau_1(x_{\text{b}}),\dots,\tau_n(x_{\text{b}})\big)\ \wedge\ \psi^*\big(x_{\text{a}},\pi_{n_1}(z_1),\dots,\pi_{n_k}(z_k),z_{k+1},\dots,z_m,x_{\text{c}}\big)\Big),$$

which has the desired form. □

Remark. Corollary 4.3 implies the quantifier elimination result in [15]: when all quotients A/nA are finite, the maps ρ_n are quantifier-free definable in the language used there.

4.2. **Preservation of distality.** In this section we prove a result on preservation of distality in pure short exact sequences. Let

$$0\to A\xrightarrow{\ \iota\ } B\xrightarrow{\ \nu\ } C\to 0$$

be a pure short exact sequence of morphisms of abelian groups. We allow here A and C to be equipped with arbitrary additional structure, and denote the respective languages of these expansions by $\mathcal{L}^*_{\text{a}}$ and $\mathcal{L}^*_{\text{c}}$. We also let $\boldsymbol{M}=\big(A,(A/nA)_{n\geq 0},B,C;\dots\big)$ be the corresponding $\mathcal{L}^*_{\text{abcq}}$-structure as in Section 4.1. In this situation we have:

Remark 4.4.
1. (1) The $\mathcal{L}^*_{\text{abcq}}$-structure $\boldsymbol{M}$ and its $\mathcal{L}^*_{\text{abc}}$-reduct (A,B,C) are bi-interpretable.
2. (2) The collection of the sorts A and A/nA ($n\geq 0$) is fully stably embedded in $\boldsymbol{M}$ (by the QE result in the previous section), and the full structure induced on it is bi-interpretable with A.
3. (3) Similarly, the sort C is fully stably embedded in $\boldsymbol{M}$.

Lemma 4.5. *$\boldsymbol{M}$ is NIP if and only if both the $\mathcal{L}^*_{\text{a}}$-structure A and the $\mathcal{L}^*_{\text{c}}$-structure C are NIP.*

Proof. The forward direction is clear. Suppose A and C are NIP. To show the $\boldsymbol{M}$ is NIP we may assume that it is a monster model of its theory. Adding a function symbol for a right-inverse of ν to the language $\mathcal{L}^*_{\text{abcq}}$, we obtain a structure that is bi-interpretable with a two-sorted structure consisting of two sorts given by A and C with their full induced structure; this implies that $\boldsymbol{M}$ is NIP, as a reduct of a NIP structure. □

Theorem 4.6. *$\boldsymbol{M}$ is distal if and only if both A and C are distal.*

Proof. The forward implication is immediate by Lemma 1.15 and Remark 4.4; we prove the converse. Suppose A and C are distal; again, we may assume that $\boldsymbol{M}$ is a monster model of its theory, and by Lemma 4.5, $\boldsymbol{M}$ is NIP. Assume towards contradiction that $\boldsymbol{M}$ is not distal; then by Remark 4.4(1), its $\mathcal{L}^*_{\text{abc}}$-reduct is also not distal, and hence satisfies condition (3) in Corollary 1.11. Thus, also using Remark 1.13, we obtain a partitioned $\mathcal{L}^*_{\text{abc}}$-formula $\varphi(x;y)$, where $|x|=1$, as well as an indiscernible sequence $(b_i)_{i\in\mathbb{Q}_\infty}$ of the same sort as x and some tuple d of the same sort as the multivariable y such that $(b_i)_{i\in\mathbb{Q}_\infty\setminus\{0\}}$ is d-indiscernible and $\boldsymbol{M}\models\varphi(b_i;d)\iff i\neq 0$. By assumption, Remark 4.4 and Lemma 1.16, the variable x is necessarily of sort B. We say that a tuple is contained in d if all its components appear as components of d.

It is easy to see from the QE (Corollary 4.3) that the formula $\varphi(x;d)$ is equivalent to a positive boolean combination of formulas of the form:
1. (1) $\psi^*\big(\nu(t_1(x,b')),\dots,\nu(t_m(x,b')),c\big)$ where b' is a tuple of sort B, c is a tuple of sort C, both contained in d, the t_k are $\mathcal{L}_{\text{b}}$-terms, and ψ^* is an $\mathcal{L}^*_{\text{c}}$-formula.
2. (2) $\theta^*\big(a,\rho_{n_1}(t_1(x,b')),\dots,\rho_{n_m}(t_m(x,b'))\big)$ where a is a tuple in A, b' is a tuple in B, both contained in d, the t_k are $\mathcal{L}_{\text{b}}$-terms, $n_k\in\mathbb{N}$, and θ^* is an $\mathcal{L}^*_{\text{aq}}$-formula, where $\mathcal{L}^*_{\text{aq}}=\mathcal{L}^*_{\text{a}}\cup\{\pi_0,\pi_1,\dots\}$.

By Remarks 1.12 and 1.13, it is enough to show that $\varphi(x;y)$ cannot be of any of these forms. Below we let i, j range over $\mathbb{Q}_\infty$ and k over $\{1,\dots,m\}$.

Suppose first that (1) holds. As ν is a group morphism, $\psi^*\big(\nu(t_1(x,b')),\dots,\nu(t_m(x,b')),c\big)$ is equivalent to a formula of the form $\psi_1^*\big(\nu(x),c'\big)$ where c' is a d-definable tuple of sort C and ψ_1^* is an $\mathcal{L}_\mathrm{c}^*$-formula. By choice of (b_i), the sequence $\big(\nu(b_i)\big)$ in C is indiscernible, $\big(\nu(b_i)\big)_{i\neq 0}$ is c'-indiscernible, and

$$\boldsymbol{M}\models\psi_1^*(\nu(b_i),c') \quad\Longleftrightarrow\quad \boldsymbol{M}\models\varphi(b_i,d) \quad\Longleftrightarrow\quad i\neq 0.$$

This contradicts distality of the structure induced on C.

Now suppose that we are in case (2). We may assume that for each k we have $r_k\in\mathbb{Z}$ and a d-definable $b_k'\in B$ with $t_k(b_i,b')=r_kb_i-b_k'$ for each i. Set $B_{n_k}=\nu^{-1}(n_kC)$. By Case (1) applied to the $\mathcal{L}_\mathrm{c}$-formulas defining n_kC and its complement, the truth value of the condition "$r_kb_i-b_k'\in B_{n_k}$" doesn't depend on i. If $r_kb_i-b_k'\notin B_{n_k}$ for some/all i, then $\rho_{n_k}(r_kb_i-b_k')=0=\rho_{n_k}(0_\mathrm{b})$ for all i by definition. Thus, replacing the term t_k by 0_b, we still have

$$\boldsymbol{M}\models\theta^*\big(a,\rho_{n_1}(t_1(b_i,b')),\dots,\rho_{n_m}(t_m(b_i,b'))\big)\iff i\neq 0.$$

Hence we may assume that $r_kb_i-b_k'\in B_{n_k}$ for all i. Repeating this argument for each k one by one, we may reduce to the case that $r_kb_i-b_k'\in B_{n_k}$ for all i, k. As B_{n_k} is a subgroup of B, we have

$$r_kb_i-r_kb_j=(r_kb_i-b_k')-(r_kb_j-b_k')\in B_{n_k}\qquad\text{for all } i,\ j.$$

Let $b_i^k:=r_kb_i-r_kb_\infty\in B_{n_k}$ and $b^k:=b_k'-r_kb_\infty$. Note that

$$b_i^k-b^k=r_kb_i-r_kb_\infty-(b_k'-r_kb_\infty)=r_kb_i-b_k'\in B_{n_k}$$

and hence $b^k\in B_{n_k}$. As ρ_{n_k} restricts to a group morphism $B_{n_k}\to A/n_kA$, we have

$$\rho_{n_k}(r_kb_i-b_k')=\rho_{n_k}(b_i^k-b^k)=\rho_{n_k}(b_i^k)-\rho_{n_k}(b^k)\qquad\text{for all } i.$$

Let $\beta_i:=(\beta_i^1,\dots,\beta_i^m)$ and $\beta:=(\beta^1,\dots,\beta^m)$ where $\beta_i^k:=\rho_{n_k}(b_i^k)$, $\beta^k:=\rho_{n_k}(b^k)$, and let $x_1,\dots,x_m$ be distinct variables with x_k of sort A/n_kA. Consider the $\mathcal{L}_\mathrm{aq}^*$-formula

$$\theta_1^*(x_1,\dots,x_m,a,\beta):=\theta^*(a,x_1-\beta^1,\dots,x_m-\beta^m).$$

We then have:

- $(\beta_i)_{i\in\mathbb{Q}}$ is indiscernible (by construction, as $(b_i)_{i\in\mathbb{Q}}$ is b_∞-indiscernible),
- $(\beta_i)_{i\in\mathbb{Q}\setminus\{0\}}$ is $a\gamma$-indiscernible (by construction, as $(b_i)_{i\in\mathbb{Q}\setminus\{0\}}$ is $ab_\infty b_1'\dots b_m'$-indiscernible),

and, unwinding, for every $i\in\mathbb{Q}$, in $\boldsymbol{M}$ we have

$$\begin{aligned}\models\theta_1^*(\beta_i,a,\beta)\quad&\Longleftrightarrow\quad\models\theta^*\big(a,\beta_i^1-\beta^1,\dots,\beta_i^m-\beta^m\big)\\&\Longleftrightarrow\quad\models\theta^*\big(a,\rho_{n_1}(b_i^1)-\rho_{n_1}(b^1),\dots,\rho_{n_m}(b_i^m)-\rho_{n_m}(b^m)\big)\\&\Longleftrightarrow\quad\models\theta^*\big(a,\rho_{n_1}(r_1b_i-b_1'),\dots,\rho_{n_m}(r_mb_i-b_m')\big)\quad\Longleftrightarrow\quad i\neq 0.\end{aligned}$$

This contradicts distality of the $\mathcal{L}_\mathrm{aq}^*$-structure A. □

Remark. In this subsection we assumed that the $\mathcal{L}_\mathrm{ac}^*$-structure $(A,C,R_0,R_1,\dots)$ expanding the $\mathcal{L}_\mathrm{ac}$-structure (A,C) is obtained by combining separate expansions of the $\mathcal{L}_\mathrm{a}$-structure A and of the $\mathcal{L}_\mathrm{c}$-structure C. Let now $(A,C)^\circ$ be an arbitrary expansion of (A,C), and denote its language by $\mathcal{L}_\mathrm{ac}^\circ$ and the corresponding $\mathcal{L}_\mathrm{abcq}^\circ$-structure by $\boldsymbol{M}^\circ$. A straightforward adaption of the proofs shows that Lemma 4.5 and Theorem 4.6 remain true: $\boldsymbol{M}^\circ$ is NIP (distal) iff $(A,C)^\circ$ is NIP (distal, respectively).

4.3. **A variant for abelian monoids.** For later use, we now consider a slight variant of Corollary 4.3 for abelian groups augmented by absorbing elements. Let $(A, 0, +)$ be an abelian monoid. An element ∞ of A is said to be *absorbing* if $\infty + a = \infty$ for all $a \in A$. (Clearly there is at most one absorbing element.) For example, if R is a commutative ring, then $(R, 1, \cdot\,)$ is an abelian monoid with absorbing element 0. If A is an abelian group and $\infty \notin A$ is a new element, then $A_\infty := A \cup \{\infty\}$ with the group operation $+$ of A extended to a binary operation on A_∞ such that

$$a + \infty = \infty + a = \infty \qquad \text{for all } a \in A_\infty$$

is an abelian monoid with absorbing element ∞. In this case we also extend $a \mapsto -a\colon A \to A$ to a map $A_\infty \to A_\infty$ by setting $-\infty := \infty$. Every morphism $f\colon A \to B$ of abelian groups extends uniquely to a monoid morphism $f_\infty\colon A_\infty \to B_\infty$. Here is a special case of this construction:

Notation. Given a commutative ring R and a subgroup G of the multiplicative group $R^\times$ of units of R we let $R/G := (R^\times/G)_\infty$. In this case we always denote the absorbing element of R/G by 0, so the residue morphism $R^\times \to R^\times/G$ extends to a surjective monoid morphism $R \to R/G$ which maps $0 \in R$ to $0 \in R/G$.

Let now

$$0 \to A \xrightarrow{\iota} B \xrightarrow{\nu} C \to 0$$

be a pure short exact sequence of abelian groups. We redefine the languages introduced at the beginning of this section as follows:

- $\mathcal{L}_{\rm ac} = \{0_{\rm a}, +_{\rm a}, -_{\rm a}, \infty_{\rm a}, 0_{\rm c}, +_{\rm c}, -_{\rm c}, \infty_{\rm c}\}$, the language of the pair (A_∞, C_∞);
- $\mathcal{L}_{\rm b} = \{0_{\rm b}, +_{\rm b}, -_{\rm b}, \infty_{\rm b}\}$, the language of B_∞;
- $\mathcal{L}_{\rm abc} = \mathcal{L}_{\rm ac} \cup \mathcal{L}_{\rm b} \cup \{\iota_\infty, \nu_\infty\}$, the language of the three-sorted structure $(A_\infty, B_\infty, C_\infty)$.

We denote the extension of $\pi_n\colon A \to A/nA$ to a morphism $A_\infty \to (A/nA)_\infty$ also by π_n, and now introduce $\rho_n\colon B_\infty \to (A/nA)_\infty$ by defining $\rho_n(b) \in A/nA$ for $b \in \nu^{-1}(nC)$ as before and declaring $\rho_n(b) := \infty$ for $b \in B_\infty \setminus \nu^{-1}(nC)$. Thus $\rho_0\colon B_\infty \to A_\infty$ agrees with the inverse of ι on $\iota(A)$ and is constant ∞ on $B_\infty \setminus \iota(A)$. We let

$$\mathcal{L}_{\rm abcq} = \mathcal{L}_{\rm abc} \cup \{\rho_0, \rho_1, \dots, \pi_0, \pi_1, \dots\}, \qquad \mathcal{L}_{\rm acq} = \mathcal{L}_{\rm ac} \cup \{\pi_0, \pi_1, \dots\},$$

and we let $T^\infty_{\rm abcq}$ be the theory of all $\mathcal{L}_{\rm abcq}$-structures arising from a pure exact sequence of abelian groups as above. The $\mathcal{L}_{\rm abcq}$-terms of the form $\rho_n\big(t(x_{\rm b})\big)$ or $\nu\big(t(x_{\rm b})\big)$, for a term $t(x_{\rm b})$ in the sublanguage $\{0_{\rm b}, +_{\rm b}, -_{\rm b}\}$ of $\mathcal{L}_{\rm b}$, are called *special*.

Proposition 4.7. *In $T^\infty_{\rm abcq}$ every $\mathcal{L}_{\rm abc}$-formula $\phi(x_{\rm a}, x_{\rm b}, x_{\rm c})$ is equivalent to a formula*

$$\phi_{\rm acq}\big(x_{\rm a}, \sigma_1(x_{\rm b}), \dots, \sigma_m(x_{\rm b}), x_{\rm c}\big)$$

where the σ_i are special terms and $\phi_{\rm acq}$ is a suitable $\mathcal{L}_{\rm acq}$-formula.

Mutatis mutandis, the proof of this proposition is similar to that of Theorem 4.2. (Main change: B_0 is the subgroup of B generated by those entries of b which do not equal ∞, and similarly for B_0'.) Next, let $\mathcal{L}^*_{\rm ac}$ be the language of an expansion $(A_\infty, C_\infty, R_0, R_1, \dots)$ of the $\mathcal{L}_{\rm ac}$-structure (A_∞, C_∞), let $\mathcal{L}^*_{\rm abc} = \mathcal{L}_{\rm abc} \cup \mathcal{L}^*_{\rm ac}$ be the language of $(A_\infty, B_\infty, C_\infty, R_0, R_1, \dots)$, and $\mathcal{L}^*_{\rm acq} = \mathcal{L}_{\rm acq} \cup \mathcal{L}^*_{\rm ac}$. As in the proof of Corollary 4.3, the preceding proposition implies:

Corollary 4.8. *In $T^\infty_{\rm abcq}$ every $\mathcal{L}^*_{\rm abc}$-formula $\phi^*(x_{\rm a}, x_{\rm b}, x_{\rm c})$ is equivalent to a formula*

$$\phi^*_{\rm acq}\big(x_{\rm a}, \sigma_1(x_{\rm b}), \dots, \sigma_m(x_{\rm b}), x_{\rm c}\big)$$

where the σ_i are special terms and $\phi^*_{\rm acq}$ is a suitable $\mathcal{L}^*_{\rm acq}$-formula.

Remark 4.9. In the previous corollary one may assume that no special terms of the form $\rho_1\big(t(x_{\rm b})\big)$ appear among the σ_j. Since $\rho_n(b-b')=\rho_n\big(b+(n-1)b'\big)$ for $n\geq 2$ and $b,b'\in B_\infty$, we can also arrange that the terms $\rho_n\big(t(x_{\rm b})\big)$, $n\geq 2$ appearing among the σ_j do not involve the function symbol $-_{\rm b}$. Moreover, since ν is a group morphism on its proper domain of definition, we can achieve that none of the terms of the form $\nu\big(t(x_{\rm b})\big)$ appearing as some σ_j involve $-_{\rm b}$.

4.4. **Weakly pure exact sequences.** Consider a sequence

$$0\to A\xrightarrow{\iota}B\xrightarrow{\nu}C\to 0 \tag{4.1}$$

of morphisms of abelian groups where ι is injective, ν is surjective, and $\ker\nu\subseteq\operatorname{im}\iota$, and let $\delta:=\nu\circ\iota\colon A\to C$. Note that with $\overline{\nu}$ denoting the composition of ν with the natural surjection

$$c\mapsto\overline{c}:=c+\operatorname{im}\delta\colon C\to\overline{C}:=C/\operatorname{im}\delta,$$

we obtain a short exact sequence

$$0\to A\xrightarrow{\iota}B\xrightarrow{\overline{\nu}}\overline{C}\to 0,$$

which we call the short exact sequence associated to our given sequence (4.1).

Lemma 4.10. *Suppose the short exact sequence associated to* (4.1) *as well as the short exact sequence* $0\to\ker A\xrightarrow{\subseteq}A\xrightarrow{\delta}\operatorname{im}\delta\to 0$ *both split. Then with* $A_1:=\ker\delta$, $B_1:=\operatorname{im}\delta$, *and* $C_1:=\operatorname{coker}\delta=\overline{C}$, *we have a commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\ \iota\ } & B & \xrightarrow{\ \nu\ } & C & \longrightarrow & 0\\
 & & {\scriptstyle\cong}\big\downarrow{\scriptstyle f_A} & & {\scriptstyle\cong}\big\downarrow{\scriptstyle f_B} & & {\scriptstyle\cong}\big\downarrow{\scriptstyle f_C} & & \\
0 & \longrightarrow & A_1\oplus B_1 & \longrightarrow & A_1\oplus B_1\oplus C_1 & \longrightarrow & B_1\oplus C_1 & \longrightarrow & 0
\end{array}$$

where the second arrow on the bottom row is the natural inclusion and the third arrow the natural projection.

Proof. Take group morphisms $s\colon B_1\to A$ and $t\colon C_1\to B$ such that $\delta\circ s=\operatorname{id}_{B_1}$ and $\overline{\nu}\circ t=\operatorname{id}_{C_1}$. Since ν induces an isomorphism $B/\operatorname{im}\iota\to C/\operatorname{im}\delta$, we have $g(b):=b-t(\overline{\nu}(b))\in\operatorname{im}\iota$. One checks that f_A, f_B, f_C defined by

$$f_A(a)=\big(a-s(\delta(a))\big)+\delta(a),\quad f_B(b)=f_A\big(\iota^{-1}(g(b))\big)+\overline{\nu}(b),\quad f_C(c)=\big(c-\nu(t(\overline{c}))\big)+\overline{c}$$

for $a\in A$, $b\in B$, $c\in C$ have the required properties. □

We say that (4.1) is **weakly pure exact** if $\operatorname{im}\iota$ is a pure subgroup of B and $\ker\nu$ is a pure subgroup of $\operatorname{im}\iota$. Thus every pure short exact sequence is weakly pure exact; moreover, if (4.1) is weakly pure exact, then its associated short exact sequence is pure.

