Problem Set 4
Due March 29
Model Theory

1. [Optional] We say that an $\mathcal{L}$-structure $\mathcal{M} \vDash T$ is an **existentially closed** model of $T$ if $\mathcal{M}$ is existentially closed in every $\mathcal{N} \vDash T$ with $\mathcal{M} \subseteq \mathcal{N}$.

   a) Suppose that $T$ is a $\forall \exists$-theory. Show that for every model $\mathcal{M}$ of $T$ there exists a model $\mathcal{M}^* \vDash T$ with $\mathcal{M} \subseteq \mathcal{M}^*$, having the following property: for every quantifier-free $\mathcal{L}$-formula $\varphi(x, y_1, \ldots, y_m)$ and $a_1, \ldots, a_m \in M$, if there exists some $\mathcal{N} \vDash T$ with $\mathcal{M}^* \subseteq \mathcal{N}$ and $b \in N$ such that $\mathcal{N} \models \varphi(b, a_1, \ldots, a_m)$, then there exists $c \in M^*$ such that $\mathcal{M}^* \models \varphi(c, a_1, \ldots, a_m)$.

   b) Use (a) to show that for any model $\mathcal{M}$ of a $\forall \exists$-theory $T$ there is some existentially closed $\mathcal{N} \vDash T$ with $\mathcal{M} \subseteq \mathcal{N}$ and $|N| = \max \{|M|, |\mathcal{L}|, |\mathcal{N}_0\}$.

2. Let $T$ be an $\mathcal{L}$-theory.

   a) Show that if $\mathcal{M} \subseteq \mathcal{N}$ are models of $T$ and $\mathcal{M}$ is an existentially closed model of $T$, then there is $\mathcal{M}_1 \vDash T$ such that $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{M}_1$ with $\mathcal{M} \preceq \mathcal{M}_1$. (Hint: Diagram Lemma.)

   b) Show that $T$ is model-complete if and only if every model of $T$ is existentially closed. (Hint for $\Leftarrow$: suppose that $\mathcal{M}_0 \subseteq \mathcal{N}_0$ are models of $T$; use (a) to build a chain $\mathcal{M}_0 \subseteq \mathcal{N}_0 \subseteq \mathcal{M}_1 \subseteq \mathcal{N}_1 \subseteq \mathcal{M}_2 \subseteq \cdots$ of models of $T$ such that $\mathcal{M}_i \preceq \mathcal{M}_{i+1}$ and $\mathcal{N}_i \preceq \mathcal{N}_{i+1}$.)

3. Let $T$ be an $\mathcal{L}$-theory.

   a) Show that $T$ admits quantifier-elimination if and only if for all $\mathcal{M} \vDash T$ the $\mathcal{L}(\mathcal{M})$-theory $T \cup \text{Diag}(\mathcal{M})$ is complete.

   b) Show that $T$ is model-complete if and only if for all $\mathcal{M} \vDash T$, the $\mathcal{L}(\mathcal{M})$-theory $T \cup \text{Diag}(\mathcal{M})$ is complete. (This explains the origin of the term “model-complete.”)

4. An $\mathcal{L}$-structure $\mathcal{M}$ is called **ultra-homogeneous** if every isomorphism between finitely generated substructures of $\mathcal{M}$ can be extended to an automorphism of $\mathcal{M}$.

   a) Let $\mathcal{M}$ be a finite $\mathcal{L}$-structure. Show that $\text{Th}(\mathcal{M})$ admits quantifier-elimination if and only if $\mathcal{M}$ is ultra-homogeneous.

   b) Show that the finite abelian group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ (construed as a structure in the language $\mathcal{L} = \{0, +, -\}$) is not ultra-homogeneous.

5. Let $K$ be a field and let $\mathcal{L}$ be the language of vector spaces over $K$, consisting of the binary function symbol $+$, the unary function symbol $-$, a constant symbol $0$, and for each $a \in K$ a unary function symbol $\mu_a$. We construe each $K$-vector space $V$ as an $\mathcal{L}$-structure by interpreting $+$, $-$ and $0$ as usual and $\mu_a$ by scalar multiplication $v \mapsto a \cdot v$ by $a$. Let $T$ be the theory of infinite $K$-vector spaces in this language. Show that $T$ admits quantifier elimination, and use this to show that $T$ is complete.

6. Let $\mathcal{L} = \{<\}$ and let $T$ be a satisfiable theory containing the axioms for linearly ordered sets which contain at least two elements. Show that if $T$ admits quantifier-elimination, then $\text{Mod}(T) = \text{Mod}(\text{DLO})$. 

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