1. Find the (absolute) maximum and minimum of

$$f(x) = \frac{\sin x + \cos x}{2}$$

on the interval $[0, \pi]$, showing your work.

$$f'(x) = \frac{1}{2}(\cos x - \sin x) = 0$$

$$\cos x = \sin x \qquad x = \pi/4$$

$$f(\pi/4) = \frac{\sin(\pi/4) + \cos(\pi/4)}{2} = \frac{\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}}{2} = \frac{\sqrt{2}}{2}$$

max

$$f(0) = \frac{0 + 1}{2} = \frac{1}{2}$$

$$f(\pi) = \frac{0 + (-1)}{2} = -\frac{1}{2} \quad \text{min}$$

2. Use the Mean Value Theorem to prove that if $f(1) = 3$ and $f'(x) < -1$ for all $x \geq 0$, then $f(4)$ is negative.

By the MVT

$$\frac{f(4) - f(1)}{4 - 1} = f'(c) < -1$$

for some c in [1, 4]

$$\frac{f(4) - 3}{3} < -1$$

$$f(4) - 3 < -3$$

$$f(4) < 0$$

3. A function is defined by

$$f(x) = \frac{x}{\sqrt{2x^2+1}}.$$

[10 points] (a) Find the horizontal asymptotes of $f(x)$, if it has any.

[10 points] (b) Given that

$$f'(x) = \frac{1}{\sqrt{(2x^2+1)^3}},$$

determine the intervals on which $f(x)$ is concave up and the intervals on which $f(x)$ is concave down.

(a) $\lim_{x\to\infty} \frac{x}{\sqrt{2x^2+1}} \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{2+1/x^2}} = 1/\sqrt{2}$

$\lim_{x\to-\infty} \frac{x}{\sqrt{2x^2+1}} = \lim_{x\to\infty} \frac{-x}{\sqrt{2(-x)^2+1}}$

$= -\lim_{x\to\infty} \frac{x}{\sqrt{2x^2+1}} = -\frac{1}{\sqrt{2}}$

(b) $f'(x) = (2x^2+1)^{-3/2}$

$f''(x) = -\frac{3}{2}(2x^2+1)^{-5/2}(4x) = \frac{-6x}{(2x^2+1)^{5/2}}$

Concave up on $(-\infty, 0)$

Concave down on $(0, \infty)$

4. Given that

$$f''(x) = \sin x + 6x,$$

$f'(\pi) = 1$ and $f(\pi) = 0$, find $f(x)$.

$$f'(x) = \int \sin x + 6x \, dx$$

$$= -\cos x + 3x^2 + C$$

$$f'(\pi) = -\cos(\pi) + 3\pi^2 + C = 1$$

$$-(-1) + 3\pi^2 + C = 1 \qquad C = -3\pi^2$$

$$f(x) = \int -\cos x + 3x^2 - 3\pi^2 \, dx$$

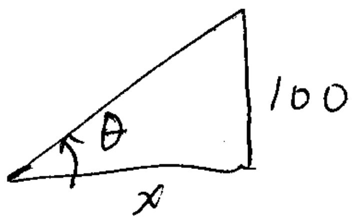
$$= -\sin x + x^3 - 3\pi^2 x + D$$

$$f(\pi) = -\sin(\pi) + \pi^3 - 3\pi^2(\pi) + D = 0$$

$$0 + \pi^3 - 3\pi^3 + D = 0 \qquad D = 2\pi^3$$

$$f(x) = -\sin x + x^3 - 3\pi^2 x + 2\pi^3$$

5. A kite 100 feet above the ground moves horizontally at a rate of 8 feet per second. At what rate is the angle θ between the kite string and the (horizontal) ground changing when the length of the string is 200 feet? (Note: Think of the kite string as a straight line from a point on the ground to the kite.)