Lemma 4.11. *Suppose* $\overline{C}=C/\operatorname{im}\delta$ *and* $\operatorname{im}\delta$ *are both torsion-free; then* (4.1) *is weakly pure exact.*

Proof. Let $b\in B$ and $n\geq 1$ with $nb\in\operatorname{im}\iota$. Take $a\in A$ with $\iota(a)=nb$; then $n\nu(b)=\delta(a)\in\operatorname{im}\delta$ and hence $\nu(b)\in\operatorname{im}\delta$ (since $\overline{C}$ is torsion-free), so $\nu(b)=\nu(\iota(a'))$ where $a'\in A$; then $b-\iota(a')\in\ker\nu\subseteq\operatorname{im}\iota$ and hence $b\in\operatorname{im}\iota$. This shows that $\operatorname{im}\iota$ is a pure subgroup of B. Next, let $a\in A$ and $n\geq 1$ with $n\iota(a)\in\ker\nu$; then $n\delta(a)=0$ and thus $\delta(a)=0$ (since $\operatorname{im}\delta$ is torsion-free), that is, $\iota(a)\in\ker\nu$. Therefore $\ker\nu$ is a pure subgroup of $\operatorname{im}\iota$. □

A variant of Theorem 4.2 holds for weakly pure exact sequences. To make this precise, view each weakly pure exact sequence (4.1) as an $\mathcal{L}_{\rm abc}$-structure in the natural way. For each n let $\pi_n\colon A\to A/nA$ be the natural surjection, define $\rho_n\colon B\to A/nA$ according to the pure exact sequence associated to (4.1), and expand the $\mathcal{L}_{\rm abc}$-structure (4.1) to a structure in the language $\mathcal{L}_{\rm abcd} := \mathcal{L}_{\rm abcq} \cup \{\delta\}$ in the natural way. Let $T_{\rm abcd}$ be the theory of $\mathcal{L}_{\rm abcd}$-structures

$$(A, B, C, \pi_0, \pi_1, \dots, \rho_0, \rho_1, \dots, \delta)$$

which arise from a weakly pure exact sequence (4.1) in this way. Let $\mathcal{L}_{\rm acd}$ be the sublanguage $\mathcal{L}_{\rm acq} \cup \{\delta\}$ of $\mathcal{L}_{\rm abcd}$. We then have:

Theorem 4.12. *In $T_{\rm abcd}$ every $\mathcal{L}_{\rm abc}$-formula $\phi(x_{\rm a}, x_{\rm b}, x_{\rm c})$ is equivalent to a formula*

$$\phi_{\rm acd}\big(x_{\rm a}, \sigma_1(x_{\rm b}), \dots, \sigma_m(x_{\rm b}), x_{\rm c}\big)$$

where the σ_i are special terms and $\phi_{\rm acd}$ is a suitable $\mathcal{L}_{\rm acd}$-formula.

Proof. The proof is similar to the proof of Theorem 4.2 with the following modifications. Let $\boldsymbol{M} = (A, B, C, \dots)$ and $\boldsymbol{M}' = (A', B', C', \dots)$ be models of $T_{\rm abcd}$, and with the same notational conventions as in the proof of Theorem 4.2, assume that we are given a, b, c in $\boldsymbol{M}$ and a', b', c' in $\boldsymbol{M}'$ such that the type of $a\sigma(b)c$ in the $\mathcal{L}_{\rm acd}$-reduct $\boldsymbol{M}_{\rm acd} = (A, C, \delta)$ of $\boldsymbol{M}$ is the same as the type of $a'\sigma(b')c'$ in the $\mathcal{L}_{\rm acd}$-reduct $\boldsymbol{M}'_{\rm acd} = (A', C', \delta')$ of $\boldsymbol{M}'$; we need to show that then abc and $a'b'c'$ have the same type in $\boldsymbol{M}$ and in $\boldsymbol{M}'$, respectively.

Assuming, as we may, that $\boldsymbol{M}$, $\boldsymbol{M}'$ are sufficiently saturated, we first show that a given isomorphism $\boldsymbol{M}_{\rm acd} \to \boldsymbol{M}'_{\rm acd}$ extends to an isomorphism $\boldsymbol{M} \to \boldsymbol{M}'$. For this, by Lemma 4.10 we may assume that $B = A_1 \oplus B_1 \oplus C_1$, where $A = A_1 \oplus B_1$, $C = B_1 \oplus C_1$ and ι and ν are the natural injection and the natural projection; then $\delta(a_1 + b_1) = b_1$ for $a_1 \in A_1$, $b \in B_1$. Similarly with A', B', C', etc. in place of A, B, C, etc. If the isomorphisms $f_A \colon A \to A'$ and $f_C \colon C \to C'$ are compatible with δ, δ', then they have the form

$$\begin{aligned} f_A(a_1 + b_1) &= f(a_1 + b_1) + g(b_1) \\ f_C(b_1 + c_1) &= \big(g(b_1) + h_1(c_1)\big) + h_2(c_1) \qquad (a_1 \in A_1,\ b_1 \in B_1,\ c_1 \in C_1) \end{aligned}$$

for group morphims $f \colon A \to A_1'$, $g \colon B_1 \to B_1'$, $h_1 \colon C_1 \to B_1'$, and $h_2 \colon C_1 \to C_1'$. Then

$$(a_1 + b_1 + c_1) \mapsto f(a_1 + b_1) + \big(g(b_1) + h_1(c_1)\big) + h_2(c_1) \qquad (a_1 \in A_1,\ b_1 \in B_1,\ c_1 \in C_1)$$

is a group isomorphism $f_B \colon B \to B'$, and (f_A, f_B, f_C) is an isomorphism between the $\mathcal{L}_{\rm abc}$-reducts of $\boldsymbol{M}$ and $\boldsymbol{M}'$, which gives rise to an isomorphism $\boldsymbol{M} \to \boldsymbol{M}'$ of $\mathcal{L}_{\rm abcd}$-structures as required.

Therefore, as in the proof of Theorem 4.2 we can assume $\boldsymbol{M} = \boldsymbol{M}'$, $a = a'$, $c = c'$, $\sigma(b) = \sigma(b')$, and it suffices to show that there is an automorphism of $\boldsymbol{M}$ which is the identity on A and C and sends b to b'. Let B_0, B_0' and the group isomorphism $f_0 \colon B_0 \to B_0'$ be as in the proof of Theorem 4.2. Identifying $\overline{C} = \operatorname{coker} \delta$ with C_1 in the natural way, the short exact sequence associated to our given weakly pure exact sequence is

$$0 \to A = A_1 \oplus B_1 \xrightarrow{\iota} B = A_1 \oplus B_1 \oplus C_1 \xrightarrow{\overline{\nu}} C_1 \to 0$$

where ι is the natural inclusion and $\overline{\nu}$ the natural projection. In particular $A_1 = \ker \overline{\nu}$, and since $\overline{\nu}(b_0) = \overline{\nu}(f_0(b_0))$, we have $f_0(b_0) - b_0 \in A_1$ for each $b_0 \in B_0$. Set

$$A_0 := B_0 \cap A = B_0' \cap A, \qquad C_0 := \overline{\nu}(B_0) = \overline{\nu}(B_0') \subseteq C_1.$$

As in the proof of Theorem 4.2 we see that we have a morphism $h_0 \colon C_0 \to A_1$ satisfying

$$f_0(b_0) = b_0 + h_0\big(\overline{\nu}(b_0)\big) \qquad \text{for all } b_0 \in B_0.$$

Now h_0 is a partial morphism $C_1 \to A$, and thus also a partial morphism $C_1 \to A_1$ since A_1 is pure in A. Extend h_0 to a group morphism $h \colon C_1 \to A$; then $b \mapsto b + h(\overline{\nu}(b))$ defines an automorphism of B which, together with the identity on all other sorts, is an automorphism of $\boldsymbol{M}$ fixing A and C and mapping b to b' as desired. □

The theorem above yields a quantifier elimination result for arbitrary expansions of $\mathcal{L}_{\rm acd}$ just as in Corollary 4.3. We also have a version of Theorem 4.12 for abelian monoids, just like Proposition 4.7. To formulate this, redefine the languages $\mathcal{L}_{\rm ac}$, $\mathcal{L}_{\rm b}$, and $\mathcal{L}_{\rm abc}$ as in Section 4.3. Given a weakly pure exact sequence (4.1), denote the extension of $\pi_n\colon A\to A/nA$ to a morphism $A_\infty\to (A/nA)_\infty$ by π_n. We define $\rho_n\colon B_\infty\to (A/nA)_\infty$ by defining $\rho_n(b)\in A/nA$ for $b\in\overline{\nu}^{-1}(n\overline{C})=nB+\iota(A)$ as before and $\rho_n(b):=\infty$ for $b\in B_\infty\setminus\big(nB+\iota(A)\big)$. With $\mathcal{L}_{\rm abcd}$, $\mathcal{L}_{\rm acd}$ as before, let $T^\infty_{\rm abcd}$ be the theory of all $\mathcal{L}_{\rm abcd}$-structures which arise this way from a weakly pure exact sequence (4.1). Then Theorem 4.12 goes through, with a similar proof, and implies a version with additional structure on the $\mathcal{L}_{\rm ac}$-structure (A,C) as in Corollary 4.8.

4.5. **Connection to abelian structures.** In this subsection we generalize Theorems 4.2 and 4.12 to pure exact sequences of abelian structures in the sense of Fisher [31]; for this we use a well-known generalization of the Baur-Monk quantifier simplification for modules to the case of abelian structures. (This is not used later in the paper.) Recall that an *abelian structure* is an S-sorted structure $\boldsymbol{A}=\big((A_s);(R_i),(f_j)\big)$ where for each sort $s\in S$, among the primitives of $\boldsymbol{A}$ are distinguished a constant $0_s\in A_s$, a unary function $-_s\colon A_s\to A_s$, and a binary function $+_s\colon A_s\times A_s\to A_s$, such that the (one-sorted) structure $(A_s;0_s,-_s,+_s)$ is an abelian group, and all other relations $R_i\subseteq A_{s_1}\times\cdots\times A_{s_m}$ are subgroups and all functions $f_j\colon A_{s_1}\times\cdots\times A_{s_n}\to A_s$ are group morphisms. Also recall that given a language $\mathcal{L}$, the set of positive primitive (p.p.) $\mathcal{L}$-formulas is the closure of the set of atomic $\mathcal{L}$-formulas under conjunction and existential quantification. Let now $\mathcal{L}$ be the language of an abelian structure $\boldsymbol{A}$ as above. For each p.p. $\mathcal{L}$-formula $\phi(x)$,

$$\phi^{\boldsymbol{A}}=\big\{a\in A_x:\boldsymbol{A}\models\phi(a)\big\}$$

is a subgroup of A_x. Given two p.p. $\mathcal{L}$-formulas $\phi(x)$, $\psi(x)$ where x is a single variable of sort $s\in S$, we set

$$\dim^{\geq n}_{\phi,\psi}:=\exists x_1\cdots\exists x_n\left(\bigwedge_{1\leq i\leq n}\phi(x_i)\wedge\bigwedge_{1\leq i<j\leq n}\neg\psi(x_i-x_j)\right),$$

so $\boldsymbol{A}\models\dim^{\geq n}_{\phi,\psi}$ iff $|\phi^{\boldsymbol{A}}/(\phi\wedge\psi)^{\boldsymbol{A}}|\geq n$; the $\mathcal{L}$-sentences $\dim^{\geq n}_{\phi,\psi}$ are called *dimension sentences.* The following is a version of the Baur-Monk Theorem for abelian structures [70].

Proposition 4.13. *Each $\mathcal{L}$-formula is equivalent, in the theory of abelian $\mathcal{L}$-structures, to a boolean combination of p.p. $\mathcal{L}$-formulas and dimension sentences.*

We call a family of p.p. $\mathcal{L}$-formulas **fundamental** (for $\boldsymbol{A}$) if every p.p. $\mathcal{L}$-formula is equivalent in $\boldsymbol{A}$ to a conjunction of formulas $\phi(t(x))$ where ϕ is fundamental and t is a tuple of $\mathcal{L}$-terms. For example, it is well-known that if $\boldsymbol{A}$ is just an abelian group, then the formulas of the form $n|x$ for $n=0,2,3,\dots$ form a fundamental family [40, A.2.1].

Let now $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ be abelian $\mathcal{L}$-structures. Let $\iota\colon\boldsymbol{A}\to\boldsymbol{B}$ be a morphism of $\mathcal{L}$-structures. Recall that ι is said to be an embedding if ι is injective and for each relation symbol R of $\mathcal{L}$ we have $R^{\boldsymbol{A}}=\iota^{-1}(R^{\boldsymbol{B}})$; as a consequence, $\phi^{\boldsymbol{A}}\subseteq\iota^{-1}(\phi^{\boldsymbol{B}})$ for each p.p. $\mathcal{L}$-formula $\phi(x)$. We say that such an embedding ι is **pure** if $\phi^{\boldsymbol{A}}=\iota^{-1}(\phi^{\boldsymbol{B}})$ for each p.p. $\mathcal{L}$-formula $\phi(x)$. If $\boldsymbol{A}$ is a substructure of $\boldsymbol{B}$ and the natural inclusion $\boldsymbol{A}\to\boldsymbol{B}$ is a pure embedding, then $\boldsymbol{A}$ is said to be a pure substructure of $\boldsymbol{B}$. A morphism $\nu\colon\boldsymbol{B}\to\boldsymbol{C}$ is said to be a **projection** if ν is surjective and $R^{\boldsymbol{C}}=\nu(R^{\boldsymbol{B}})$ for every relation symbol R of $\mathcal{L}$, and such a projection ν is said to be **pure** if $\phi^{\boldsymbol{C}}=\nu(\phi^{\boldsymbol{B}})$ for each p.p. $\mathcal{L}$-formula $\phi(x)$.

In the following, we assume for notational simplicity that our language $\mathcal{L}$ is one-sorted, and we denote the structures $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ by A, B, C, respectively.

Lemma 4.14. *Let $0\to A\xrightarrow{\iota}B\xrightarrow{\nu}C\to 0$ be a short exact sequence of morphisms of $\mathcal{L}$-structures, where ι is an embedding and ν is a projection. Then ι is pure iff ν is pure.*

Proof. First assume that ι is pure. Consider a p.p. $\mathcal{L}$-formula $\phi(x) = \exists x' \bigwedge_{i=1}^n R_i\big(t_i(x,x')\big)$, where each t_i is a tuple of $\mathcal{L}$-terms and each R_i is a relation symbol of $\mathcal{L}$ or an equation between components of t_i, and let $c \in C_x$ with $C \models \phi(c)$. Take $c' \in C_{x'}$ such that $C \models \bigwedge_i R_i\big(t_i(c,c')\big)$, and let b, b' be preimages of c, c', respectively, under ν. Since ν is a projection, we can take appropriate tuples a_i in A such that $B \models \bigwedge_i R_i\big(t_i(b,b') + \iota(a_i)\big)$. Since ι is pure, there are $a \in A_x$, $a' \in A_{x'}$ such that $A \models \bigwedge_i R_i\big(t_i(a,a') + a_i\big)$. This implies $B \models \bigwedge_i R_i\big(t_i(b-\iota(a), b'-\iota(a'))\big)$. So $b - \iota(a)$ is a preimage of c under ν satisfying ϕ. This shows that ν is pure.

For the converse assume that ν is pure, and let $a \in A_x$ where $\iota(a)$ satisfies a p.p.-formula $\phi(x)$ as above. So there is $b' \in B_{x'}$ such that $B \models \bigwedge_i R_i\big(t_i(\iota(a), b')\big)$. Therefore $C \models \bigwedge_i R_i\big(t_i(0, \nu(b')\big)$ and by assumption there is $a' \in A_{x'}$ such that $B \models \bigwedge_i R_i\big(t_i(0, b' - \iota(a')\big)$. This implies $B \models \bigwedge_i R_i\big(t_i(\iota(a,a'))\big)$. So $A \models \bigwedge_i R_i\big(t_i(a,a')\big)$ since ι is an embedding, and a satisfies ϕ. □

A short exact sequence $0 \to A \xrightarrow{\iota} B \xrightarrow{\nu} C \to 0$ of morphisms of $\mathcal{L}$-structures where ι is a pure embedding and ν is a pure projection is called **pure**.

Remark. If B is the direct sum of the abelian $\mathcal{L}$-structures A and C (defined in the obvious way), then the resulting sequence $A \xrightarrow{\iota} B \xrightarrow{\nu} C$ is pure exact. All pure exact sequences where A is $|\mathcal{L}|^+$-saturated are of this form.

Lemma 4.15. *Let $\nu\colon B \to C$ be a pure projection, $\phi(x,x')$ be a p.p. $\mathcal{L}$-formula, $b \in B_x$, and $c' \in C_{x'}$. Then the following are equivalent:*

1. *There is $b' \in B_{x'}$ such that $B \models \phi(b,b')$ and $\nu(b') = c'$;*
2. *$B \models \exists x' \phi(b,x')$ and $C \models \phi\big(\nu(b), c'\big)$.*

Proof. The direction (1) ⇒ (2) is clear; we only use that ν is morphism. For the converse assume (2). Take $b'_0 \in B_{x'}$ such that $B \models \phi(b,b'_0)$. Since ν is a pure projection, there are $b_1 \in B_x$ and $b'_1 \in B_{x'}$ such hat $\nu(b_1) = \nu(b)$, $\nu(b'_1) = c'$ and $B \models \phi(b_1, b'_1)$. So $B \models \phi(b - b_1, b'_0 - b'_1)$. By the last lemma, $A := \ker \nu$ is (the underlying set of) a pure substructure of B. Since $b - b_1 \in A$, purity gives an $a' \in A_{x'}$ such that $B \models \phi(b - b_1, a')$. So we have $B \models \phi(b, b')$ for $b' = b'_1 + a'$. We see now that $\nu(b') = c'$, and (1) holds. □

We now consider a sequence

$$0 \to A \xrightarrow{\iota} B \xrightarrow{\nu} C \to 0 \tag{4.2}$$

of morphisms of abelian $\mathcal{L}$-structures. We let $\mathcal{L}_{\mathrm{a}}$, $\mathcal{L}_{\mathrm{b}}$, $\mathcal{L}_{\mathrm{c}}$ be pairwise disjoint copies of $\mathcal{L}$ (for A, B, C, respectively), introduce a three-sorted language $\mathcal{L}_{\mathrm{abc}} = \mathcal{L}_{\mathrm{a}} \cup \mathcal{L}_{\mathrm{b}} \cup \mathcal{L}_{\mathrm{c}} \cup \{\iota, \nu\}$, and view (A,B,C) as an $\mathcal{L}_{\mathrm{abc}}$-structure in the natural way. This $\mathcal{L}_{\mathrm{abc}}$-structure (A,B,C) is also abelian, hence Proposition 4.13 applies to (A,B,C). (As a consequence, (A,B,C) is stable [40, A.1.13].) Let the multivariables x_{a}, x_{b}, x_{c} be of sort A, B and C, respectively, and similarly with y in place of x.

4.5.1. *Pure exact sequences.* In this subsection we assume that the sequence (4.2) is pure exact. Furthermore we consider an arbitrary expansion $(A,C)^*$ of the reduct (A,C) of (A,B,C) with language $\mathcal{L}^*_{\mathrm{ac}}$, and we let $\mathcal{L}^*_{\mathrm{abc}} := \mathcal{L}^*_{\mathrm{ac}} \cup \mathcal{L}_{\mathrm{b}}$. Unless mentioned otherwise, in the following, "equivalent" means "equivalent in the $\mathcal{L}^*_{\mathrm{abc}}$-structure (A,B,C)". By an **ac-existential quantification** of an $\mathcal{L}^*_{\mathrm{abc}}$-formula ψ we mean a formula of the form $\exists x_{\mathrm{a}} \exists x_{\mathrm{c}}\, \psi$, for some multivariables x_{a}, x_{c}.

Lemma 4.16. *Every p.p. $\mathcal{L}^*_{\mathrm{abc}}$-formula $\phi^*_{\mathrm{abc}}(x_{\mathrm{a}}, x_{\mathrm{b}}, x_{\mathrm{c}})$ is equivalent to an ac-existential quantification of a formula*

$$\phi_{\mathrm{b}}\big(\iota(x_{\mathrm{a}}), x_{\mathrm{b}}\big) \wedge \phi^*_{\mathrm{ac}}\big(x_{\mathrm{a}}, \nu(x_{\mathrm{b}}), x_{\mathrm{c}}\big),$$

where ϕ_{b} is a p.p. $\mathcal{L}_{\mathrm{b}}$-formula and ϕ^*_{ac} is a p.p. $\mathcal{L}^*_{\mathrm{ac}}$-formula.

