MIDTERM EXAM 1

Math 31B, Fall Quarter 2008

Integration and Infinite Series

October 22, 2008

ANSWERS

Problem 1. Compute the following integrals.

1.
$$\int \sin y \cdot \cos y \, dy$$

2.
$$\int \frac{2e^{2z}}{e^{2z}+1}\, dz$$
(You might have to use substitution twice.)

(10+10 points.)

Answer:

1. Integration by parts: Take $u = \sin y$, $v' = \cos y$. Then $u' = \cos y$, $v = \sin y$, hence
$$\int \sin y \cdot \cos y \, dy = \sin^2 y - \int \sin y \cdot \cos y \, dy$$
and therefore
$$\int \sin y \cdot \cos y \, dy = \frac{1}{2}\sin^2 y + C.$$

2. We use the substitution rule with $w = e^{2z}$, so $\frac{dw}{dz} = 2e^{2z}$. Thus
$$\int \frac{2e^{2z}}{e^{2z}+1}\, dz = \int \frac{dw}{w+1}.$$
Now we use the substitution rule again, this time with $u = w + 1$, so $\frac{du}{dw} = 1$. Hence
$$\int \frac{dw}{w+1} = \int \frac{1}{u}\, du = \ln|u| + C = \ln|w+1| + C = \ln(e^{2z}+1) + C$$

Problem 2.

1. For which real values a with $0 < a < 1$ is
$$\lim_{x\to\infty} \frac{(\ln x)^a}{x} = 0?$$
Justify your answer.

2. Evaluate the limit
$$\lim_{x\to 0} (1+2x)^{2/x}.$$

(10+10 points.)

Answer:

1. For $0 < a < 1$ we have
$$\lim_{x\to\infty} (\ln x)^a = \lim_{x\to\infty} x = \infty.$$
So by l'Hôpital's Rule we have
$$\lim_{x\to\infty} \frac{(\ln x)^a}{x} = \lim_{x\to\infty} \frac{a(\ln x)^{a-1}\cdot(1/x)}{1} = a \lim_{x\to\infty} \frac{(\ln x)^{a-1}}{x}.$$
Now if $a < 1$ then $a - 1 < 0$ and hence
$$a \lim_{x\to\infty} \frac{(\ln x)^{a-1}}{x} = a \left(\lim_{x\to\infty} (\ln x)^{a-1}\right)\left(\lim_{x\to\infty} \frac{1}{x}\right) = a\cdot 0\cdot 0 = 0.$$
Therefore
$$\lim_{x\to\infty} \frac{(\ln x)^a}{x} = 0$$
for all a with $0 < a < 1$.

2. We have
$$\ln(1+2x)^{2/x} = \frac{2}{x}\ln(1+2x) = \frac{2\ln(1+2x)}{x}.$$
Now as $x \to 0$ both numerator and denominator of this fraction approach 0, so l'Hôpital's Rule applies and we obtain
$$\lim_{x\to 0} \frac{2\ln(1+2x)}{x} = \lim_{x\to 0} \frac{2/(1+2x)\cdot 2}{1} = 4.$$
Hence
$$\lim_{x\to 0} (1+2x)^{2/x} = e^4.$$

Problem 3.
How large should N be to guarantee that the error in the midpoint rule approximation M_N to

$$\int_0^4 \sqrt{x+1}\,dx$$

is accurate to within 0.1? (20 points.)

Answer:
Let $f(x) = \sqrt{x+1}$. Then

$$f'(x) = \frac{1}{2}(x+1)^{-1/2}, \qquad f''(x) = -\frac{1}{4}(x+1)^{-3/2}.$$

Note that f'' is increasing and $f''(x) < 0$ on the interval $0 \le x \le 4$. So

$$|f''(x)| \le |f''(0)| = \frac{1}{4} \qquad \text{for } 0 \le x \le 4,$$

hence we can choose $K_2 = \frac{1}{4}$. We now use the formula for the midpoint error:

$$\text{Error}(M_N) = \left| \int_0^2 \sqrt{x+1}\,dx - M_N \right| \le \frac{K_2(b-a)^3}{24N^2} = \frac{(1/4)\cdot 4^3}{24N^2} = \frac{2}{3N^2}.$$

We'd like to have $\text{Error}(M_N) \le 0.1$, and this is certainly the case if

$$\frac{2}{3N^2} \le 0.1 = \frac{1}{10},$$

or equivalently

$$N^2 \ge \frac{20}{3}.$$

So $N = 3$ is enough.

Problem 4. Evaluate

1. $$\int \cos^3 x \, dx.$$

2. $$\int_1^e x \ln(x^2) \, dx.$$

(15+10 points.)

Answer:

1. There are two possibilities. We can either use the reduction formula
$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$
to get
$$\begin{aligned}\int \cos^3 x \, dx &= \frac{1}{3} \cos^2 x \sin x + \frac{2}{3} \int \cos x \, dx \\ &= \frac{1}{3} \cos^2 x \sin x + \frac{2}{3} \sin x + C,\end{aligned}$$
or use the identity $\cos^2 x = 1 - \sin^2 x$ and the substitution $u = \sin x$ to write
$$\begin{aligned}\int \cos^3 x \, dx &= \int (1 - \sin^2 x) \cos x \, dx \\ &= \int (1 - u^2) \, du \\ &= u - \frac{u^3}{3} + C = \sin x - \frac{\sin^3 x}{3} + C.\end{aligned}$$

2. Integration by parts: take $u = \ln x$ (so $u' = 1/x$) and $v' = x$ (so $v = x^2/2$). Then
$$\int x \ln x^2 \, dx = 2 \int x \ln x \, dx = x^2 \ln x - \int (1/x) x^2, dx = x^2(\ln x - 1/2)$$
and hence
$$\int_1^e x \ln(x^2) \, dx = \frac{1}{2}(e^2 + 1).$$

Problem 5.

The atmospheric pressure $P(h)$ (in pounds per square inch) at a height h (in miles) above sea level on earth satisfies the differential equation

$$P' = -kP \qquad \text{for some positive constant } k.$$

Measurements with a barometer show that

$$P(0) = 14.7 \qquad \text{and} \qquad P(10) = 2.$$

Find k. (10 points.)

Answer:

We have

$$P(h) = P_0 e^{-kh}$$

where $P_0 = P(0) = 14.7$. Since $P(10) = 2$ we have

$$2 = 14.7 \cdot e^{-10k}$$

and hence, taking ln on both sides of the equation and solving for k:

$$k = -\frac{1}{10} \ln\left(\frac{2}{14.7}\right).$$

Problem 6. Find

$$\int \frac{1}{x^2-4}\,dx.$$

(5 points.)

Answer:
We have the factorization

$$x^2-4=(x-2)(x+2).$$

We set up the partial fraction decomposition:

$$\frac{1}{x^2-4}=\frac{A}{x-2}+\frac{B}{x+2}.$$

Clearing the denominators yields

$$1=A\cdot(x+2)+B\cdot(x-2).$$

Substitution of $x=2$ yields $A=1/4$, and substitution of $x=-2$ gives $B=-1/4$. Thus

$$\begin{aligned}\int \frac{1}{x^2-4}\,dx &= A\int \frac{1}{x-2}\,dx + B\int \frac{1}{x+2}\,dx \\ &= \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + C.\end{aligned}$$