HOUR EXAM II

Math 180, Fall 2003

Calculus I

November 14, 2003

ANSWERS

Problem 1.

Find the derivatives of the following functions. Indicate which rules of differentiation you used. You do not need to simplify after taking derivatives.

$$f(x) = e^2 + 2\sqrt{xe^x} \qquad\qquad g(x) = \frac{\cos(5x)}{5x}$$

Answer:
We have, using the Chain and Product Rules:

$$f'(x) = (e^2)' + (2\sqrt{xe^x})' = 2\frac{1}{2}(xe^x)^{-1/2} \cdot (1 \cdot e^x + x \cdot e^x) = \frac{e^x(x+1)}{\sqrt{xe^x}}.$$

Using the Quotient Rule we get

$$g'(x) = \frac{-\sin(5x) \cdot 5 \cdot 5x - \cos(5x) \cdot 5}{(5x)^2} = \frac{-5x\sin(5x) - \cos(5x)}{5x^2}.$$

Problem 2.

Consider the function

$$f(x) = \frac{1}{1+x^2}.$$

Find the local linearization of f at $x = 1$.

Answer:
We have

$$f'(x) = -\frac{2x}{(1+x^2)^2},$$

so $f'(1) = -\frac{1}{2}$. Since $f(1) = \frac{1}{2}$, the local linearlization of f at 1 is

$$y = -\frac{1}{2}(x-1) + \frac{1}{2} = -\frac{1}{2}x + 1.$$

Problem 3. Find the following limits:

$$\lim_{x\to\infty} \frac{x^2}{e^{3x}} \qquad\qquad \lim_{x\to 0} \frac{x}{\cos x}$$

If you use L'Hopital's Rule, make sure that its hypotheses are satisfied!

Answer:
For the first limit, we may use L'Hopital's Rule, since $x^2 \to \infty$ and $e^{3x} \to \infty$ as $x \to \infty$. We get

$$\lim_{x\to\infty} \frac{x^2}{e^{3x}} = \lim_{x\to\infty} \frac{2x}{3e^{3x}}.$$

Now $2x \to \infty$ and $3e^{3x} \to \infty$ as $x \to \infty$, so we may use L'Hopital's Rule again:

$$\lim_{x\to\infty} \frac{2x}{3e^{3x}} = \lim_{x\to\infty} \frac{2}{9e^{3x}} = 0,$$

since $9e^{3x} \to \infty$ as $x \to \infty$. In the second limit, the hypotheses of L'Hopital's Rule are not satisfied. We have $\frac{x}{\cos x} \to \frac{0}{1} = 0$ as $x \to 0$.

Problem 4.

If you have 200 feet of fencing and you want to enclose a rectangular area up against a long straight wall of length ℓ, what is the largest area you can enclose?

Answer:
Let us denote the length of the other side of the rectangle by x. Then the area of the rectangle is given by $A = x\ell$. We also have the requirement that its circumference, minus the length of the wall, equal 200, that is, $2x + \ell = 200$, or in other words, $\ell = 200 - 2x$. Hence $A(x) = x\ell = x(200 - 2x)$. We have $A'(x) = 200 - 4x$, which equals 0 exactly if $x = 50$. So $x = 50$ is a critical point of A, and it is a global maximum of A since $A''(x) = -4 < 0$ for all x. The corresponding area is $A(50) = 50(200 - 2 \cdot 50) = 5000$ (square feet).

Problem 5.

Consider the equation

$$\ln(xy) + y^2 = x$$

for *positive* x, y.

1. Find $\frac{dy}{dx}$.
2. How many points (x, y) are there where the tangent line is horizontal?

Answer:
Using implicit differentiation we get

$$\frac{1}{xy}\left(y + x\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 1$$

and solving for $\frac{dy}{dx}$ yields

$$\frac{dy}{dx} = \frac{y(x-1)}{x(2y^2+1)}.$$

Now the tangent line at (x, y) is horizontal exactly if $\frac{dy}{dx} = 0$, and this is the case if $x = 1$. So we need to investigate whether there exists a point with coordinates $(1, y)$, $y > 0$, on the graph defined by the given equation, that is, whether $\ln y + y^2 = 1$ for some $y > 0$. The function $\ln y + y^2$ has derivative $1/y + 2y > 0$ for all $y > 0$, hence is strictly increasing. For $y = 1$ we get $\ln 1 + 1^2 = 1$. So there is exactly one point, namely $(1, 1)$, at which the tangent line is horizontal.

Problem 6. (Extra credit— 10 bonus points, but no partial credit.)

Find the derivative of the function

$$f(x) = x^x,$$

defined for all $x > 0$. (Hint: remember that $a = e^{\ln a}$ for every $a > 0$.)

Answer:
By the hint we get $x^x = e^{\ln x^x}$. By the rules for computing with ln we also have $\ln x^x = x \ln x$, so $f(x) = x^x = e^{x \ln x}$. By the Chain Rule followed by an application of the Product Rule we get

$$\begin{aligned} f'(x) &= e^{x \ln x} \cdot (x \ln x)' \\ &= e^{x \ln x} \cdot (1 \cdot \ln x + x \cdot 1/x) \\ &= x^x(\ln x + 1). \end{aligned}$$