Proof. Recall that each p.p. formula is equivalent to an existential quantification of a *basic* formula, i.e., a conjunction of atomic formulas. Since ν is a morphism of $\mathcal{L}$-structures and $\nu \circ \iota = 0$, every term $\nu(t)$ can be replaced by a sum of terms $\nu(x_{\mathrm{b}})$. So every basic formula is equivalent to a formula $\psi_{\mathrm{b}}\big(\iota(t), x_{\mathrm{b}}\big) \wedge \psi^*_{\mathrm{ac}}\big(x_{\mathrm{a}}, \nu(x_{\mathrm{b}}), x_{\mathrm{c}}\big)$, where ψ_{b} is a basic $\mathcal{L}_{\mathrm{b}}$-formula, ψ^*_{ac} is a basic $\mathcal{L}^*_{\mathrm{ac}}$-formula, and t is a tuple of $\mathcal{L}^*_{\mathrm{ac}}$-terms in x_{a}, $\nu(x_{\mathrm{b}})$, and x_{c}. We can replace t by existentially quantified multivariables x'_{a} of sort A and add the equations $x'_{\mathrm{a}} = t$. Thus we may assume that our p.p. formula has the form

$$\exists y_{\mathrm{b}}\big(\psi_{\mathrm{b}}(\iota(x_{\mathrm{a}}), x_{\mathrm{b}}, y_{\mathrm{b}}) \wedge \psi^*_{\mathrm{ac}}(x_{\mathrm{a}}, \nu(x_{\mathrm{b}}), \nu(y_{\mathrm{b}}), x_{\mathrm{c}})\big).$$

This formula in turn is equivalent to

$$\exists y_{\mathrm{c}}\Big(\theta(x_{\mathrm{a}}, x_{\mathrm{b}}, y_{\mathrm{c}}) \wedge \psi^*_{\mathrm{ac}}(x_{\mathrm{a}}, \nu(x_{\mathrm{b}}), y_{\mathrm{c}}, x_{\mathrm{c}})\Big) \quad \text{where } \theta := \exists y_{\mathrm{b}}\big(\psi_{\mathrm{b}}(\iota(x_{\mathrm{a}}), x_{\mathrm{b}}, y_{\mathrm{b}}) \wedge \nu(y_{\mathrm{b}}) = y_{\mathrm{c}}\big),$$

and by Lemma 4.15, θ is equivalent to

$$\exists y_{\mathrm{b}} \psi_{\mathrm{b}}\big(\iota(x_{\mathrm{a}}), x_{\mathrm{b}}, y_{\mathrm{b}}\big) \wedge \psi_{\mathrm{c}}\big(0, \nu(x_{\mathrm{b}}), y_{\mathrm{c}}\big),$$

where ψ_{c} is the $\mathcal{L}_{\mathrm{c}}$-copy of ψ_{b}. □

For a p.p. $\mathcal{L}$-formula $\phi(x)$ let A_ϕ be the quotient group A_x/ϕ^A and $\pi_\phi \colon A_x \to A_\phi$ be the natural surjection. Define the map $\rho_\phi \colon B_x \to A_\phi$ on $\nu^{-1}(\phi^C)$ as the composition of the maps

$$\nu^{-1}(\phi^C) = \phi^B + \iota(A_x) \to \big(\phi^B + \iota(A_x)\big)/\phi^B \xrightarrow{\sim} \iota(A_x)/\big(\phi^B \cap \iota(A_x)\big) \xrightarrow{\sim} A_\phi,$$

and identically zero outside $\nu^{-1}(\phi^C)$. The following lemma is clear from the definitions.

Lemma 4.17. *Let $a \in A_x$, $b \in B_x$. Then $\iota(a) + b \in \phi^B$ iff $\pi_\phi(a) + \rho_\phi(b) = 0$ and $\nu(b) \in \phi^C$.*

We now fix a family of p.p. $\mathcal{L}$-formulas which is fundamental for B. We expand $(A, C)^*$ by a new sort A_ϕ together with the corresponding projection map π_ϕ, for every fundamental $\mathcal{L}$-formula ϕ. Let

$$\mathcal{L}^*_{\mathrm{acq}} := \mathcal{L}^*_{\mathrm{ac}} \cup \{\pi_\phi : \phi \text{ fundamental}\}$$

be the language of this expansion. We call terms of the form $\rho_\phi\big(t(x_{\mathrm{b}})\big)$ or $\nu(x_{\mathrm{b}})$ for a fundamental ϕ and a tuple t of $\mathcal{L}_{\mathrm{b}}$-terms *special*.

Lemma 4.18. *Every p.p. $\mathcal{L}^*_{\mathrm{abc}}$-formula $\phi^*_{\mathrm{abc}}(x_{\mathrm{a}}, x_{\mathrm{b}}, x_{\mathrm{c}})$ is equivalent to a formula*

$$\phi^*_{\mathrm{acq}}\big(x_{\mathrm{a}}, \sigma_1(x_{\mathrm{b}}), \dots, \sigma_m(x_{\mathrm{b}}), x_{\mathrm{c}}\big)$$

where the σ_i are special terms and ϕ^*_{acq} is a suitable p.p. $\mathcal{L}^*_{\mathrm{acq}}$-formula.

Proof. By Lemma 4.16 it suffices to prove this for formulas $\phi^*_{\mathrm{abc}}(x_{\mathrm{a}}, x_{\mathrm{b}}) = \phi_{\mathrm{b}}\big(t_{\mathrm{b}}(\iota(x_{\mathrm{a}}), x_{\mathrm{b}})\big)$ where ϕ_{b} is fundamental and t_b is a tuple of $\mathcal{L}_{\mathrm{b}}$-terms. We may arrange that $t_{\mathrm{b}}\big(\iota(x_{\mathrm{a}}), x_{\mathrm{b}}\big) = \iota\big(r_{\mathrm{a}}(x_{\mathrm{a}})\big) + s_{\mathrm{b}}(x_{\mathrm{b}})$ for a tuple r_{a} of $\mathcal{L}_{\mathrm{a}}$-terms and a tuple s_{b} of $\mathcal{L}_{\mathrm{b}}$-terms. Let ϕ_{c} and s_{c} be the $\mathcal{L}_{\mathrm{c}}$-copies of ϕ_{b} and s_{b}, respectively; then by Lemma 4.17, $\phi^*_{\mathrm{abc}}(x_{\mathrm{a}}, x_{\mathrm{b}})$ is equivalent to $\pi_\phi\big(r_{\mathrm{a}}(x_{\mathrm{a}})\big) + \rho_\phi\big(s_{\mathrm{b}}(x_{\mathrm{b}})\big) = 0 \wedge \phi_{\mathrm{c}}\big(s_{\mathrm{c}}(\nu(x_{\mathrm{b}}))\big)$. □

We now obtain versions of Theorem 4.2 and Corollary 4.3 for our pure exact sequence (4.2):

Theorem 4.19. *Every $\mathcal{L}_{\mathrm{abc}}$-formula $\phi(x_{\mathrm{a}}, x_{\mathrm{b}}, x_{\mathrm{c}})$ is equivalent to a formula*

$$\phi_{\mathrm{acq}}\big(x_{\mathrm{a}}, \sigma_1(x_{\mathrm{b}}), \dots, \sigma_m(x_{\mathrm{b}}), x_{\mathrm{c}}\big)$$

where the σ_i are special terms and ϕ_{acq} is a suitable $\mathcal{L}_{\mathrm{acq}}$-formula.

Proof. By Proposition 4.13, every $\mathcal{L}_{\mathrm{abc}}$-formula is equivalent to a boolean combination of p.p. $\mathcal{L}_{\mathrm{abc}}$-formulas. Now apply Lemma 4.18 to the trivial expansion of (A, C). □

Corollary 4.20. *Every $\mathcal{L}^*_{\mathrm{abc}}$-formula $\phi^*(x_{\mathrm{a}}, x_{\mathrm{b}}, x_{\mathrm{c}})$ is equivalent to a formula*

$$\phi^*_{\mathrm{acq}}\big(x_{\mathrm{a}}, \sigma_1(x_{\mathrm{b}}), \dots, \sigma_m(x_{\mathrm{b}}), x_{\mathrm{c}}\big)$$

where the σ_i are special terms and ϕ^*_{acq} a suitable $\mathcal{L}^*_{\mathrm{acq}}$-formula.

Proof. This follows from the theorem like Corollary 4.3 follows from Theorem 4.2. □

4.5.2. *Weakly pure exact sequences.* In this subsection we assume that (4.2) is **weakly pure exact**, i.e., ι a pure embedding, ν a pure projection, and $\operatorname{im}\iota \subseteq \ker\nu$. As in Section 4.4 let $\delta := \nu \circ \iota$. The pair (A, C) is then an abelian $\mathcal{L}_{\mathrm{acd}}$-structure, where $\mathcal{L}_{\mathrm{acd}} = \mathcal{L}_{\mathrm{ac}} \cup \{\delta\}$. Let $(A, C)^*$ be an expansion of (A, C) with language $\mathcal{L}^*_{\mathrm{acd}}$, let $\mathcal{L}^*_{\mathrm{abcd}} := \mathcal{L}^*_{\mathrm{acd}} \cup \mathcal{L}_{\mathrm{b}}$. "Equivalent" now means "equivalent in the $\mathcal{L}^*_{\mathrm{abcd}}$-structure (A, B, C)", and we define ac-existential quantifications as in the previous subsection. We have then the following generalization of Lemma 4.16:

Lemma 4.21. *Every p.p. $\mathcal{L}^*_{\mathrm{abcd}}$-formula $\phi^*_{\mathrm{abcd}}(x_{\mathrm{a}}, x_{\mathrm{b}}, x_{\mathrm{c}})$ is equivalent to an ac-existential quantification of a formula*

$$\phi_{\mathrm{b}}\big(\iota(x_{\mathrm{a}}), x_{\mathrm{b}}\big) \wedge \phi^*_{\mathrm{acd}}\big(x_{\mathrm{a}}, \nu(x_{\mathrm{b}}), x_{\mathrm{c}}\big),$$

where ϕ_{b} is a p.p. $\mathcal{L}_{\mathrm{b}}$-formula and ϕ^*_{acd} is a p.p. $\mathcal{L}^*_{\mathrm{acd}}$-formula.

Proof. The proof is the same as the proof of Lemma 4.16, except that terms $\nu(\iota(t))$ are not replaced by 0 but by $\delta(t)$. Note that we use here, in Lemma 4.15, that ν is a pure projection. □

Let $\overline{C} := \operatorname{coker}\delta = C/\operatorname{im}\delta$ equipped with its induced structure under the natural surjection $c \mapsto \overline{c}\colon C \to \overline{C}$. This surjection $c \mapsto \overline{c}$ is a pure projection; composition with ν yields a pure projection $\overline{\nu}\colon B \to \overline{C}$ as in Section 4.4. The natural inclusion $\ker\delta \to A$ is a pure embedding. Moreover, $\ker\overline{\nu} = A$, and the short exact sequence

$$0 \to A \xrightarrow{\iota} B \xrightarrow{\overline{\nu}} \overline{\nu} \to 0$$

of morphisms of $\mathcal{L}$-structures associated to (4.2) is pure exact. We define for every p.p. $\mathcal{L}$-formula $\phi(x)$ the map $\rho_\phi\colon B_x \to A_\phi = A_x/\phi^A$ as in the last subsection but according to the pure exact sequence associated to (4.2) displayed above. Lemma 4.17 then becomes:

Lemma 4.22. *Let $a \in A_x$, $b \in B_x$; then $\iota(a) + b \in \phi^B$ iff $\pi_\phi(a) + \rho_\phi(b) = 0$ and $\delta(a) + \nu(b) \in \phi^C$.*

Proof. The direction from left to right is clear since $\iota(a) + b \in \phi^B$ implies $\nu(\iota(a) + b) \in \phi^C$. The converse follows from Lemma 4.17 since $\delta(a) + \nu(b) \in \phi^C$ implies $\overline{\nu}(b) \in \phi^{\overline{C}}$. □

As in the last subsection we fix now a family of p.p. $\mathcal{L}$-formulas which is fundamental for B and expand $(A, C)^*$ by the new sorts A_ϕ for every fundamental ϕ together with the projection map π_ϕ. Let $\mathcal{L}^*_{\mathrm{acdq}}$ be the language of the resulting expansion. Lemma 4.18 is now:

Lemma 4.23. *Every p.p. $\mathcal{L}^*_{\mathrm{abcd}}$-formula $\phi^*_{\mathrm{abcd}}(x_{\mathrm{a}}, x_{\mathrm{b}}, x_{\mathrm{c}})$ is equivalent to a formula*

$$\phi^*_{\mathrm{acdq}}\big(x_{\mathrm{a}}, \sigma_1(x_{\mathrm{b}}), \dots, \sigma_m(x_{\mathrm{b}}), x_{\mathrm{c}}\big)$$

where the σ_i are special terms and ϕ^*_{acdq} is a suitable p.p. $\mathcal{L}^*_{\mathrm{acdq}}$-formula.

Proof. As the proof Lemma 4.18, except that $\phi_{\mathrm{b}}(t_{\mathrm{b}}(\iota(x_{\mathrm{a}}), x_{\mathrm{b}}))$ is equivalent to

$$\pi_\phi\big(r_{\mathrm{a}}(x_{\mathrm{a}})\big) + \rho_\phi\big(s_{\mathrm{b}}(x_{\mathrm{b}})\big) = 0 \wedge \phi_{\mathrm{c}}\big(\delta(r_{\mathrm{a}}(x_{\mathrm{a}})) + s_{\mathrm{c}}(\nu(x_{\mathrm{b}}))\big) \qquad \square.$$

As in the last subsection we can conclude:

Corollary 4.24. *Every $\mathcal{L}^*_{\mathrm{abcd}}$-formula $\phi^*(x_{\mathrm{a}}, x_{\mathrm{b}}, x_{\mathrm{c}})$ is equivalent to a formula*

$$\phi^*_{\mathrm{acdq}}\big(x_{\mathrm{a}}, \sigma_1(x_{\mathrm{b}}), \dots, \sigma_m(x_{\mathrm{b}}), x_{\mathrm{c}}\big)$$

where the σ_i are special terms and ϕ^*_{acdq} a suitable $\mathcal{L}^*_{\mathrm{acdq}}$-formula.

Remarks.

(1) There is always a fundamental family of p.p. $\mathcal{L}$-formulas, namely the set of *all* p.p. $\mathcal{L}$-formulas. So, by the previous corollary and following the proofs of Lemma 4.5 and Theorem 4.6, we see that a weakly pure exact sequence (A, B, C) of abelian $\mathcal{L}$-structures with an expansion $(A, C)^*$ of (A, C, δ) is NIP (or distal) if and only if $(A, C)^*$ is NIP (or distal).

(2) If (A, C, δ) comes from a weakly pure exact sequence, then $\delta\colon A \to \operatorname{im}\delta$ is a pure projection and the natural inclusion $\operatorname{im}\delta \to C$ a pure embedding. The converse is may be true, but we know it only if $\ker\delta$ is a direct summand of A or $\operatorname{im}\delta$ is a direct summand of C.

5. Eliminating Field Quantifiers in Henselian Valued Fields

In this section we discuss two frameworks for elimination of field quantifiers in henselian valued fields of characteristic zero construed as multi-sorted structures. The first one is the familiar RV (leading term) setting, for which we use [32] as our reference. Here the additional sorts are quotients of the multiplicative group of the underlying field by groups of higher 1-units. (See Sections 5.1–5.3.) In our second context we instead use, besides the value group, certain imaginary sorts obtained from quotient rings of the valuation ring, and employ the results of Section 4 to prove the relevant elimination theorems. In the equicharacteristic zero case, which we treat first, this setting simplifies even more, to quotients of the multiplicative group of the residue field; see Section 5.4 below. Each of these various settings has advantages that make it more convenient for some tasks rather than others; in this spirit, the elimination theorems from the present section are applied in combination to prove our main theorem in the next section.

5.1. **Quantifier elimination in henselian valued fields.** *Throughout this section we fix a valued field K of characteristic zero.* We let $v\colon K^\times \to \Gamma = v(K^\times)$ be the valuation of K, and $\mathcal{O}$ its valuation ring. As in Section 4.3 we consider the abelian monoid $\Gamma_\infty := \Gamma \cup \{\infty\}$ with absorbing element $\infty \notin \Gamma$, and extend the ordering of Γ to a total ordering on Γ_∞ with $\gamma < \infty$ for all $\gamma \in \Gamma$; as usual we denote the extension of v to a monoid morphism $K \to \Gamma_\infty$ also by v. Let γ, δ range over $\Gamma^{\geq 0}$. Let

$$\mathfrak{m}_\delta := \{x \in K : vx > \delta\},$$

so $\mathfrak{m}_\delta$ is an ideal of $\mathcal{O}$ with $\mathfrak{m}_\gamma \subseteq \mathfrak{m}_\delta$ if $\gamma \geq \delta$. The maximal ideal of $\mathcal{O}$ is $\mathfrak{m} := \mathfrak{m}_0$, and its residue field is $\boldsymbol{k} := \mathcal{O}/\mathfrak{m}$. Let also

$$\mathrm{RV}_\delta := K/(1+\mathfrak{m}_\delta), \qquad \mathrm{RV}_\delta^\times := \mathrm{RV}_\delta \setminus \{0\},$$

with residue morphism $\mathrm{rv}_\delta\colon K \to \mathrm{RV}_\delta$. Thus for $a \in K^\times$ we have $\mathrm{rv}_\delta(a) = a(1+\mathfrak{m}_\delta) \in \mathrm{RV}_\delta^\times$, and rv_δ sends $0 \in K$ to the absorbing element 0 of RV_δ. We write

$$\mathrm{RV} := \mathrm{RV}_0 = K/(1+\mathfrak{m}), \qquad \mathrm{rv} := \mathrm{rv}_0\,.$$

For $a \in \mathcal{O} \setminus \mathfrak{m}$, the element $a(1+\mathfrak{m})$ of $\mathrm{RV}^\times$ only depends on the coset $a+\mathfrak{m}$, and we hence obtain a group embedding $\boldsymbol{k}^\times \to \mathrm{RV}^\times$ which sends the element $a+\mathfrak{m}$ of $\boldsymbol{k}^\times$ to $a(1+\mathfrak{m}) \in \mathrm{RV}^\times$. Together with the group morphism $v_{\mathrm{rv}}\colon \mathrm{RV}^\times \to \Gamma$ induced by the valuation $v\colon K^\times \to \Gamma$, this group embedding fits into a pure short exact sequence

$$1 \to \boldsymbol{k}^\times \to \mathrm{RV}^\times \xrightarrow{v_{\mathrm{rv}}} \Gamma \to 0.$$

We denote the extension of v_{rv} to a morphism $\mathrm{RV} \to \Gamma_\infty$ of monoids by the same symbol. Besides the induced multiplication, RV_δ also inherits a partially defined addition from K via the ternary relation

$$(5.1) \qquad \oplus_\delta\,(r,s,t) \quad\Longleftrightarrow\quad \exists x,y,z \in K\big(r = \mathrm{rv}_\delta(x) \wedge s = \mathrm{rv}_\delta(y) \wedge t = \mathrm{rv}_\delta(z) \wedge x+y=z\big).$$

For $\gamma \geq \delta$ we also have a natural surjective monoid morphism $\mathrm{rv}_{\gamma\to\delta}\colon \mathrm{RV}_\gamma \to \mathrm{RV}_\delta$.

It turns out that for what follows, not all of RV_δ will be needed. Therefore, from now on we let γ and δ (possibly with decorations) range over $\{0\}$ if $\operatorname{char}\boldsymbol{k} = 0$, and over the set $v(p^{\mathbb{N}}) := \{v(p^n) : n \geq 0\}$ if $\operatorname{char}\boldsymbol{k} = p > 0$. We introduce a many-sorted structure $\boldsymbol{K}$ whose sorts are K and the sets RV_δ, equipped with the following primitives:

- (K1) the ring primitives on K;
- (K2) on each sort RV_δ, the monoid primitives and the partial addition relation $\oplus_\delta$ defined above;
- (K3) for each δ, the map $\mathrm{rv}_\delta\colon K \to \mathrm{RV}_\delta$; and
- (K4) for each $\gamma \geq \delta$, the maps $\mathrm{rv}_{\gamma\to\delta}\colon \mathrm{RV}_\gamma \to \mathrm{RV}_\delta$.

We also denote by RV_* the structure with underlying sorts RV_δ and primitives listed under (K2) and (K4) above, with associated language $\mathcal{L}_{\mathrm{RV}_*}$.

Remark 5.1. The relation $v_{\mathrm{rv}}(x) \leq v_{\mathrm{rv}}(y)$ on RV is definable in RV_* [32, Proposition 2.8(1)]. Namely,

$$v_{\mathrm{rv}}(x) \leq 0 \iff \neg \oplus_0 (x, 1, 1), \qquad v_{\mathrm{rv}}(x) = 0 \iff v_{\mathrm{rv}}(x) \leq 0 \wedge \exists y\big(x \cdot y = 1 \wedge v_{\mathrm{rv}}(y) \leq 0\big)$$

and hence

$$v_{\mathrm{rv}}(x) = v_{\mathrm{rv}}(y) \iff \exists z\big(v_{\mathrm{rv}}(z) = 0 \wedge x = y \cdot z\big), \qquad v_{\mathrm{rv}}(x) < v_{\mathrm{rv}}(y) \iff x \neq 0 \wedge \oplus_0(x, y, x).$$

Hence the multiplicative group $\ker v_{\mathrm{rv}} \cong \boldsymbol{k}^\times$ is definable in RV_*. As a consequence the ordered abelian group $\Gamma = v(K^\times)$ is interpretable in RV_*, and using $\oplus_0$ it follows that the field $\boldsymbol{k}$ is also interpretable in RV_*.

Remark 5.2. Our valued field viewed as a structure $(K, \mathcal{O})$ in the language of rings expanded by a unary predicate for the valuation ring $\mathcal{O}$ of K is bi-interpretable with $\boldsymbol{K}$ (regardless of the characteristic of $\boldsymbol{k}$). Hence $(K, \mathcal{O})$ is distal, respectively has a distal expansion, iff $\boldsymbol{K}$ has the corresponding property, by Fact 1.14(1).

Fact 5.3 (Flenner [32, Propositions 4.3 and 5.1])**.** *Suppose K is henselian.*

1. *Let $S \subseteq K$ be A-definable in $\boldsymbol{K}$, for some parameter set A in $\boldsymbol{K}$. Then there are $a_1, \dots, a_m \in K \cap \operatorname{acl}(A)$ and an $\operatorname{acl}(A)$-definable $D \subseteq \mathrm{RV}_{\delta_1} \times \cdots \times \mathrm{RV}_{\delta_m}$, for some $\delta_1, \dots, \delta_m$, such that*
$$S = \big\{x \in K : \big(\mathrm{rv}_{\delta_1}(x - a_1), \dots, \mathrm{rv}_{\delta_m}(x - a_m)\big) \in D\big\};$$
2. *RV_* is fully stably embedded (i.e., the structure on RV_* induced from $\boldsymbol{K}$, with parameters, is precisely the one described above).*

Fact 5.3 is uniform in K; moreover, it continues to hold if we add arbitrary additional structure on RV_*; see the discussion before [32, Proposition 4.3].

Remarks 5.4.

1. Among the primitives of RV_* we have the projections $\mathrm{rv}_{\gamma\to\delta}$ $(\gamma \geq \delta)$; thus in Fact 5.3 we may assume that $\delta_1 = \cdots = \delta_m = \delta$, after possibly modifying D and taking $\delta := \max\{\delta_1, \dots, \delta_m\}$.
2. Note that for any $x \in K$, $y \in K^\times$, we have $\mathrm{rv}_\delta(x) = \mathrm{rv}_\delta(y)$ iff $v(x - y) > vy + \delta$; hence for any $z \in K$ and $x, y \in K \setminus \{z\}$, $\mathrm{rv}_\delta(x - z) = \mathrm{rv}_\delta(y - z)$ iff $v(x - y) > v(y - z) + \delta$.

5.2. **The finitely ramified case.** For later use we analyze the kernels of the group morphisms

$$\mathrm{rv}_{\gamma\to\delta}\colon \mathrm{RV}_\gamma^\times \to \mathrm{RV}_\delta^\times \qquad (\gamma \geq \delta).$$

In the following well-known lemma and its corollary we assume that we have a generator π for the maximal ideal: $\pi\mathcal{O} = \mathfrak{m}$.

Lemma 5.5. *Suppose $n \geq 1$. Then the map*

$$\varphi\colon 1 + \pi^n\mathcal{O} \to \mathcal{O}/\pi\mathcal{O} = \boldsymbol{k}, \qquad \varphi(1 + \pi^n a) := a + \pi\mathcal{O} \text{ for } a \in \mathcal{O}$$

is a surjective group morphism from the multiplicative abelian group $1 + \pi^n\mathcal{O}$ to the additive abelian group $\boldsymbol{k}$ with kernel $1 + \pi^{n+1}\mathcal{O}$. Thus, as abelian groups, $(1 + \pi^n\mathcal{O})/(1 + \pi^{n+1}\mathcal{O}) \cong \boldsymbol{k}$.

We leave the proof of Lemma 5.5 to the reader; an easy induction on r based on this lemma yields:

Corollary 5.6. *Suppose $\boldsymbol{k}$ is finite. Then $|(1+\pi^n\mathcal{O})/(1+\pi^{n+r}\mathcal{O})| = |\boldsymbol{k}|^r$ for each $n \geq 1$ and $r \in \mathbb{N}$.*

We now obtain our desired result:

Lemma 5.7. *Suppose K is finitely ramified with finite residue field $\boldsymbol{k} = \mathcal{O}/\mathfrak{m}$ of characteristic p. Then for each n, the kernel of the group morphism*

$$\mathrm{rv}_{v(p^{n+1})\to v(p^n)} \colon \mathrm{RV}^{\times}_{v(p^{n+1})} \to \mathrm{RV}^{\times}_{v(p^n)}$$

is finite.

Proof. Take $\pi \in \mathcal{O}$ with $\mathfrak{m} = \pi\mathcal{O}$; then $p = \pi^e u$ where $e \in \mathbb{N}$, $e \geq 1$, $u \in \mathcal{O}^{\times}$. By Corollary 5.6,

$$(1 + p^n\mathfrak{m})/(1 + p^{n+1}\mathfrak{m}) = (1 + \pi^{en+1}\mathcal{O})/(1 + \pi^{(en+1)+e}\mathcal{O})$$

is finite, as required. □

We also need additive versions of the results above. In the following lemma and its corollary, we again assume that π satisfies $\pi\mathcal{O} = \mathfrak{m}$:

Lemma 5.8. *The map*

$$\pi^n a \mapsto a + \pi\mathcal{O} \colon \pi^n\mathcal{O} \to \mathcal{O}/\pi\mathcal{O} = \boldsymbol{k}$$

is a surjective group morphism from the additive abelian group $\pi^n\mathcal{O}$ to the additive abelian group $\boldsymbol{k}$ with kernel $\pi^{n+1}\mathcal{O}$. Thus $\pi^n\mathcal{O}/\pi^{n+1}\mathcal{O} \cong \boldsymbol{k}$.

Corollary 5.9. *Suppose $\boldsymbol{k}$ is finite. Then $|\pi^n\mathcal{O}/\pi^{n+r}\mathcal{O}| = |\boldsymbol{k}|^r$ for each $r \in \mathbb{N}$.*

Now given a prime p and some n we let $R_{p^n} := \mathcal{O}/p^n\mathfrak{m}$ (so $R_{p^0} = \boldsymbol{k}$). In the same way as Corollary 5.6 gave rise to Lemma 5.7, from the previous corollary we obtain:

Lemma 5.10. *Suppose K is finitely ramified with finite residue field of characteristic p. Then for each n, the kernel of the natural surjective group morphism $R_{p^{n+1}} \to R_{p^n}$ is finite. (Hence R_{p^n} is finite for each n.)*

5.3. **NIP for RV_*.** *In this subsection K is henselian, and the structure $\boldsymbol{K}$ and its reduct RV_* are as introduced in Section 5.1.* We allow RV_* to be equipped with additional structure, and equip its expansion $\boldsymbol{K}$ with the corresponding additional structure. Recall that then, by part (2) of Fact 5.3 and the remark following it, RV_* is fully stably embedded in $\boldsymbol{K}$. As a warm-up to the proof of Proposition 6.1 below, we show a version of Fact 2.17:

Proposition 5.11. *Suppose $\boldsymbol{k}$ is finite or of characteristic zero. Then $\boldsymbol{K}$ is NIP if and only if K is finitely ramified and RV_* is NIP.*

Here the forward direction is obvious by Remark 5.2, Fact 2.18, and the fact that NIP is preserved under reducts. The proof of the converse relies on an analysis of indiscernible sequences in valued fields, with the distinction of cases similar to [15] or [10, Section 7.2]. (A similar case distinction is at the heart of the proof of Proposition 6.1.) Given a linearly ordered set I we let $I_\infty := I \cup \{\infty\}$ where ∞ is a new element, equipped with the extension of the ordering $\leq$ of I to the linear ordering on I_∞, also denoted by $\leq$, such that $i < \infty$ for all $i \in I$. Recall that I^* denotes the set I equipped with the reversed ordering $\geq$. In the two lemmas and their corollary below we let $(a_i)_{i\in I}$ be an indiscernible sequence of singletons of the field sort in $\boldsymbol{K}$ where I does not have a largest or smallest element. For the first lemma see [9]. (Also compare with Lemma 2.11 above.)

Lemma 5.12. *Exactly one of the following cases occurs:*

(1) *$v(a_i - a_j) < v(a_j - a_k)$ for all $i < j < k$ in I (we say that (a_i) is* pseudocauchy*);*

(2) $v(a_i - a_j) > v(a_j - a_k)$ *for all* $i < j < k$ *in* I *(so the sequence* $(a_i)_{i\in I^*}$ *is pseudocauchy); or*
(3) $v(a_i - a_j) = v(a_j - a_k)$ *for all* $i < j < k$ *in* I *(we refer to such a sequence* (a_i) *as a* fan*).*

Note that if $(a_i)_{i\in I}$ is pseudocauchy and $a_\infty \in K$ is such that $(a_i)_{i\in I_\infty}$ is indiscernible, then $(a_i)_{i\in I_\infty}$ is also pseudocauchy, and similarly with "fan" in place of "pseudocauchy".

Lemma 5.13. *Suppose* $(a_i)_{i\in I}$ *is pseudocauchy, and let* $a_\infty \in K$ *such that* $(a_i)_{i\in I_\infty}$ *is indiscernible. Then the sequence* $i \mapsto v(a_\infty - a_i)$ *is strictly increasing.*

Proof. Since $(a_i)_{i\in I_\infty}$ remains pseudocauchy, if $i < j$ are in I, then $v(a_j - a_i) < v(a_\infty - a_j)$ and so $v(a_\infty - a_i) = v\big(a_\infty - a_j + (a_j - a_i)\big) = v(a_j - a_i) < v(a_\infty - a_j)$. □

Corollary 5.14. *Suppose* K *is finitely ramified. Then with* $(a_i)_{i\in I}$ *and* a_∞ *as in Lemma 5.13,*

$$v(a_\infty - a_i) > v(a_\infty - a_j) + \delta \quad \text{for all } \delta \text{ and } i > j \text{ in } I. \tag{5.2}$$

Proof. Assume that we have some δ such that

$$v(a_\infty - a_i) \leq v(a_\infty - a_j) + \delta \quad \text{for some } i > j \text{ in } I;$$

then by δ-indiscernibility (as $\delta \in \operatorname{dcl}(\emptyset)$),

$$v(a_\infty - a_i) \leq v(a_\infty - a_j) + \delta \quad \text{for all } i > j \text{ in } I,$$

so for each j the interval $\big[v(a_\infty - a_j), v(a_\infty - a_j) + \delta\big]$ in Γ is infinite, contradicting finite ramification. □

Now suppose $\boldsymbol{k}$ is finite or of characteristic zero, K is finitely ramified, and RV_* is NIP. To show that $\boldsymbol{K}$ is NIP we may assume that $\boldsymbol{K}$ is a monster model of its theory. Suppose $\boldsymbol{K}$ is not NIP. Then there is an indiscernible sequence $(a_i)_{i\in\mathbb{Z}}$ of elements of the field sort of $\boldsymbol{K}$ and a definable $S \subseteq K$ such that $i \in \mathbb{Z}$ is even iff $a_i \in S$. By Fact 5.3 and the remark following it we may choose $b = (b_1, \dots, b_m) \in K^m$, some δ, as well as a definable subset D of RV_δ^m, such that for $a \in K$:

$$a \in S \qquad \Longleftrightarrow \qquad \big(\mathrm{rv}_\delta(a - b_1), \dots, \mathrm{rv}_\delta(a - b_m)\big) \in D.$$

By Lemma 5.12, one of the following three cases occurs.

Case 1: $(a_i)_{i\in\mathbb{Z}}$ *is pseudocauchy.* Using saturation take some $a_\infty \in K$ such that $(a_i)_{i\in\mathbb{Z}_\infty}$ is indiscernible. Let i, j range over $\mathbb{Z}$, and let $k \in \{1, \dots, m\}$. Suppose first that $v(b_k - a_\infty) > v(a_\infty - a_j)$ for all j. Using (5.2) we then obtain $v(b_k - a_\infty) > v(a_\infty - a_j) + \delta$ and hence $\mathrm{rv}_\delta(b_k - a_j) = \mathrm{rv}_\delta(a_\infty - a_j)$, for all j. Now suppose $v(b_k - a_\infty) \leq v(a_\infty - a_j)$ for some j; then $v(b_k - a_\infty) + \delta < v(a_\infty - a_i)$ for all $i > j$, and hence $\mathrm{rv}_\delta(b_k - a_i) = \mathrm{rv}_\delta(b_k - a_\infty)$ for $i > j$. Permuting the components of b, we can thus arrange that we have some $l \in \{1, \dots, m+1\}$ and some j such that for $i > j$ and $k = 1, \dots, m$ we have

$$\mathrm{rv}_\delta(b_k - a_i) = \begin{cases} \mathrm{rv}_\delta(a_\infty - a_i) & \text{if } k < l \\ \mathrm{rv}_\delta(b_k - a_\infty) & \text{otherwise.} \end{cases}$$

Put $r_i := \mathrm{rv}_\delta(a_i - a_\infty)$ for $i > j$ and $s_k := \mathrm{rv}_\delta(a_\infty - b_k)$ for $k = l, \dots, m$. The sequence $(r_i)_{i>j}$ is indiscernible, and for $i > j$ we have

$$(r_i, \dots, r_i, s_l, \dots, s_m) \in D \qquad \Longleftrightarrow \qquad i \text{ is even},$$

in contradiction with RV_* being NIP.

Case 2: $(a_i)_{i\in\mathbb{Z}^*}$ *is pseudocauchy.* Then we apply Case 1 to the sequence $(a_{-i})_{i\in\mathbb{Z}}$ in place of $(a_i)_{i\in\mathbb{Z}}$.

Case 3: $(a_i)_{i\in\mathbb{Z}}$ *is a fan.* Note that then $\boldsymbol{k}$ necessarily is infinite, hence $\operatorname{char} \boldsymbol{k} = 0$ by hypothesis, so $\delta = 0$. Let i, j range over $\mathbb{Z}$ and k over $\{1, \dots, m\}$, and let γ be the common value of $v(a_i - a_j)$ for all $i \neq j$. Let $c \in K$ and j be given; if $\gamma < v(c - a_j)$, then $\gamma = v(c - a_i)$ for all $i \neq j$, whereas if $\gamma > v(c - a_j)$ then $v(c - a_i) = v(c - a_j) < \gamma$ for each $i \neq j$. Hence we can choose an even j

such that for each k we either have $\gamma > v(b_k - a_i)$ for all $i \geq j$ or $\gamma = v(b_k - a_i)$ for all $i \geq j$. Now if $\gamma > v(b_k - a_j)$, then $\operatorname{rv}(b_k - a_i) = \operatorname{rv}(b_k - a_j)$ for $i > j$, whereas if $\gamma = v(b_k - a_j)$, then $\operatorname{rv}(b_k - a_i) = \operatorname{rv}(b_k - a_j) \oplus \operatorname{rv}(a_j - a_i)$ for $i > j$. Hence by reindexing the components of b we can arrange that we have some $l \in \{1, \dots, m+1\}$ such that with $r_i := \operatorname{rv}(a_i - a_j)$ for $i > 0$ and $s_k := \operatorname{rv}(a_j - b_k)$ for $k = 1, \dots, m$, for $i > j$ and $k = 1, \dots, m$:

$$\operatorname{rv}(a_i - b_k) = \begin{cases} r_i \oplus s_k & \text{if } k < l \\ s_k & \text{otherwise.} \end{cases}$$

The sequence $(r_i)_{i>j}$ is indiscernible, and for $i > j$ we have

$$(r_i \oplus s_1, \dots, r_i \oplus s_{l-1}, s_l, \dots, s_m) \in D \qquad \Longleftrightarrow \qquad i \text{ is even,}$$

in contradiction with RV_* being NIP. □

5.4. **A quantifier elimination in equicharacteristic zero.** We use the quantifier elimination result for pure short exact sequences from Section 4 to prove a variant of the QE result of Flenner, already used earlier, in the equicharacteristic zero case. As above we extend the valuation $v\colon K^\times \to \Gamma$ to a monoid morphism $K \to \Gamma_\infty$, also denoted by v, with $v(0) = \infty$. Recall that $\Gamma_\infty = \Gamma \cup \{\infty\}$ where $\gamma < \infty$ for all $\gamma \in \Gamma$ and $\gamma + \infty = \infty + \gamma = \infty$ for all $\gamma \in \Gamma_\infty$. We also extend the residue morphism

$$a \mapsto \operatorname{res}(a) := a + \mathfrak{m} \colon \mathcal{O} \to \boldsymbol{k} = \mathcal{O}/\mathfrak{m}$$

to K by setting $\operatorname{res}(a) := 0$ for $a \in K \setminus \mathcal{O}$. *In the rest of this subsection $\boldsymbol{k}$ has characteristic zero.*

We consider K as a three-sorted structure with sorts $\boldsymbol{k}$, K, Γ_∞ in the language

$$\mathcal{L}_{\mathrm{rkg}} = \mathcal{L}_{\mathrm{r}} \cup \mathcal{L}_{\mathrm{k}} \cup \mathcal{L}_{\mathrm{g}} \cup \{v, \operatorname{res}\}$$

where

$$\mathcal{L}_{\mathrm{r}} = \{0_{\mathrm{r}}, 1_{\mathrm{r}}, +_{\mathrm{r}}, -_{\mathrm{r}}, \cdot_{\mathrm{r}}\}, \quad \mathcal{L}_{\mathrm{k}} = \{0_{\mathrm{k}}, 1_{\mathrm{k}}, +_{\mathrm{k}}, -_{\mathrm{k}}, \cdot_{\mathrm{k}}\}, \quad \mathcal{L}_{\mathrm{g}} = \{0_{\mathrm{g}}, +_{\mathrm{g}}, <, \infty\}.$$

For our quantifier elimination result we expand $(\boldsymbol{k}, \Gamma_\infty)$ by a new sort $\boldsymbol{k}/(\boldsymbol{k}^\times)^n$ for every $n \geq 2$, together with the natural surjections $\pi^n\colon \boldsymbol{k} \to \boldsymbol{k}/(\boldsymbol{k}^\times)^n$. Let

$$\mathcal{L}_{\mathrm{rgq}} = \mathcal{L}_{\mathrm{r}} \cup \mathcal{L}_{\mathrm{g}} \cup \{\pi^2, \pi^3, \dots\}$$

be the language of this expansion.

Define, for every n, a map $\operatorname{res}^n\colon K \to \boldsymbol{k}/(\boldsymbol{k}^\times)^n$ in the following way: If $v(a) \notin n\Gamma$, set $\operatorname{res}^n(a) := 0$. Otherwise, let b be any element of K with $nv(b) = v(a)$ and set $\operatorname{res}^n(a) := \pi^n \operatorname{res}(a \cdot b^{-n})$. This does not depend on the choice of b since $nv(c) = v(a)$ implies that $b \cdot c^{-1}$ has value 0, so is a unit in $\mathcal{O}$ and $\operatorname{res}(a \cdot c^{-n}) = \operatorname{res}(a \cdot b^{-n}) \cdot \operatorname{res}(b \cdot c^{-1})^n$. One verifies easily that the restriction of res^n to $v^{-1}(n\Gamma)$ is a group morphism $v^{-1}(n\Gamma) \to \boldsymbol{k}^\times/(\boldsymbol{k}^\times)^n$. We identify $\boldsymbol{k}$ with $\boldsymbol{k}/(\boldsymbol{k}^\times)^0$ in the natural way, so $\operatorname{res} = \operatorname{res}^0$. We also extend the multiplicative inverse function $a \mapsto a^{-1}\colon K^\times \to K^\times$ to a function $K \to K$ by setting $0^{-1} := 0$, and let

$$\mathcal{L}_{\mathrm{rkgq}} := \mathcal{L}_{\mathrm{rkg}} \cup \{\,^{-1}, \pi^2, \pi^3, \dots, \operatorname{res}^2, \operatorname{res}^3, \dots\}.$$

Let the multivariables x_{r}, x_{k}, x_{g} be of sort $\boldsymbol{k}$, K, and Γ_∞, respectively. We call $\mathcal{L}_{\mathrm{rkgq}}$-terms of the form $v\big(p(x_{\mathrm{k}})\big)$, $\operatorname{res}\big(p(x_{\mathrm{k}})q(x_{\mathrm{k}})^{-1}\big)$ or $\operatorname{res}^n\big(p(x_{\mathrm{k}})\big)$ (where $n \geq 2$), for polynomials p, q with integer coefficients, *special*. We have the following analogue of Theorem 4.2:

Theorem 5.15. *In the theory of henselian valued fields with residue field of characteristic zero, viewed as $\mathcal{L}_{\mathrm{rkgq}}$-structures in the natural way, every $\mathcal{L}_{\mathrm{rkg}}$-formula $\phi(x_{\mathrm{r}}, x_{\mathrm{k}}, x_{\mathrm{g}})$ is equivalent to a formula*

$$\phi_{\mathrm{rgq}}\big(x_{\mathrm{r}}, \sigma_1(x_{\mathrm{k}}), \dots, \sigma_m(x_{\mathrm{k}}), x_{\mathrm{g}}\big)$$

where the σ_i are special terms and ϕ_{rgq} is a suitable $\mathcal{L}_{\mathrm{rgq}}$-formula.

In the proof we make use of Flenner's quantifier elimination theorem, already stated in Section 5.1 above. For convenience let us slightly paraphrase this result, in the case of equicharacteristic zero. Recall that in this case the structure RV_* has a single new (interpretable) sort

$$\mathrm{RV} = K/(1+\mathfrak{m}),$$

which comes equipped with the binary operation $\cdot_{\mathrm{rv}}$ which gives RV the structure of an abelian monoid and makes the natural projection $\mathrm{rv}\colon K \to \mathrm{RV}$ a monoid morphism. Note that $0_{\mathrm{RV}} := \mathrm{rv}(0)$ is an absorbing element of RV and $\mathrm{RV}^\times := \mathrm{RV} \setminus \{0_{\mathrm{RV}}\} = K^\times/(1+\mathfrak{m})$ is a group. The projection rv and the valuation $v\colon K \to \Gamma_\infty$ also induce morphisms $\iota\colon \boldsymbol{k} \to \mathrm{RV}$ and $\nu\colon \mathrm{RV} \to \Gamma_\infty$ of abelian monoids, which give rise to a pure short exact sequence

$$1 \to \boldsymbol{k}^\times \to \mathrm{RV}^\times \to \Gamma \to 0 \tag{5.3}$$

of abelian groups. Let

$$\mathcal{L}_{\mathrm{rv}} = \mathcal{L}_{\mathrm{r}} \cup \mathcal{L}_{\mathrm{g}} \cup \{\cdot_{\mathrm{rv}}, \iota, \nu\}$$

be the language of the structure $(\boldsymbol{k}, \mathrm{RV}, \Gamma_\infty)$, and let

$$\mathcal{L}_{\mathrm{rkg,rv}} := \mathcal{L}_{\mathrm{rkg}} \cup \{\mathrm{rv}, \cdot_{\mathrm{rv}}, \iota, \nu\} = \mathcal{L}_{\mathrm{r}} \cup \mathcal{L}_{\mathrm{k}} \cup \mathcal{L}_{\mathrm{g}} \cup \{v, \mathrm{res}, \mathrm{rv}, \cdot_{\mathrm{rv}}, \iota, \nu\}.$$

Now Flenner's result [32, Proposition 4.3] is:

Fact 5.16. *In the theory of henselian valued fields with residue field of characteristic zero, formulated in the language $\mathcal{L}_{\mathrm{rkg,rv}}$, every $\mathcal{L}_{\mathrm{rkg}}$-formula $\phi(x_{\mathrm{r}}, x_{\mathrm{k}}, x_{\mathrm{g}})$ is equivalent to a formula*

$$\phi_{\mathrm{rv}}\big(x_{\mathrm{r}}, \mathrm{rv}(q_1(x_{\mathrm{k}})), \dots, \mathrm{rv}(q_k(x_{\mathrm{k}})), x_{\mathrm{g}}\big)$$

where the q_i are polynomials with integer coefficients and ϕ_{rv} is a suitable $\mathcal{L}_{\mathrm{rv}}$-formula.

Actually, Flenner's result is a bit stronger, allowing variables ranging over the RV-sort; moreover, Fact 5.16 also works for arbitrary expansions of the $\mathcal{L}_{\mathrm{rv}}$-structure $(\boldsymbol{k}, \mathrm{RV}, \Gamma_\infty)$. (See the discussion preceding [32, Proposition 4.3].)

We now apply the material of Section 4.3 to the short exact sequence (5.3). Let $\phi(x_{\mathrm{r}}, x_{\mathrm{k}}, x_{\mathrm{g}})$ be an $\mathcal{L}_{\mathrm{rkg}}$-formula and take $q_1, \dots, q_k$ and ϕ_{rv} as in Fact 5.16. Corollary 4.8 and Remark 4.9 applied to ϕ_{rv} show that $\phi(x_{\mathrm{r}}, x_{\mathrm{k}}, x_{\mathrm{g}})$ is equivalent to a formula

$$\phi_{\mathrm{rgq}}\big(x_{\mathrm{r}}, \sigma_1(x_{\mathrm{k}}), \dots, \sigma_m(x_{\mathrm{k}}), x_{\mathrm{g}}\big)$$

where the σ_j are terms of the form

$$\rho_0\big(\mathrm{rv}(q_1(x_{\mathrm{k}}))^{e_1} \cdots \mathrm{rv}(q_k(x_{\mathrm{k}}))^{e_k}\big) \qquad (e_1, \dots, e_k \in \mathbb{Z})$$

or

$$\rho_n\big(\mathrm{rv}(q_1(x_{\mathrm{k}}))^{e_1} \cdots \mathrm{rv}(q_k(x_{\mathrm{k}}))^{e_k}\big) \qquad (e_1, \dots, e_k \in \mathbb{N},\ n \geq 2)$$

or

$$\nu\big(\mathrm{rv}(q_1(x_{\mathrm{k}}))^{e_1} \cdots \mathrm{rv}(q_k(x_{\mathrm{k}}))^{e_k}\big) \qquad (e_1, \dots, e_k \in \mathbb{N}),$$

and ϕ_{rgq} is a suitable $\mathcal{L}_{\mathrm{rgq}}$-formula. Here the maps $\rho_n\colon \mathrm{RV} \to \boldsymbol{k}/(\boldsymbol{k}^\times)^n$ are as defined in Section 4.3. Since rv is a monoid morphism, for each appropriate tuple a of the field sort and $e_1, \dots, e_k \in \mathbb{Z}$ we have

$$\mathrm{rv}(q_1(a))^{e_1} \cdots \mathrm{rv}(q_k(a))^{e_k} = \mathrm{rv}\big(p(a)q(a)^{-1}\big) \qquad \text{where } p = \prod_{e_j \geq 0} q_j^{e_j} \text{ and } q = \prod_{e_j < 0} q_j^{-e_j}.$$

We have $\nu \circ \mathrm{rv} = v$. Recall that ρ_n is identically zero outside $\nu^{-1}(n\Gamma)$, hence $\rho_n \circ \mathrm{rv} = \mathrm{res}^n$. Thus each term σ_j is special. This finishes the proof of Theorem 5.15. □

Remarks.

(1) Suppose K_{rgq} is equipped with additional structure, and we equip its expansion to an $\mathcal{L}_{\mathrm{rkgq}}$-structure with the corresponding additional structure. The theorem above then remains true in this setting; this is shown just as in Corollary 4.8. As a consequence, K_{rgq} is fully stably embedded in the $\mathcal{L}_{\mathrm{rkgq}}$-structure K, and the induced structure on K_{rgq} is the given one.

(2) Suppose now that $\boldsymbol{k}$ and Γ_∞ come equipped with additional structure, and the $\mathcal{L}_{\mathrm{rkgq}}$-structure K is expanded by these structures on its sorts $\boldsymbol{k}$ and Γ_∞; then the sorts $\boldsymbol{k}$, Γ_∞ are fully stably embedded in K, with the induced structure on these sorts just the given ones.

We finish this subsection with observing that the structure RV_* introduced in Section 5.1 is only ostensibly richer than the structure $(\boldsymbol{k}, \mathrm{RV}, \Gamma_\infty)$ of RV viewed as pure short exact sequence:

Lemma 5.17. *The $\mathcal{L}_{\mathrm{rv}}$-structure $(\boldsymbol{k}, \mathrm{RV}, \Gamma_\infty)$ and the $\mathcal{L}_{\mathrm{RV}_*}$-structure RV_* are bi-interpretable.*

To see this note that the relation $\oplus = \oplus_0$ on RV introduced in (5.1) is definable in $(\boldsymbol{k}, \mathrm{RV}, \Gamma_\infty)$: for $a, b, c \in \mathrm{RV}^\times$ we have

$$\begin{aligned} \oplus(a,b,c) \quad \Longleftrightarrow \quad & \Big[\nu(a) = \nu(b) \ \&\ \exists y \in \boldsymbol{k}\big(\iota(y) \cdot_{\mathrm{rv}} a = b \ \&\ \iota(1+y) \cdot_{\mathrm{rv}} a = c\big)\Big] \ \vee \\ & \Big[\nu(a) > \nu(b) \ \&\ b = c\Big] \ \vee \ \Big[\nu(b) > \nu(a) \ \&\ a = c\Big]. \end{aligned}$$

Conversely, Remark 5.1 shows that $\boldsymbol{k}$, Γ_∞ and the morphisms ι, ν are interpretable in RV_*. Note that in this lemma we may allow $\boldsymbol{k}$ and Γ_∞ to be equipped with additional structure, and RV_* with the corresponding structure, that is, by all relations $S \subseteq \mathrm{RV}^m$ where $S \subseteq (\ker v_{\mathrm{rv}})^m = (\boldsymbol{k}^\times)^m$ is definable in $\boldsymbol{k}$ or $S = v_{\mathrm{rv}}^{-1}(v_{\mathrm{rv}}(S))$ where $v_{\mathrm{rv}}(S) \subseteq \Gamma^m$ is definable in Γ_∞.

Corollary 5.18. *Suppose that $\boldsymbol{k}$ and Γ_∞ are equipped with additional structure; then K is NIP iff both $\boldsymbol{k}$ and Γ_∞ are NIP.*

Proof. By Proposition 5.11 and the remark preceding the corollary, K is NIP iff $(\boldsymbol{k}, \mathrm{RV}, \Gamma_\infty)$ is NIP, and by Lemma 4.5, $(\boldsymbol{k}, \mathrm{RV}, \Gamma_\infty)$ is NIP iff $\boldsymbol{k}$ and Γ_∞ are NIP. □

5.5. **A generalization.** In this subsection we put the QE result for weakly pure exact sequences from Section 4.4 to work by proving a version of Theorem 5.15 for henselian valued fields of characteristic zero with arbitrary residue field. Only Corollary 5.23 from this subsection is used later. Throughout this subsection we assume that K is henselian, and we let M, N range over $\mathbb{N}^{\geq 1}$.

Let R_N be the ring $\mathcal{O}/N\mathfrak{m}$, and extend the residue morphism

$$x \mapsto \mathrm{res}_N(x) := x + N\mathfrak{m} \colon \mathcal{O} \to R_N$$

to a map $K \to R_N$, also denoted by res_N, by setting $\mathrm{res}_N(x) := 0$ for $x \in K \setminus \mathcal{O}$. The valuation $v \colon K \to \Gamma_\infty$ induces a map $v_N \colon R_N \to \Gamma_\infty$ with

$$v_N(r) = \begin{cases} v(x) & \text{if } r = \mathrm{res}_N(x) \neq 0, \\ \infty & \text{if } r = 0. \end{cases}$$

Note that $0 \leq v_N(r) \leq v(N)$ for $r \in R_N$, $r \neq 0$. We have $R_1 = \boldsymbol{k}$. If $\operatorname{char} \boldsymbol{k} = 0$, then $R_N = \boldsymbol{k}$ and $v_N(R_N) = \{0, \infty\}$ for all N. If $\operatorname{char} \boldsymbol{k} = p > 0$, then $R_M = R_N$ if M and N are divisible by the same powers of p. If M is a multiple of N, let $\mathrm{res}^M_N \colon R_M \to R_N$ be the natural surjection; its kernel is

$$\mathrm{res}_M(N\mathfrak{m}) = \big\{r \in R_M : v_N(r) > v(N)\big\},$$

and $v_N\big(\mathrm{res}^M_N(r)\big) = v_M(r)$ for $r \in R_M$ with $\mathrm{res}^M_N(r) \neq 0$. Let

$$\mathcal{L}_{\mathrm{rng}} = \big\{+_N,\ \cdot_N\ , v_N, \mathrm{res}^M_N : N \text{ divides } M\big\} \cup \mathcal{L}_{\mathrm{g}}$$

be the language of the multi-sorted structure $K_{\mathrm{rng}} = (R_1, R_2, \dots, \Gamma_\infty)$. The pair consisting of the family of rings (R_N) and the family of morphisms $(\mathrm{res}^M_N)_{N|M}$ forms an inverse system; let $\varprojlim R_N$ denote its inverse limit. The morphisms $\mathrm{res}_N\colon \mathcal{O} \to R_N$ induce a ring morphism $\mathcal{O} \to \varprojlim R_N$ whose kernel is

$$\dot{\mathfrak{m}} := \bigcap_N N\mathfrak{m} = \big\{x \in K : v(x) > v(N) \text{ for every } N\big\},$$

and hence induces an embedding $\varphi\colon \mathcal{O}/\dot{\mathfrak{m}} \to \varprojlim R_N$. Clearly we have:

Lemma 5.19. *Suppose K_{rng} is $\aleph_1$-saturated; then φ is an isomorphism.*

We now consider K as a many-sorted structure $(K, R_1, R_2, \dots, \Gamma_\infty)$ in the language

$$\mathcal{L}_{\mathrm{rkng}} = \mathcal{L}_{\mathrm{k}} \cup \mathcal{L}_{\mathrm{rng}} \cup \{v, \mathrm{res}_1, \mathrm{res}_2, \dots\}.$$

Lemma 5.20. *Suppose n^2 divides N, and for $i = 1, 2$ let $x_i \in K$ with $v(x_i) + 2v(n) \leq v(N)$. Then with $r_i = \mathrm{res}_N(x_i)$, the following are equivalent:*

1. *$x_1 \cdot x_2^{-1} \in (K^\times)^n$;*
2. *$r_1 r^n = r_2$ or $r_1 = r_2 r^n$ for some $r \in R_N$.*

Proof. Suppose $x_1 \cdot x_2^{-1} \in (K^\times)^n$, and say $v(x_1) \leq v(x_2)$; then $x_1 z^n = x_2$ for some $z \in \mathcal{O}$, so $r_1 r^n = r_2$ for $r = \mathrm{res}_N(z)$. Conversely, suppose $r_1 r^n = r_2$ where $r \in R_N$. Take $y \in \mathcal{O}$ with $v(x_1 y^n - x_2) > v(N)$; then

$$v(x_1 x_2^{-1} y^n - 1) > v(N) - v(x_2) \geq 2v(n).$$

Hensel's Lemma (in the Newton formulation) applied to the polynomial $x_1 x_2^{-1} y^n - X^n \in \mathcal{O}[X]$ yields an $x \in K$ such that $x_1 x_2^{-1} y^n - x^n = 0$, so $x_1 x_2^{-1} \in (K^\times)^n$. □

From Lemma 5.20 we see that for N, n as in the lemma,

$$r_1 \sim^n_N r_2 \quad :\Longleftrightarrow \quad \exists s\ (r_1 s^n = r_2 \vee r_1 = r_2 s^n)$$

defines an equivalence relation on the subset

$$R^n_N = \big\{r \in R_N : v_N(r) + 2v(n) \leq v(N)\big\}$$

of R_N. For such N, n we introduce a new sort

$$S^n_N := (R^n_N/{\sim^n_N}) \cup \{0\}$$

together with the map $\pi^n_N\colon R_N \to S^n_N$ which agrees with the quotient map $R^n_N \to R^n_N/{\sim^n_N}$ on R^n_N and is 0 on $R_N \setminus R^n_N$. Let

$$\mathcal{L}_{\mathrm{rngq}} = \mathcal{L}_{\mathrm{rng}} \cup \{\pi^n_N : n^2 \text{ divides } N\}$$

be the language of the expansion $(K_{\mathrm{rng}}, S^n_N)$ of K_{rng}. Note that $(K_{\mathrm{rng}}, S^n_N)$ is interpretable in K_{rng}.

Finally, we define, for every n such that n^2 divides N, the following map $\mathrm{res}^n_N\colon K \to S^n_N$: If there is some $\gamma \in \Gamma$ such that $0 \leq v(x) - n\gamma \leq v(N) - 2v(n)$, choose $y \in K$ with $v(y) = \gamma$ and set

$$\mathrm{res}^n_N(x) = \pi^n_N\big(\,\mathrm{res}_N(x \cdot y^{-n})\big);$$

one verifies easily that this does not depend on the choice of γ and y. If there is no such γ, set $\mathrm{res}^n_N(x) := 0$. We view each henselian valued field of characteristic zero in the natural way as an $\mathcal{L}_{\mathrm{krngq}}$-structure where

$$\mathcal{L}_{\mathrm{rkngq}} := \mathcal{L}_{\mathrm{rkng}} \cup \{\mathrm{res}^n_N : n^2 \text{ divides } N\}.$$

Let the multivariables x_{r}, x_{k}, x_{g} be of sort $R_1, R_2, \ldots$, K, and Γ_∞, respectively. We call $\mathcal{L}_{\mathrm{rkngq}}$-terms of the form $v\big(p(x_{\mathrm{k}})\big)$, $\mathrm{res}^0_N\big(p(x_{\mathrm{k}})q(x_{\mathrm{k}})^{-1}\big)$ or $\mathrm{res}^n_N\big(p(x_{\mathrm{k}})\big)$ (where $n \geq 1$), for polynomials p, q with integer coefficients, *special*. We then have the following theorem.

Theorem 5.21. *In the $\mathcal{L}_{\mathrm{rkngq}}$-theory of characteristic zero henselian valued fields, every $\mathcal{L}_{\mathrm{rkng}}$-formula $\phi(x_{\mathrm{r}}, x_{\mathrm{k}}, x_{\mathrm{g}})$ is equivalent to a formula*

$$\phi_{\mathrm{rngq}}\big(x_{\mathrm{r}}, \sigma_1(x_{\mathrm{k}}), \ldots, \sigma_m(x_{\mathrm{k}}), x_{\mathrm{g}}\big)$$

where the σ_i are special terms and ϕ_{rngq} is a suitable $\mathcal{L}_{\mathrm{rngq}}$-formula.

For the proof of this theorem, suppose our valued field K (as always, of characteristic zero) is henselian. Let r, a, γ be finite tuples in K of the same sort as x_{r}, x_{k}, x_{g}, respectively. Let $\sigma_0, \sigma_1, \ldots$ list all special terms, and let $\sigma(a)$ denote the tuple $\sigma_0(a), \sigma_1(a), \ldots$. We have to show that the type of $\big(r, \sigma(a), \gamma\big)$ in K_{rng} determines the type of (r, a, γ) in the $\mathcal{L}_{\mathrm{rkngq}}$-structure K. For this we may assume that K is special of some suitable cardinality κ, e.g., $\kappa = \beth_\omega(\omega)$ (see [40, Theorem 10.4.2(c)]). The following claim is then clear (see [40, Theorems 10.4.4 and 10.4.5 (a)]:

Claim 1. *The type of $\big(r, \sigma(a), \gamma\big)$ in K_{rng} determines the isomorphism type of $\big(K_{\mathrm{rng}}, r, \sigma(a), \gamma\big)$.*

In the following we use the notation and terminology of [2, Section 3.4]. Let Δ be the smallest convex subgroup of Γ containing all $v(N)$. Let $\dot{v}\colon K^\times \to \dot{\Gamma} := \Gamma/\Delta$ be the coarsening of v by Δ, with residue field $\dot{K}$ of characteristic zero, and let $v\colon \dot{K}^\times \to \Delta$ be the corresponding specialization of v. The valuation ring of the valuation v on $\dot{K}$ is $\mathcal{O}_{\dot{K}} := \mathcal{O}/\dot{\mathfrak{m}}$, where

$$\dot{\mathfrak{m}} := \big\{x \in K : v(x) > v(N) \text{ for all } N\big\}$$

is the maximal ideal of the valuation ring

$$\dot{\mathcal{O}} := \big\{x \in K : v(x) > -v(N) \text{ for some } N\big\}$$

of $\dot{v}$, and the maximal ideal of $\mathcal{O}_{\dot{K}}$ is $\mathfrak{m}_{\dot{K}} := \mathfrak{m}/\dot{\mathfrak{m}}$. The valued field $\dot{K}$ is henselian [2, Lemma 3.4.2]. (In fact, even better: $\dot{K}$ is complete with archimedean value group; cf. the proof of Claim 2 below.) We view $\dot{K}$ as the two-sorted structure $\big(\dot{K}, \Gamma_\infty, v\big)$, with the ring structure on $\dot{K}$ and the ordered group structure on Γ, and the valuation $v\colon \dot{K}^\times \to \Delta \subseteq \Gamma$ extended to a map $\dot{K} \to \dot{\Gamma}_\infty$ as usual. The natural surjection $\mathcal{O} \to \mathcal{O}_{\dot{K}}$ induces an isomorphism

$$R_N = \mathcal{O}/N\mathfrak{m} \to \mathcal{O}_{\dot{K}}/N\mathfrak{m}_{\dot{K}} = (\mathcal{O}/\dot{\mathfrak{m}})/(N\mathfrak{m}/\dot{\mathfrak{m}}),$$

and we identify R_N with its image; note that then R_N is interpretable in $\dot{K}$, and we may view r as a tuple of elements in $\dot{K}^{\mathrm{eq}}$. The maps $\dot{\mathrm{res}}^n\colon K \to \dot{K}/(\dot{K}^\times)^n$ are defined as before Theorem 5.15, for the valuation $\dot{v}$ in place of v. Now let $\theta(a)$ be a sequence enumerating all terms of the form $\dot{\mathrm{res}}^n\big(q(a)\big)$ or $v\big(q(a)\big)$ for polynomials q with integer coefficients.

Claim 2. *The isomorphism type of $\big(K_{\mathrm{rng}}, r, \sigma(a), \gamma\big)$ determines that of $\big(\dot{K}, r, \theta(a), \gamma\big)$.*

Proof. By Lemma 5.19, since K_{rng} is $\aleph_1$-saturated, we have an isomorphism $\mathcal{O}_{\dot{K}} = \mathcal{O}/\dot{\mathfrak{m}} \xrightarrow{\cong} \varprojlim R_N$, and $\dot{K}$ is the fraction field of $\mathcal{O}_{\dot{K}}$. It remains to show that $\sigma(a)$ determines each value $\dot{\mathrm{res}}^n(b)$ where $b = q(a)$ for some polynomial q with integer coefficients. For this we may assume $\dot{v}(b) \in n\dot{\Gamma}$. Take $c \in K$ with $n\dot{v}(c) = \dot{v}(b)$, so $bc^{-n} \in \dot{\mathcal{O}}$; then with $y := \dot{\mathrm{res}}(bc^{-n}) \in \dot{K}^\times$ we have

$$\dot{\mathrm{res}}^n(b) = y \cdot (\dot{K}^\times)^n \in (\dot{K}^\times)/(\dot{K}^\times)^n,$$

where $\dot{\mathrm{res}}\colon \dot{\mathcal{O}} \to \dot{K}$ is the natural surjection. If necessary replacing b, c, y by their respective inverses, we can arrange that $0 \leq v(b) - nv(c) \leq v(M)$ for some M. Set $N := n^2M$; then $\mathrm{res}^n_N(b) \in S^n_N$ is the equivalence class of $\dot{\mathrm{res}}_N(y) \in R_N$; here $\dot{\mathrm{res}}_N\colon \mathcal{O}_{\dot{K}} \to R_N$ is the natural surjection. Now suppose $\sigma(a) = \sigma(a')$ where a' is a tuple in K of the same sort as a, and let $b' := q(a')$. Then $v(b) = v(b')$, so $n\dot{v}(c) = \dot{v}(b')$ and $0 \leq v(b') - nv(c) \leq v(M)$. Thus setting $y' := \dot{\mathrm{res}}(b'c^{-n})$, we have

$$\dot{\mathrm{res}}^n(b') = y' \cdot (\dot{K}^\times)^n \in (\dot{K}^\times)/(\dot{K}^\times)^n.$$

By hypothesis we have $\mathrm{res}^n_N(b) = \mathrm{res}^n_N(b')$ and hence $\dot{\mathrm{res}}_N(y) \sim^n_N \dot{\mathrm{res}}_N(y')$. By Lemma 5.20 applied to $\dot{K}$ in place of K we therefore obtain $y/y' \in (\dot{K}^\times)^n$ and thus $\dot{\mathrm{res}}^n(b) = \dot{\mathrm{res}}^n(b')$ as required. □

Let $\dot{\mathrm{RV}} := K/(1 + \dot{\mathfrak{m}})$ be the abelian monoid introduced in Section 5.4, with $\dot{v}$ in place of v, and let $\dot{\mathrm{rv}}\colon K \to \dot{\mathrm{RV}}$ be the natural surjection. Note that since $\dot{\mathfrak{m}} \subseteq \mathfrak{m}$, we have a natural surjective monoid morphism $\dot{\mathrm{RV}} \to \mathrm{RV} = K/(1 + \mathfrak{m})$, and we hence obtain a sequence

$$1 \to \dot{K}^\times \xrightarrow{\iota} \dot{\mathrm{RV}}^\times \xrightarrow{\nu} \Gamma \to 0 \tag{5.4}$$

of morphisms of abelian groups where ι is injective, ν is surjective, and $\ker \nu \subseteq \operatorname{im} \iota$; since $\Delta = \operatorname{im}(\nu \circ \iota)$ and Γ/Δ are both torsion-free, this sequence is weakly pure exact, by Lemma 4.11. We consider now the structure $(\dot{K}, \dot{\mathrm{RV}}, \Gamma_\infty)$ in the three-sorted language $\mathcal{L}_{\mathrm{rv}}$ (see Section 5.4), which comprises of the field $\dot{K}$, the abelian monoid structures on Γ_∞ and $\dot{\mathrm{RV}}$, and the maps ι, ν. Let $\tau(a)$ be an enumeration of all terms $\dot{\mathrm{rv}}\big(q(a)\big)$, where q ranges over polynomials with integer coefficients.

Claim 3. *The type of $\big(r, \theta(a), \gamma\big)$ in $\dot{K}$ determines the type of $\big(r, \tau(a), \gamma\big)$ in $(\dot{K}, \dot{\mathrm{RV}}, \Gamma_\infty)$.*

Proof. This follows from Theorem 4.12 applied to the weakly pure exact sequence (5.4) as in the proof of Theorem 5.15. □

Claim 4. *The type of $\big(r, \tau(a), \gamma\big)$ in $(\dot{K}, \dot{\mathrm{RV}}, \Gamma_\infty)$ determines the type of (r, a, γ) in the $\mathcal{L}_{\mathrm{rkngq}}$-structure K.*

Proof. This follows from Flenner's QE (Fact 5.16). To see this, let $(\dot{K}, \dot{\mathrm{RV}}, \dot{\Gamma}_\infty)$ be the $\mathcal{L}_{\mathrm{rv}}$-structure associated to the Δ-coarsening of the valued field K, as in Section 5.4: that is, $(\dot{K}, \dot{\mathrm{RV}}, \dot{\Gamma}_\infty)$ consists of the field $\dot{K}$, the abelian monoids $\dot{\Gamma}_\infty$, $\dot{\mathrm{RV}}$, the map $\iota\colon \dot{K} \to \dot{\mathrm{RV}}$ from above, and the composition $\dot{\nu}\colon \dot{\mathrm{RV}} \to \dot{\Gamma}_\infty$ of ν with the natural surjection $\pi\colon \Gamma_\infty \to \dot{\Gamma}_\infty$. Expand this structure by a sort for Γ_∞ as well as the primitives ν, π. Note that $\dot{\Gamma} = \Gamma/\nu(\iota(\dot{K}^\times))$ and $\dot{\nu} = \pi \circ \nu$. Hence the type of $\big(r, \tau(a), \gamma\big)$ in $(\dot{K}, \dot{\mathrm{RV}}, \Gamma_\infty)$ determines the type of $\big(r, \tau(a), \gamma\big)$ in this expanded structure $(\dot{K}, \dot{\mathrm{RV}}, \dot{\Gamma}_\infty)$. Now by Fact 5.16 and the remark following it, the type of $\big(r, \tau(a), \gamma\big)$ in $(\dot{K}, \dot{\mathrm{RV}}, \dot{\Gamma}_\infty)$ implies the type of (r, a, γ) in the Δ-coarsening of K, viewed as $\mathcal{L}_{\mathrm{rkg,rv}}$-structure in the natural way, and expanded by a sort for Γ_∞ and the primitives ν, π. This $\mathcal{L}_{\mathrm{rkg,rv}}$-structure defines the valuation v on K (as $v = \nu \circ \dot{\mathrm{rv}}$), and hence interprets K viewed as $\mathcal{L}_{\mathrm{rkngq}}$-structure. This yields the claim. □

The combinations of the four claims above completes the proof of Theorem 5.21.

Remark 5.22. Theorem 5.21 implies a quantifier elimination result for arbitrary expansions of $\mathcal{L}_{\mathrm{rngq}}$ just as in Corollary 4.8.

In the following corollary we assume that Γ_∞ comes equipped with additional structure, and the $\mathcal{L}_{\mathrm{krng}}$-structure K is expanded by this structure on its sort Γ_∞; by Remark 5.22, Γ_∞ is then stably embedded in K, and the structure induced on Γ_∞ is the given one.

Corollary 5.23. *Suppose $\boldsymbol{k}$ is finite. Then K is NIP iff K is finitely ramified and Γ_∞ is NIP.*

Proof. The forward direction is clear by earlier results. For the converse, suppose K is finitely ramified but not NIP. We may assume that K is a monster model of its theory. Then there is an indiscernible sequence $(a_i)_{i\in\mathbb{N}}$ of elements of the field sort and a definable subset $S \subseteq K$ such that for all i, we have $a_i \in S$ iff i is even. By Theorem 5.21 there are special terms $\sigma_1(x_k), \dots, \sigma_m(x_k)$ and a suitable $\mathcal{L}_{\text{rngq}}$-formula ψ (possibly involving parameters) such that for $a \in K$:

$$a \in S \iff K \models \psi\big(\sigma_1(a), \dots, \sigma_m(a)\big).$$

In particular,

$$K \models \psi\big(\sigma_1(a_i), \dots, \sigma_m(a_i)\big) \iff i \text{ is even.}$$

Since $\boldsymbol{k}$ is finite, so are R_N and hence all S_N^n, by Lemma 5.10. Hence after modifying ψ and the σ_j suitably, we can assume that each σ_j has the form $\sigma_j(x_k) = v\big(q_j(x_k)\big)$ for some polynomial $q_j(x_k)$ with integer coefficients. From [63, Lemma A.18] we obtain $\gamma_1, \dots, \gamma_m \in \Gamma$, $r_1, \dots, r_m \in \mathbb{N}$, and an indiscernible sequence (α_i) of elements of Γ such that

$$v\big(q_j(a_i)\big) = \gamma_j + r_j\alpha_i \qquad \text{for sufficiently large } i.$$

With x_g a variable of sort Γ_∞ and

$$\psi_g(x_g) := \psi\big(\gamma_1 + r_1 x_g, \dots, \gamma_m + r_m x_g\big),$$

for sufficiently large i we then have

$$K \models \psi_g(\alpha_i) \iff i \text{ is even,}$$

showing that Γ_∞ has IP. □

6. Distality in Henselian Valued Fields

The main aim of this section is to prove the theorem stated in the introduction. In Section 6.3 we consider when naming a henselian valuation on a distal field preserves distality. After some valuation-theoretic preliminaries in Section 6.4, we investigate the structure of fields with a distal expansions in Section 6.5. Using work of Johnson [46], we obtain some consequences in the dp-minimal case in Section 6.6.

6.1. **Reduction to** RV_***.** *In this subsection K is a henselian valued field of characteristic zero, and the structure $\boldsymbol{K}$ and its reduct RV_* are as introduced in Section 5.1, where RV_* may carry additional structure.* The aim of the present subsection is to prove the following:

Proposition 6.1. *$\boldsymbol{K}$ is distal if and only if K is finitely ramified and RV_* is distal.*

The "only if" part is straightforward by Lemma 1.15, full stable embeddedness of RV_* in $\boldsymbol{K}$ (see Fact 5.3(2)), and Corollary 2.19. In the rest of this subsection we assume that K is finitely ramified and RV_* is distal, and show that then $\boldsymbol{K}$ is also distal. We may assume that $\boldsymbol{K}$ is a monster model of $\text{Th}(\boldsymbol{K})$. Note that $\boldsymbol{K}$ is automatically NIP by Fact 2.1 and Proposition 5.11. Suppose towards a contradiction that $\boldsymbol{K}$ is not distal. By Corollary 1.11 there are an indiscernible sequence $(a_i)_{i\in\mathbb{Q}}$ with $a_i \in K$ and finite tuples $b = (b_1, \dots, b_n)$ in K and c in RV_*, as well as a formula $\phi(x, b, c)$, such that $(a_i)_{i\in\mathbb{Q}^{\neq}}$ is bc-indiscernible, where $\mathbb{Q}^{\neq} := \mathbb{Q} \setminus \{0\}$, but $\models \phi(a_i, b, c)$ iff $i \neq 0$. By Fact 5.3 and Remark 5.4(1), $\phi(x, b, c)$ is equivalent to a formula of the form

$$\psi\big(\text{rv}_\delta(x - b'_1), \dots, \text{rv}_\delta(x - b'_m), c'\big) \tag{6.1}$$

for some δ, some m and $b' = (b'_1, \dots, b'_m) \in K^m$, some tuple c' from RV_*, and an $\mathcal{L}_{\text{RV}_*}$-formula ψ, where in addition $b'_1, \dots, b'_m, c' \in \text{acl}(bc)$. In particular, $(a_i)_{i\in\mathbb{Q}^{\neq}}$ is $b'c'$-indiscernible, hence after replacing our original formula with this new one, we can assume that $\phi(x, b, c)$ itself is of the form (6.1) with $b = b'$. So for $i \in \mathbb{Q}$:

$$\models \psi\big(\text{rv}_\delta(a_i - b_1), \dots, \text{rv}_\delta(a_i - b_n), c\big) \iff i \neq 0. \tag{6.2}$$

As the structure induced on RV_* is distal by Fact 5.3 and $\boldsymbol{K}$ is NIP, Proposition 1.17 implies that $(a_i)_{i\in\mathbb{Q}}$ is c-indiscernible. By Lemma 5.12, the following three cases exhaust all the possibilities.

Case 1: *$(a_i)_{i\in\mathbb{Q}}$ is pseudocauchy.* Take $a_\infty \in K$ such that $(a_i)_{i\in\mathbb{Q}_\infty^{\neq}}$ is bc-indiscernible and $(a_i)_{i\in\mathbb{Q}_\infty}$ is c-indiscernible. (Such an a_∞ exists by assumption and saturation.) Then the sequence $\big(v(a_\infty - a_i)\big)_{i\in\mathbb{Q}}$ is strictly increasing. Now for each $k \in \{1,\dots,n\}$, one of the following must occur.

(a) *$v(b_k - a_\infty) > v(a_\infty - a_i)$ for all $i \in \mathbb{Q}$.* As the sequence $(a_i)_{i\in\mathbb{Q}}$ is endless, in view of (5.2) we then have $v(b_k - a_\infty) > v(a_\infty - a_i) + \delta$ and hence $\mathrm{rv}_\delta(b_k - a_i) = \mathrm{rv}_\delta(a_\infty - a_i)$, for all $i \in \mathbb{Q}$.
(b) *$v(b_k - a_\infty) < v(a_\infty - a_i)$ for all $i \in \mathbb{Q}$.* As in (a), this implies that $v(b_k - a_\infty) + \delta < v(a_\infty - a_i)$ and hence $\mathrm{rv}_\delta(b_k - a_i) = \mathrm{rv}_\delta(b_k - a_\infty)$, for all $i \in \mathbb{Q}$.
(c) *There are $i > j$ in $\mathbb{Q}$ such that $v(a_\infty - a_i) \geq v(b_k - a_\infty) \geq v(a_\infty - a_j)$.* After increasing i or decreasing j if necessary we can assume that $i, j \neq 0$. As the relation $v(x) \leq v(y)$ is $\emptyset$-definable, we obtain a contradiction with b_r-indiscernibility of $(a_i)_{i\in\mathbb{Q}_\infty^{\neq}}$.

Permuting the components of b, we can thus assume that we have some $l \in \{1,\dots,n+1\}$ such that for each $i \in \mathbb{Q}$ and $k = 1,\dots,n$ we have

$$\mathrm{rv}_\delta(a_i - b_k) = \begin{cases} \mathrm{rv}_\delta(a_i - a_\infty) & \text{if } k < l \\ \mathrm{rv}_\delta(a_\infty - b_k) & \text{otherwise.} \end{cases}$$

Set $r_i := \mathrm{rv}_\delta(a_i - a_\infty)$ for $i \in \mathbb{Q}$ as well as $s_k := \mathrm{rv}_\delta(a_\infty - b_k)$ for $k = l,\dots,n$, and $r := (r_l,\dots,r_n)$. Now the sequence $(r_i)_{i\in\mathbb{Q}}$ is indiscernible, and $(r_i)_{i\in\mathbb{Q}^{\neq}}$ is sc-indiscernible (as $(a_i)_{i\in\mathbb{Q}_\infty^{\neq}}$ is bc-indiscernible). As RV_* is distal, by Proposition 1.10 this implies that $(r_i)_{i\in\mathbb{Q}}$ is also sc-indiscernible. But then

$$\begin{aligned} \models \psi\big(\,\mathrm{rv}_\delta\,(a_1 - b_1)\,,\dots,\mathrm{rv}_\delta\,(a_1 - b_n)\,,c\big) \quad &\iff \quad \models \psi(r_1,\dots,r_1,s,c) \\ &\iff \quad \models \psi(r_0,\dots,r_0,s,c) \\ &\iff \quad \models \psi\big(\,\mathrm{rv}_\delta(a_0 - b_1),\dots,\mathrm{rv}_\delta(a_0 - b_n),c\big), \end{aligned}$$

contradicting (6.2).

Case 2: *$(a_i)_{i\in\mathbb{Q}^*}$ is pseudocauchy.* Then we apply Case 1 to the sequence $(a_{-i})_{i\in\mathbb{Q}}$ in place of $(a_i)_{i\in\mathbb{Q}}$.

Case 3: *$(a_i)_{i\in\mathbb{Q}}$ is a fan.* Note again that then $\boldsymbol{k}$ is infinite, hence $\operatorname{char}\boldsymbol{k} = 0$ by Fact 2.1, and thus $\delta = 0$. Take some a_∞ as in Case 1, and let γ be the common value of $v(a_i - a_j)$ for all $i \neq j$ in $\mathbb{Q}_\infty$. Let $k \in \{1,\dots,n\}$; then one of the following must occur.

(a) *$v(b_k - a_\infty) < \gamma$.* Then $\mathrm{rv}(b_k - a_i) = \mathrm{rv}(b_k - a_\infty)$ for all $i \in \mathbb{Q}$.
(b) *$v(b_k - a_i) > \gamma$ for some $i \in \mathbb{Q}^{\neq}$.* Then $v(b_k - a_i) > v(a_\infty - a_i)$ and $v(b_k - a_j) \leq v(a_\infty - a_j)$ for each $j \in \mathbb{Q} \setminus \{0, i\}$, contradicting b_k-indiscernibility of $(a_i)_{i\in\mathbb{Q}_\infty^{\neq}}$.
(c) *$v(b_k - a_0) > \gamma$.* Then $\mathrm{rv}(a_0 - a_\infty) = \mathrm{rv}(b_k - a_\infty)$. Note that the sequence $\big(\mathrm{rv}(a_i - a_\infty)\big)_{i\in\mathbb{Q}}$ is indiscernible, and hence not totally indiscernible, by distality and stable embeddedness of RV_*. So $\big(\mathrm{rv}(a_i - a_\infty)\big)_{i\in\mathbb{Q}^{\neq}}$ is not indiscernible over $\mathrm{rv}(a_0 - a_\infty) = \mathrm{rv}(b_k - a_\infty)$ by Corollary 1.6. But this again contradicts the b_k-indiscernibility of $(a_i)_{i\in\mathbb{Q}_\infty^{\neq}}$.
(d) *$v(b_k - a_i) = \gamma$ for all $i \in \mathbb{Q}$.* Then $\mathrm{rv}(b_k - a_\infty) = \gamma$ and thus $\mathrm{rv}(b_k - a_i) = \mathrm{rv}(b_k - a_\infty) \oplus \mathrm{rv}(a_\infty - a_i)$ for all $i \in \mathbb{Q}$.

Reindexing the components of b, we can thus assume that we have some $l \in \{1,\dots,n+1\}$ such that for $i \in \mathbb{Q}$ and $k = 1,\dots,n$, with $r_i := \mathrm{rv}(a_i - a_\infty)$ and $s_k := \mathrm{rv}(a_\infty - b_k)$:

$$\mathrm{rv}(a_i - b_k) = \begin{cases} r_i \oplus s_k & \text{if } k < l \\ s_k & \text{otherwise.} \end{cases}$$

Let $s := (s_1, \dots, s_n)$. Then $(r_i)_{i \in \mathbb{Q}}$ is indiscernible and $(r_i)_{i \in \mathbb{Q}^{\neq}}$ is sc-indiscernible, since $(a_i)_{i \in \mathbb{Q}_\infty}$ is indiscernible and $(a_i)_{i \in \mathbb{Q}^{\neq}_\infty}$ is bc-indiscernible. Hence $(r_i)_{i \in \mathbb{Q}}$ is sc-indiscernible by Proposition 1.10, as RV_* is distal. But then

$$\begin{aligned} \models \psi\big(\mathrm{rv}(a_1 - b_1), \dots, \mathrm{rv}(a_1 - b_n), c\big) \quad &\Longleftrightarrow \quad \models \psi(r_1 \oplus s_1, \dots, r_1 \oplus s_{l-1}, s_l, \dots, s_n, c) \\ &\Longleftrightarrow \quad \models \psi(r_0 \oplus s_1, \dots, r_0 \oplus s_{l-1}, s_l, \dots, s_n, c) \\ &\Longleftrightarrow \quad \models \psi\big(\mathrm{rv}(a_0 - b_1), \dots, \mathrm{rv}(a_0 - b_n), c\big), \end{aligned}$$

contradicting (6.2). This finishes the proof of Proposition 6.1. □

6.2. **Reduction of distality from RV_* to $\boldsymbol{k}$ and Γ.** *In this subsection we assume that the structure on RV_* is obtained from structures on $\boldsymbol{k}$, Γ_∞ by expanding RV_* by all relations $S \subseteq \mathrm{RV}^m$ where $S \subseteq (\ker v_{\mathrm{rv}})^m = (\boldsymbol{k}^\times)^m$ is definable in $\boldsymbol{k}$ or $S = v_{\mathrm{rv}}^{-1}(v_{\mathrm{rv}}(S))$ and $v_{\mathrm{rv}}(S) \subseteq \Gamma^m$ is definable in Γ.* We then have:

Proposition 6.2. *Suppose K is finitely ramified. Then RV_* is distal if and only if both $\boldsymbol{k}$ and Γ are.*

For the proof, it is natural to distinguish two cases.

6.2.1. $\operatorname{char} \boldsymbol{k} > 0$. Here we may assume that $\boldsymbol{k}$ is finite, by Fact 2.1. The structure induced on Γ is the given one; see the remarks preceding Corollaries 5.23. The forward direction now follows from Lemma 1.15. For the converse, suppose Γ is distal; then Γ is NIP and hence so is the structure RV_* interpretable in K, by Corollary 5.23. By Lemma 5.7, the group morphisms $\mathrm{rv}_{\gamma \to 0} \colon \mathrm{RV}^\times_\gamma \to \mathrm{RV}^\times_0 = \mathrm{RV}^\times$ have finite fibers; moreover, since $v_{\mathrm{rv}} \colon \mathrm{RV}^\times \to \Gamma$ has kernel $\boldsymbol{k}^\times$, this group morphism also has finite fibers. Hence each element of RV_* is algebraic over Γ. As Γ is distal, applying Corollary 1.26 we conclude that RV_* is distal.

6.2.2. $\operatorname{char} \boldsymbol{k} = 0$. In this case, we note that RV_* is bi-interpretable with the pure short exact sequence $1 \to \boldsymbol{k}^\times \to \mathrm{RV}^\times \to \Gamma \to 0$, in the sense of Section 4.1, where $\boldsymbol{k}$, Γ carry the given additional structure. But then the conclusion holds by Theorem 4.6. □

Combining Propositions 6.1 and 6.2 with Remark 5.2 finishes the proof of the main theorem.

6.3. **When naming a henselian valuation preserves distality.** Let $(K, \mathcal{O})$ be a henselian valued field with residue field $\boldsymbol{k}$ and value group Γ. The following is [42, Theorem A]:

Fact 6.3. *If $\boldsymbol{k}$ is* not *separably closed, then $\mathcal{O}$ is definable in the Shelah expansion K^{Sh} of the field K.*

Together with Lemma 1.30 this immediately implies:

Corollary 6.4. *If the field K has a distal expansion and $\boldsymbol{k}$ is not separably closed, then the valued field $(K, \mathcal{O})$ has a distal expansion.*

Our main theorem allows us to treat the case of separably closed residue field:

Corollary 6.5. *Suppose $\boldsymbol{k}$ is separably closed. Then the valued field $(K, \mathcal{O})$ has a distal expansion if and only if Γ has a distal expansion and $\boldsymbol{k}$ has characteristic zero.*

Proof. Note that $\boldsymbol{k}$ is necessarily infinite, and if $\boldsymbol{k}$ has characteristic zero, then $\boldsymbol{k}$ is algebraically closed, hence has distal expansion: just add a predicate for a maximal proper subfield of $\boldsymbol{k}$. Now the claim follows from our main theorem. □

In view of Conjecture 3.16 we expect that in order for $(K, \mathcal{O})$ to have a distal expansion, we only need to require that $\boldsymbol{k}$ has a distal expansion. Before we turn to discussing our conjectural classification of fields with distal expansion, we recall some definitions and basic facts about canonical valuations.

6.4. **Canonical valuations.** *In this subsection we let K be a field.* We collect some notions and basic facts used in the next subsection. Let $\mathcal{O}_1$, $\mathcal{O}_2$ be valuation rings K. One says that $\mathcal{O}_2$ is *coarser* than $\mathcal{O}_1$, and that $\mathcal{O}_1$ is *finer* than $\mathcal{O}_2$, if $\mathcal{O}_1 \subseteq \mathcal{O}_2$, that is, if $\mathcal{O}_2$ is the valuation ring of a coarsening of $(K, \mathcal{O}_1)$.

Let now H be the set of henselian valuation rings of K, and let H_c be the subset of H consisting of those valuation rings with separably closed residue field. Then $H \setminus H_c$ is linearly ordered by inclusion. If $H_c \neq \emptyset$, then H_c contains a coarsest valuation ring $\mathcal{O}_c$ of K; this valuation ring is (strictly) finer than every valuation ring in $H \setminus H_c$. If $H_c = \emptyset$, then there is a finest henselian valuation ring of K, which we also denote by $\mathcal{O}_c$. We refer to [28, Theorem 4.4.2] for these facts. The valuation ring $\mathcal{O}_c$ is called the *canonical henselian valuation ring* of the field K.

Let now p be a prime. We denote by $K(p)$ the compositum of all finite normal field extensions $L|K$ of p-power degree. If $K(p) = K$, then K is called *p-closed.*

Lemma 6.6. *Suppose K is separably closed and $p \neq \operatorname{char} K$; then K is p-closed.*

Proof. If $\operatorname{char} K = 0$, then K is algebraically closed, and if $\operatorname{char} K = q > 0$ then the degree of each finite field extension of K is a power of q. □

Following [47, Section 9.5] we say that K is *p-corrupted* if no finite extension of K is p-closed; as a consequence of a theorem of Becker [4], one has (see [47, Lemma 9.5.2]):

Lemma 6.7 (Johnson)**.** *Every perfect field which is neither algebraically closed nor real closed has a finite p-corrupted extension.*

A valuation ring $\mathcal{O}$ of K is said to be *p-henselian* if only one valuation ring of $K(p)$ lies over $\mathcal{O}$. Let H_p be the set of p-henselian valuation rings of K, and let H^p_c be the subset of H^p consisting of those valuation rings with p-closed residue field. As before, $H^p \setminus H^p_c$ is linearly ordered by inclusion. If $H^p_c \neq \emptyset$, then H^p_c contains a coarsest valuation ring $\mathcal{O}^p_c$ of K, which is then finer than every valuation ring in $H^p \setminus H^p_c$. If $H^p_c = \emptyset$, then there is a finest p-henselian valuation ring of K, also denoted by $\mathcal{O}^p_c$. One calls $\mathcal{O}^p_c$ the *canonical p-henselian valuation ring* of K. See [44], which also contains a proof of the following fact:

Proposition 6.8 (Jahnke-Koenigsmann)**.** *If K is not orderable and contains all pth roots of unity, then $\mathcal{O}^p_c$ is $\emptyset$-definable in K.*

Here we recall that K is said to be *orderable* if there is an ordering on K making K an ordered field.

6.5. **Distal fields.** *In this subsection K is a field.* The following is commonly attributed to Shelah:

Conjecture 6.9. *If K is NIP, then K is finite, separably closed, real closed, or admits a non-trivial henselian valuation.*

This conjecture has numerous consequences; for example, by [38, Proposition 6.3], it implies that every NIP valued field is henselian. In [42, Theorem B] it is shown that if K is NIP and $\mathcal{O}$ is a henselian valuation ring of K, then the valued field $(K, \mathcal{O})$ is also NIP. Hence if Conjecture 6.9 holds, then every valuation ring on a NIP field is henselian, and its residue field is NIP. Moreover, under Conjecture 6.9, any two (externally) definable valuation rings on a NIP field are comparable [38, Corollary 5.4]. In Theorem 6.12 below we show that Conjecture 6.9 also gives rise to a classification of all fields admitting a distal expansion. We first note that the non-trivial henselian valuation stipulated in Conjecture 6.9 may be taken to be $\emptyset$-definable, by results in [42, 43] (see also [38, Corollary 7.6]):

Lemma 6.10. *Suppose Conjecture 6.9 holds, and suppose K is infinite and NIP; then K is separably closed or real closed, or K has an $\emptyset$-definable non-trivial henselian valuation ring.*

Proof. Suppose K is neither separably closed nor real closed; so according to Conjecture 6.9, K has a non-trivial henselian valuation. If K has such a valuation with residue field which is separably closed or real closed, then by [43, Theorem 3.10 and Corollary 3.11, respectively], there is an $\emptyset$-definable non-trivial henselian valuation ring of K. Hence we may assume that the residue field of each henselian valuation on K is not separably closed and not real closed. In particular, the residue field $\boldsymbol{k}$ of $\mathcal{O} := \mathcal{O}_{\mathrm{c}}$ is neither separably closed nor real closed. Hence $\mathcal{O}$ is the finest henselian valuation ring of K; in particular, $\boldsymbol{k}$ does not have a non-trivial henselian valuation. Now $\boldsymbol{k}$ is NIP, and so by Conjecture 6.9 applied to $\boldsymbol{k}$, this field is finite. Its absolute Galois group is non-universal, so $\mathcal{O}$ is $\emptyset$-definable by [43, Theorem 3.15 and Observation 3.16]. □

We also recall that every infinite field with a distal expansion has characteristic zero.

Corollary 6.11. *Suppose Conjecture 6.9 holds, and K is infinite and has a distal expansion. Then K is algebraically closed or real closed, or K has an $\emptyset$-definable non-trivial henselian valuation ring $\mathcal{O}$ whose residue field*

1. *is finite, or*
2. *is a field of characteristic zero with a distal expansion.*

Proof. Suppose K is neither algebraically closed nor real closed; then by Lemma 6.10 we can take an $\emptyset$-definable non-trivial henselian valuation ring $\mathcal{O}$ of K. Let $\boldsymbol{k}$ be the residue field of $\mathcal{O}$; then $\boldsymbol{k}$ also has a distal expansion by the forward direction in our main theorem; in particular, if $\operatorname{char} \boldsymbol{k} > 0$, then $\boldsymbol{k}$ is finite. □

In connection with option (1) in Corollary 6.11 recall that if $(K, \mathcal{O})$ is an infinite NIP henselian valued field of characteristic zero with finite residue field, then $(K, \mathcal{O})$ has a specialization which is p-adically closed of finite p-rank, for some prime p. (Remark 2.20.) We do not know whether we can upgrade (2) in this corollary to "is algebraically closed of characteristic zero, or real closed" (even while simultaneously weakening the condition that $\mathcal{O}$ be $\emptyset$-definable in K to $\mathcal{O}$ being externally definable, say). Instead we show:

Theorem 6.12. *Suppose Conjecture 6.9 holds, and K is NIP and does not define a valuation ring whose residue field is infinite of positive characteristic; then K has a henselian valuation ring, type-definable over the empty set, whose residue field is algebraically closed of characteristic zero, real closed, or finite.*

Before we give the proof of Theorem 6.12, we establish analogues of two results from [47] (9.5.4 and 9.5.7, respectively):

Lemma 6.13. *Suppose Conjecture 6.9 holds and K is NIP, non-orderable, and contains all p-th roots of unity, where p is a prime. Let $\mathcal{O} = \mathcal{O}_{\mathrm{c}}^{p}$ be the canonical p-henselian valuation ring of K; then $\mathcal{O}$ is $\emptyset$-definable, and its residue field is finite, has characteristic p, or is p-closed.*

Proof. Proposition 6.8 yields the $\emptyset$-definability of $\mathcal{O}$. Suppose the residue field $\boldsymbol{k}$ of $\mathcal{O}$ is infinite, has characteristic $\neq p$, and is not p-closed. Then by Lemma 6.6, $\boldsymbol{k}$ cannot be separably closed; since K is non-orderable, $\boldsymbol{k}$ is also not real closed. Hence by Conjecture 6.9 we may equip $\boldsymbol{k}$ with a non-trivial henselian valuation ring; let $\boldsymbol{k} \to \boldsymbol{k}_1$ be the corresponding place. Composition of the places $K \to \boldsymbol{k} \to \boldsymbol{k}_1$ then gives rise to a henselian valuation ring $\mathcal{O}_1$ of K with residue field $\boldsymbol{k}_1$ such that $\boldsymbol{k}$ is a specialization of $(K, \mathcal{O}_1)$, and then $\mathcal{O}_1$ is a strictly finer p-henselian valuation ring than $\mathcal{O}$, a contradiction. □

Lemma 6.14. *Suppose Conjecture 6.9 holds, and suppose K is infinite NIP, and the residue field of each $\emptyset$-definable valuation ring of K has characteristic zero. Let $\mathcal{O}_\infty$ be the intersection of all $\emptyset$-definable valuation rings of K. Then $\mathcal{O}_\infty$ is a valuation ring of K whose residue field is algebraically closed of characteristic zero or real closed.*

Proof. The hypothesis and the remarks following Conjecture 6.9 yield that the set of all valuation rings of K is linearly ordered by inclusion; in particular, $\mathcal{O}_\infty$ is a valuation ring of K. As in the proof of [47, Theorem 9.5.7(2)] one also sees that $\mathcal{O}_\infty$ equals the intersection of all definable valuation rings of K. Let $\boldsymbol{k}_\infty$ be the residue field of $\mathcal{O}_\infty$. We have $\operatorname{char} \boldsymbol{k}_\infty = 0$, since otherwise some $\emptyset$-definable valuation ring $\mathcal{O} \supseteq \mathcal{O}_\infty$ of K would have residue field $\boldsymbol{k}$ with $\operatorname{char} \boldsymbol{k} = \operatorname{char} \boldsymbol{k}_\infty > 0$ [47, Remark 9.5.6]. Towards a contradiction suppose that $\boldsymbol{k}_\infty$ is neither algebraically closed nor real closed. By Lemma 6.7 we then obtain a prime p and a finite p-corrupted extension $\boldsymbol{l}$ of $\boldsymbol{k}_\infty$. Let $v_\infty \colon K^\times \to \Gamma_\infty$ denote the valuation associated to $\mathcal{O}_\infty$. Choose a finite field extension L of K which contains all $4p$-th roots of unity and such that the residue field of the unique valuation w_∞ on L extending v_∞ contains $\boldsymbol{l}$, and hence is not p-closed. Lemma 6.13 yields a valuation w on L which is $\emptyset$-definable (that is, its valuation ring is $\emptyset$-definable in the field L) and not a coarsening of w_∞. Let v be the restriction of w to a valuation on K; then v is definable, hence a coarsening of v_∞, say $v = (v_\infty)_\Delta$ where Δ is a convex subgroup of Γ_∞. Let Δ_L be the convex hull of Δ in the value group of w_∞. The restriction of the Δ_L-coarsening $(w_\infty)_{\Delta_L}$ of w_∞ to K is v. But v is henselian, so $w = (w_\infty)_{\Delta_L}$ is a coarsening of w_∞, a contradiction. □

Now Theorem 6.12 follows easily: If K has an $\emptyset$-definable valuation ring with residue field of positive characteristic, then this residue field is finite by hypothesis, and we are done. Thus we may assume that the residue field of every $\emptyset$-definable valuation ring of K has characteristic zero. Then Lemma 6.14 yields a henselian valuation ring $\mathcal{O}_\infty$, type-definable over $\emptyset$, whose residue field is algebraically closed of characteristic zero, or real closed. □

Corollary 6.15. *Suppose Conjectures 3.16 and 6.9 hold, and K is NIP; then the following are equivalent:*

1. *K has a distal expansion;*
2. *K does not interpret an infinite field of positive characteristic;*
3. *K does not define a valuation ring whose residue field is infinite of positive characteristic;*
4. *K has a henselian valuation ring whose residue field is algebraically closed of characteristic zero, real closed, or finite.*

Proof. The implications (1) ⇒ (2) ⇒ (3) are clear (using Fact 2.1), and (3) ⇒ (4) follows from Theorem 6.12. To show (4) ⇒ (1), suppose K has characteristic zero. If $\mathcal{O}$ is a henselian valuation ring of K whose residue field $\boldsymbol{k}$ is algebraically closed of characteristic zero, real closed, or finite, then $\boldsymbol{k}$ has a distal expansion, and after choosing a distal expansion of the value group of $\mathcal{O}$, our main theorem yields that $(K, \mathcal{O})$ has a distal expansion, which is also a distal expansion of K. □

We also note a consequence of Theorem 6.12 for ordered fields. In [23], a field is defined to be *almost real closed* if it has a henselian valuation ring with real closed residue field.

Corollary 6.16. *Suppose Conjecture 6.9 holds, and K is orderable and has a distal expansion; then K is almost real closed.*

Proof. Equip K with an ordering making it an ordered field; it is well-known that then every henselian valuation ring of K is convex, and hence its residue field is orderable. Now use Theorem 6.12. □

Based on Theorem 6.12 we conjecture:

Conjecture 6.17. *Suppose K has a distal expansion; then K has a henselian valuation ring whose residue field is algebraically closed of characteristic zero, real closed, or finite.*

6.6. **Distality in the** dp**-minimal case.** In this subsection we show that for dp-minimal K, the conclusion of Corollary 6.15 holds even without assuming Conjectures 3.16 and 6.9; this relies again on work of Johnson [47]. We first recall a few facts about dp-minimal fields and related structures.

Fact 6.18.

(1) *Every* dp*-minimal expansion of an ordered abelian group is distal* (*by Fact 1.5*).
(2) *Every* dp*-minimal valued field is henselian* [46, 45].

Combining Fact 6.18 and the main theorem of this paper, we get:

Corollary 6.19. *A* dp*-minimal valued field is distal* (*has a distal expansion*) *if and only if its residue field is distal* (*respectively, has a distal expansion*).

A dp-minimal (pure) field can fail to admit a distal expansion only in the most obvious way:

Corollary 6.20. *Let K be an infinite* dp*-minimal field; then the following are equivalent:*

(1) *K has a distal expansion;*
(2) *K does not interpret an infinite field of positive characteristic;*
(3) *K does not define a valuation ring whose residue field is infinite of positive characteristic;*
(4) *K has a henselian valuation ring whose residue field is algebraically closed of characteristic zero, real closed, or finite.*

Proof. As in the proof of Corollary 6.15, the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are clear. For (3) $\Rightarrow$ (4), we argue as in the proof of the corresponding implication in Theorem 6.12: If K has an $\emptyset$-definable valuation ring with residue field of positive characteristic, then (4) holds. Otherwise, let $\mathcal{O}_\infty$ be the intersection of all $\emptyset$-definable valuation rings of K; by [47, Theorem 9.1.4], $\mathcal{O}_\infty$ is a henselian valuation ring of K (with $\mathcal{O}_\infty = K$ if K admits no $\emptyset$-definable non-trivial valuations) whose residue field $\boldsymbol{k}_\infty$ is algebraically closed, real closed, or finite. Moreover, $\operatorname{char} \boldsymbol{k}_\infty = 0$ by [47, Theorem 9.4.18(3), Remark 9.5.6]. Finally, (4) $\Rightarrow$ (1) is shown as in the proof of (4) $\Rightarrow$ (1) in Corollary 6.15, using Facts 3.1 and 6.18 in place of Conjecture 3.16. □

Note that there are indeed dp-minimal fields of characteristic zero without distal expansions.

Example. Let $\mathbb{Q}_p^{\text{unr}}$ be the maximal unramified extension of the valued field $\mathbb{Q}_p$. Its value group is $\mathbb{Z}$ and its residue field is the algebraic closure $\mathbb{F}_p^{\text{a}}$ of $\mathbb{F}_p$. Let $\mathcal{O}_K$ be the unique valuation ring of $K = \mathbb{Q}_p^{\text{unr}}\big(p^{1/p}, p^{1/p^2}, \dots\big)$ lying over that of $\mathbb{Q}_p^{\text{unr}}$. Its value group $\bigcup_n \frac{1}{p^n}\mathbb{Z}$ is archimedean (hence regular) but non-divisible, and $(K, \mathcal{O}_K)$ is henselian; so it follows from [41, Theorem 5] that $\mathcal{O}_K$ is $\emptyset$-definable in the field K. Hence K is a field of characteristic zero which is dp-minimal by [47, Theorem 9.1.5, 1(c)] but has no distal expansion since it interprets an infinite field of characteristic p.

7. Distality in Expansions of Fields by Operators

In this section we use a "forgetful functor" approach to show that various expansions of distal fields by operators remain distal. Most of the results of this section were obtained and circulated in 2014. We have learned that recently some of them were observed independently in [21].

7.1. **An abstract distality criterion.** We fix a language $\mathcal{L}$ and a complete $\mathcal{L}$-theory $T = \operatorname{Th}(\boldsymbol{M})$. As usual all variables here are assumed to be (finite) multivariables. Recall that by Fact 1.8, T is distal if and only if every partitioned $\mathcal{L}$-formula $\varphi(x;y)$ has a strong honest definition in T, i.e., there is a formula $\psi(x; y_1, \dots, y_N)$, where $y_1, \dots, y_N$ are disjoint multivariables (for some $N \in \mathbb{N}$), each of the same sort as y, such that for all $a \in M_x$ and finite subsets B of M_y with $|B| \geq 2$, there are $b_1, \dots, b_N \in B$ such that $\psi(x; b_1, \dots, b_n)$ isolates $\operatorname{tp}_\varphi(a|B)$:

(1) $a \in \psi(M_x; b_1, \dots, b_N)$; and
(2) for all $b \in B$, either

$$\psi(M_x; b_1, \dots, b_N) \subseteq \varphi(M_x; b) \quad \text{or} \quad \psi(M_x; b_1, \dots, b_N) \cap \varphi(M_x; b) = \emptyset.$$

We also consider an extension $\mathcal{L}(\mathfrak{F})$ of the language $\mathcal{L}$ by a set $\mathfrak{F}$ of new function symbols. We assume that $\mathcal{L}(\mathfrak{F})$ has the same sorts as $\mathcal{L}$, and we consider $\mathfrak{F}$ itself as a language by declaring the sorts of $\mathfrak{F}$ to be those of $\mathcal{L}$. Finally, we let $T(\mathfrak{F})$ be a complete $\mathcal{L}(\mathfrak{F})$-theory extending T.

Proposition 7.1. *Suppose T is distal and the following conditions hold:*

1. *$T(\mathfrak{F})$ has quantifier elimination;*
2. *all function symbols in $\mathfrak{F}$ are unary; and*
3. *for every $\mathcal{L}(\mathfrak{F})$-term $t(x)$ there are an $\mathcal{L}$-term s in n variables of the appropriate sorts and $\mathfrak{F}$-terms $t_1(x), \dots, t_n(x)$ such that $T \vdash t(x) = s\big(t_1(x), \dots, t_n(x)\big)$.*

Then $T(\mathfrak{F})$ is distal.

Proof. Fix a model $\boldsymbol{M}$ of $T(\mathfrak{F})$, and let $\varphi(x;y)$ be a partitioned $\mathcal{L}(\mathfrak{F})$-formula; we show that $\varphi(x;y)$ has a strong honest definition in $T(\mathfrak{F})$. By assumption (1), we may assume that $\varphi(x;y)$ is quantifier-free. Then by assumptions (2) and (3) there are an $\mathcal{L}$-formula $\varphi'(x';y')$ as well as $\mathfrak{F}$-terms $s_1(x), \dots, s_m(x)$ and $t_1(y), \dots, t_n(y)$, such that for all $a \in M_x$, $b \in M_y$ we have

$$\boldsymbol{M} \models \varphi(a,b) \iff \boldsymbol{M} \models \varphi'\big(s(a), t(b)\big),$$

where

$$s(a) := \big(s_1(a), \dots, s_m(a)\big) \quad \text{and} \quad t(b) := \big(t_1(b), \dots, t_n(b)\big).$$

Suppose $y = (y_1, \dots, y_k)$ where $k = |y|$; we can assume that the terms $t_1, \dots, t_n$ contain the terms $y_1, \dots, y_k$; thus $b \mapsto t(b)\colon M_y \to M_{y'}$ is injectve. By distality of T, take a strong honest definition $\psi'(x'; y'_1, \dots, y'_N)$ for $\varphi'(x';y')$ in T, where $y'_1, \dots, y'_N$ are disjoint new multivariables of the same sort as y'; thus for all $a' \in M_{x'}$ and any finite subset B' of $M_{y'}$ with $|B'| \geq 2$, there are $b'_1, \dots, b'_N \in B'$ such that

1. $\boldsymbol{M} \models \psi'(a'; b'_1, \dots, b'_N)$; and
2. for all $b' \in B'$, either

$$\psi'(M_{x'}; b'_1, \dots, b'_N) \subseteq \varphi'(M_{x'}; b') \quad \text{or} \quad \psi'(M_{x'}; b'_1, \dots, b'_N) \cap \varphi'(M_{x'}; b') = \emptyset.$$

We claim that

$$\psi(x; y_1, \dots, y_N) := \psi'\big(s(x); t(y_1), \dots, t(y_N)\big)$$

where $y_1, \dots, y_N$ are disjoint new multivariables of the same sort as y, is a strong honest definition for $\varphi(x;y)$ in $T(\mathfrak{F})$. Let $a \in M_x$, and let $B \subseteq M_y$ be finite with $|B| \geq 2$. Set $a' := s(a)$, $B' := t(B) \subseteq M_{y'}$ (so $|B'| = |B| \geq 2$), and take $b_1, \dots, b_N \in B$ such that (1) and (2) above hold with $b'_i := t(b_i)$, for $i = 1, \dots, N$. Then $\boldsymbol{M} \models \psi(a; b_1, \dots, b_N)$, and $\psi(x; b_1, \dots, b_N)$ isolates $\operatorname{tp}_\varphi(a|B)$, as required. □

In a similar way as the preceding proposition, one shows:

Lemma 7.2. *Suppose T is distal and for every partitioned $\mathcal{L}(\mathfrak{F})$-formula $\varphi(x;y)$, where $|x| = 1$, there is a partitioned $\mathcal{L}$-formula $\varphi'(x;z)$ and a tuple of $\mathcal{L}(\mathfrak{F})$-terms $t(y)$ of length $|z|$ such that*

$$T \vdash \varphi(x;y) \leftrightarrow \varphi'\big(x; t(y)\big).$$

Then $T(\mathfrak{F})$ is distal.

Proof. Let $\varphi(x;y)$ be a partitioned $\mathcal{L}(\mathfrak{F})$-formula, where $|x| = 1$; by Proposition 1.9 it is enough to show that $\varphi(x;y)$ has a strong honest definition in $T(\mathfrak{F})$. By our hypothesis we can assume $\varphi(x;y) = \varphi'\big(x; t(y)\big)$ where $\varphi'(x;y')$ is an $\mathcal{L}$-formula and $t(y) = \big(t_1(y), \dots, t_n(y)\big)$ is an appropriate tuple of $\mathcal{L}(\mathfrak{F})$-terms whose components contain the terms $y_1, \dots, y_k$ for $y = (y_1, \dots, y_k)$. Distality of T yields a strong honest definition $\psi'(x; y'_1, \dots, y'_N)$ for $\varphi'(x;y')$ in T, where $y'_1, \dots, y'_N$ are disjoint new multivariables of the same sort as y'. Then $\psi(x; y_1, \dots, y_N) := \psi'\big(x; t(y_1), \dots, t(y_N)\big)$ is a strong honest definition for $\varphi(x;y)$ in $T(\mathfrak{F})$. □

In practice, condition (3) in Proposition 7.1 is easily verified whenever T is a relational expansion of the theory of fields, and the functions symbols in $\mathfrak{F}$ are interpreted as derivations in models of $T(\mathfrak{F})$. We now give several applications of these criteria.

7.2. **Transseries.** In this subsection we assume that the reader is familiar with [2, Chapter 16]. Consider the language

$$\mathcal{L}_{\Lambda\Omega} = \{0,\ 1,\ +,\ -,\ \cdot,\ \partial,\ \iota,\ \leq,\ \preccurlyeq,\ \Lambda,\ \Omega\}$$

introduced there. The $\mathcal{L}_{\Lambda\Omega}$-theory T^{nl} of ω-free newtonian Liouville closed H-fields eliminates quantifiers [2, Theorem 16.0.1] and has two completions: $T^{\mathrm{nl}}_{\mathrm{small}}$, of which the differential field $\mathbb{T}$ of logarithmic-exponential transseries is a model, and $T^{\mathrm{nl}}_{\mathrm{large}}$. Both completions are distal:

Corollary 7.3. *The $\mathcal{L}_{\Lambda\Omega}$-theories $T^{\mathrm{nl}}_{\mathrm{small}}$ and $T^{\mathrm{nl}}_{\mathrm{large}}$ are distal.*

Proof. Let $\mathcal{L} := \mathcal{L}_{\Lambda\Omega} \setminus \{\partial\}$ (so $\mathcal{L}(\partial) = \mathcal{L}_{\Lambda\Omega}$), let $T(\partial) = T^{\mathrm{nl}}_{\mathrm{small}}$, and let T be the $\mathcal{L}$-theory of $\mathbb{T}$. Each model of T is a real closed ordered field K, viewed as a structure in the language $\{0, 1, +, -, \cdot, \iota, \leq\}$ in the natural way, equipped with a convex dominance relation $\preccurlyeq$ and interpretations of the unary relation symbols Λ and Ω as certain convex subsets of K. By Baisalov-Poizat [3], the theory of each expansion of an o-minimal structure by convex subsets of its domain is weakly o-minimal, hence distal; in particular, T is distal. (Alternatively, we could use Fact 1.29.) Proposition 7.1 (and the quotient rule for derivations) implies that $T^{\mathrm{nl}}_{\mathrm{small}} = T(\partial)$ is distal. The argument for $T^{\mathrm{nl}}_{\mathrm{large}}$ is similar. □

Combining Fact 2.1 with the preceding corollary shows that no infinite field of positive characteristic is interpretable in $\mathbb{T}$. We venture the following:

Conjecture 7.4. *The only infinite fields interpretable in $\mathbb{T}$ are $\mathbb{T}$, $\mathbb{R}$, and their respective algebraic closures $\mathbb{T}[i]$, $\mathbb{C} = \mathbb{R}[i]$.*

7.3. **Other distal differential fields.** Proposition 7.1 can be used to show that many other theories of interest are distal as well. In general, whenever T is the theory of an expansion of a differential field (perhaps with several derivations) by relations and constants, and we know that

(1) T has QE, and
(2) the reduct of T to the language without derivations is distal,

then Proposition 7.1 implies that T itself is distal. In the literature, one finds many theories which satisfy these conditions. For instance:

Corollary 7.5. *The following theories are distal:*

(1) CODF*, the model completion of the theory of ordered differential fields from* [65]*;*
(2) CODF_m*, the model completion of the theory of ordered differential fields with m commuting derivations from* [58, 67]*;*
(3) $\mathrm{pCDF}_{d,m}$*, the model completion of the theory of p-valued fields of p-rank d with m commuting derivations from* [67]*.*

The fact that CODF is NIP was first shown (also using the "forgetful functor") in [53], and generalized to CODF_m in [36]. The paper [33] considers a generalization of CODF_m: Given a complete, model complete o-minimal theory T expanding the theory of real closed ordered fields, the theory whose models are models of T equipped with m commuting derivations which satisfy the Chain Rule with respect to the continuously differentiable definable functions in T has a model completion T_m, and if T has quantifier elimination and a universal axiomatization, then T_m has quantifier elimination [33, Theorem 6.8]. (Note that the latter hypothesis on T can always be achieved by expanding the language by function symbols for all $\emptyset$-definable functions and expanding T accordingly.) Our criterion implies that then T_m is distal; this has also been observed in [33, Proposition 6.10].

The topological fields with generic valuations considered in [36] are also distal. For example, let $\mathcal{L} = \{0, 1, +, -, \cdot, \leq, \preccurlyeq\}$ and let OVF be the $\mathcal{L}$-theory of ordered fields equipped with a non-trivial convex dominance relation; its model completion is RCVF, the theory of real closed valued fields (see [2, Section 3.6]). By [36, Corollary 6.4], the $\mathcal{L}(\partial)$-theory whose models are the expansions of models of OVF by a derivation ∂, has a model completion; this model completion is distal because RCVF is weakly o-minimal. In [56] it is shown that the $\mathcal{L}$-theory of pre-H-fields with gap 0 has a model completion. Here, a *pre-H-field* is a model of the universal part of the theory T^{nl} from Section 7.2, and such a pre-H-field has gap 0 if it satisfies the $\mathcal{L}$-sentence $\forall y(y' \preccurlyeq y \to y \preccurlyeq 1)$. This model completion has quantifier elimination [56, Theorem 7.2, Corollary 7.4], and its distality follows in the same way as above from distality of RCVF. (In [56, Theorem 7.6] it is already shown that this model completion is NIP.)

As pointed out in the introduction, definable relations in a theory which has a distal expansion satisfy strong combinatorial bounds [11, 17]. This is often used in incidence combinatorics in a more explicit form, e.g., the proof of the Szemerédi-Trotter Theorem over the field of complex numbers (which is a stable structure) relies on interpreting the field $\mathbb{C}$ in the distal field of reals in the usual way [66, 72]. Corollary 7.5 implies a qualitative analog for the *stable* theories $\mathrm{DCF}_{0,m}$ of differentially closed fields of characteristic 0 with m commuting derivations. For this we need the following facts [64, 68]:

Fact 7.6. *If $K \models \mathrm{CODF}$, then the differential field extension $K[\mathrm{i}]$ of K (where $\mathrm{i}^2 = -1$) is a differentially closed field of characteristic 0, i.e., $K[\mathrm{i}] \models \mathrm{DCF}_0$. More generally, if $K \models \mathrm{CODF}_m$, then $K[\mathrm{i}] \models \mathrm{DCF}_{0,m}$.*

This immediately yields (see Lemma 1.28):

Corollary 7.7. *The theory $\mathrm{DCF}_{0,m}$ has a distal expansion.*

Problem 7.8. By [8, Lemma 4.5.9], the theory CODF is not strongly dependent. Does DCF_0 admit a strongly dependent distal expansion?

7.4. **Henselian valued fields with analytic structure.** We finish by showing that the forgetful functor argument (in the form of Lemma 7.2) also allows us to extend the main theorem from the introduction to the *analytic expansions* of henselian valued fields introduced in [20]; for this we rely on some arguments from [57, Section 5]. We need to recall the relevant definitions from [20].

We fix a noetherian commutative ring A and an ideal $I \neq A$ of A such that A is separated and complete for its I-adic topology. Let $A\langle X\rangle = A\langle X_1, \dots, X_m\rangle$ be the ring of power series in the distinct indeterminates $X_1, \dots, X_m$ with coefficients in A whose coefficients I-adically converge to 0, and set $A_{m,n} := A\langle X\rangle[[Y]]$ where $X = (X_1, \dots, X_m)$ and $Y = (Y_1, \dots, Y_n)$ are disjoint tuples of distinct indeterminates over A. We expand the (one-sorted) language of valued fields to a language $\mathcal{L}_A$ by introducing a unary function symbol ι as well as an $(m + n)$-ary function symbol for each element of $A_{m,n}$ (which we denote by the same symbol). We let T_A be the $\mathcal{L}_A$-theory whose models are the $\mathcal{L}_A$-structures expanding a valued field $(K, \mathcal{O})$ of characteristic zero, such that with $\mathfrak{m}$ = maximal ideal of $\mathcal{O}$:

- (A1) ι is interpreted by the map $K \to K$ with $a \mapsto 1/a$ if $a \neq 0$ and $0 \mapsto 0$;
- (A2) each function symbol $f \in A_{m,n}$ is interpreted by a function $f^K : K^m \times K^n \to K$ which is identically zero outside of $\mathcal{O}^m \times \mathfrak{m}^n$ and such that $f^K(\mathcal{O}^m \times \mathfrak{m}^n) \subseteq \mathcal{O}$;
- (A3) the map $f \mapsto f^K$ is a ring morphism from A_{m+n} to the ring of functions $K^m \times K^n \to K$;
- (A4) each $f \in A_{m,n}$, viewed as an element of $A_{m,n+1}$ under the natural inclusion $A_{m,n} \subseteq A_{m,n+1}$, is interpreted as a function $K^m \times K^{n+1} \to K$ which does not depend on the last coordinate, and similarly for the inclusion $A_{m,n} \subseteq A_{m+1,n}$;
- (A5) each $a \in I \subseteq A = A_{0,0}$ is interpreted by a constant function with value in $\mathfrak{m}$;

(A6) for $a = (a_1, \dots, a_m) \in \mathcal{O}^m$ and $b = (b_1, \dots, b_n) \in \mathfrak{m}^n$ we have $X_i^K(a,b) = a_i$ $(i = 1, \dots, m)$ and $Y_j^K(a,b) = b_j$ $(j = 1, \dots, n)$.

The valued field underlying each model of T_A is automatically henselian; see [57, Proposition 3.5].

Let now $K \models T_A$, and as in Section 5.1 expand K to a multi-sorted structure $\boldsymbol{K}$ whose sorts are K (called the field sort below) and the sets RV_δ (called the RV-sorts below), with the primitives specified in (K1)–(K4). Let $\boldsymbol{K}_*$ be an expansion of $\boldsymbol{K}$ obtained by imposing additional structure on the reduct RV_* of $\boldsymbol{K}$, including,

(A7) for each $u \in A_{m+n}$, the function $u_\delta^K \colon \mathrm{RV}_\delta^{m+n} \to \mathrm{RV}_\delta$ satisfying $u_\delta^K\big(\mathrm{rv}_\delta(a)\big) = \mathrm{rv}_\delta\big(u^K(a)\big)$ for $a \in K^{m+n}$.

(See [57, Corollary 3.9].) Let also $\mathcal{L}$ be the reduct of the language $\mathcal{L}_*$ of $\boldsymbol{K}_*$ obtained by removing all symbols listed under (A1)–(A7) above. The following is a consequence of [57, Corollary 5.5] (a generalization of a theorem in [26]):

Proposition 7.9. *Let $\varphi(x,y,r)$ be an $\mathcal{L}_*$-formula where the multivariables x, y, are of the field sort with $|x| = 1$, and r is of the* RV*-sort. Then there exists an $\mathcal{L}$-formula $\varphi'(x,z,r)$ and an appropriate tuple of $\mathcal{L}_A$-terms $t(y)$ such that $\boldsymbol{K}_* \models \varphi(x,y,r) \leftrightarrow \varphi'\big(x,t(y),r\big)$.*

We now use this result to show a variant of our main theorem:

Corollary 7.10. *Let $K \models T_A$; if the valued field underlying K is distal (has a distal expansion), then the $\mathcal{L}_A$-structure K is distal (has a distal expansion, respectively).*

Proof. Suppose first that the valued field underlying K has a distal expansion; by the forward direction of our main theorem, this valued field is finitely ramified, and its value group Γ and residue field $\boldsymbol{k}$ have a distal expansion. Consider now the structure $\boldsymbol{K}_*$ introduced before Proposition 7.9, where we equip RV_* with the functions (A7) as well as the structure coming from the distal expansions of Γ and $\boldsymbol{k}$ as explained at the beginning of Section 6.2. By Propositions 6.1 and 6.2, the $\mathcal{L}$-reduct $\boldsymbol{K}$ of $\boldsymbol{K}_*$ is distal. Now Lemma 7.2 and Proposition 7.9 yield that the expansion $\boldsymbol{K}_*$ of K is distal. This shows that if the valued field underlying K has a distal expansion, then so does K. Note that if we follow this argument when the valued field underlying K itself is distal, then the distal structure $\boldsymbol{K}_*$ we obtain in this way is bi-interpretable with the $\mathcal{L}_A$-structure K. □

Example. Let $\boldsymbol{k}$ be a distal field of characteristic zero and $A = \mathbb{Z}[[t]]$, $I = tA$. Then the valued field $K = \boldsymbol{k}(\!(t)\!)$ of Laurent series with coefficients in $\boldsymbol{k}$ can be expanded to a model of T_A in a unique way such that $t \in A$ is interpreted by $t \in K$; by the previous corollary, this $\mathcal{L}_A$-structure K is distal.

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