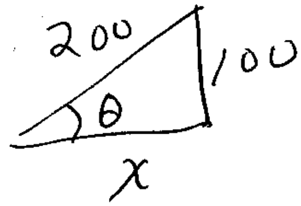
$$\frac{dx}{dt} = 8 \qquad \frac{d\theta}{dt} = ?$$

$$\tan\theta = \frac{100}{x} = 100x^{-1}$$

$$\sec^2\theta \frac{d\theta}{dt} = -100x^{-2}\frac{dx}{dt}$$

$$\sec^2\theta \frac{d\theta}{dt} = \frac{-800}{x^2}$$

$$\frac{d\theta}{dt} = -\frac{800}{x^2}\cos^2\theta$$

if 

$$\cos\theta = \frac{x}{200}$$

$$\frac{d\theta}{dt} = -\frac{800}{x^2}\left(\frac{x}{200}\right)^2 = -\frac{800}{(200)^2}$$

$$\left(= -\frac{1}{50}\right)$$

(12) 1. Find the absolute maximum and minimum of $y = x^4 - 2x^2$ on the interval $[-2, 1]$.

Solution. $y' = 4x^3 - 4x = 4x(x-1)(x+1)$, which is zero at $x = -1, 0, 1$. Therefore, the absolute maximum and minimum must occur at $x = -2, -1, 0$ and/or 1. The values of y at these points are 8,-1,0 and -1 respectively. Therefore, the maximum value is 8, and occurs at $x = -2$, while the minimum value is -1, and occurs at $x = \pm 1$.

(13) 2. Find the dimensions of the rectangle of maximum area that can be inscribed in a semicircle of radius 1.

Solution. Use the upper half of the unit circle centered at the origin, and let (x, y) be the coordinates of the point in the first quadrant at which the rectangle meets the circle. Then $x^2 + y^2 = 1$ and the area of the rectangle is $A(x) = 2xy = 2x\sqrt{1-x^2}$, so we must maximize $A(x)$ for $0 \le x \le 1$. Computing the derivative gives $A'(x) = 2(1-2x^2)/\sqrt{1-x^2}$. Since $A(0) = A(1) = 0$, the maximum must occur at the interior critical point, $x = 1/\sqrt{2}$. So, the inscribed rectangle of largest area has dimensions $\sqrt{2} \times 1/\sqrt{2}$.

(10) 3. (a) Find

$$\int (\sin x + \cos(3x+2))dx.$$

Solution. $-\cos x + \frac{1}{3}\sin(3x+2) + C$.

(b) Find the function f that satisfies $f'(t) = t^{-3/2}$ and $f(4) = 1$.

Solution. Antidifferentiating gives $f(t) = -2/\sqrt{t} + C$. The initial condition gives $C = 2$, so $f(t) = -2/\sqrt{t} + 2$.

(15) 4. Let $g(x) = x^5 - 5x + 2$.

(a) Find the critical points, and determine in each case whether it is local maximum, local minimum, or neither.

Solution. $g'(x) = 5x^4 - 5$ and $g''(x) = 20x^3$. Therefore the critical points are $x = \pm 1$. $g''(-1) = -20$ and $g''(1) = 20$, so by the second derivative test, -1 is a local maximum and $+1$ is a local minimum.

(b) Find the inflection points and the intervals on which the graph is concave up or concave down.

Solution. Since g'' changes sign only at $x = 0$, this is the only inflection point. g is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$.

(11) 5. The base x of the right triangle in the figure below increases at rate 5 cm/sec, while the height remains constant at 20 cm. At what rate is the angle θ changing when $x = 20$ cm?

Solution. The variables θ and x are related by

$$\tan\theta = 20/x,$$

so by the chain rule,

$$\sec^2\theta \frac{d\theta}{dt} = -\frac{20}{x^2}\frac{dx}{dt}.$$

When $x = 20$, $\theta = \pi/4$ and $\sec^2\theta = 2$. Therefore,

$$\frac{d\theta}{dt} = -\frac{1}{8}\ rad/sec.$$

(14) 6. Given that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad and \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6},$$

evaluate

$$(a)\ \sum_{j=3}^{50} j(j-1) = \sum_{j=1}^{50} j(j-1) - 2 = \frac{(50)(51)(101)}{6} - \frac{(50)(51)}{2} - 2 = 41640.$$

$$(b)\ \lim_{n\to\infty} \sum_{j=1}^{n} \frac{j}{n^2} = \lim_{n\to\infty} \frac{1}{n^2}\frac{n(n+1)}{2} = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}.$$

(10) 7. (a) Compute the Riemann sum corresponding to the integral of $f(x) = 2x - 1$ over the interval $[1, 3]$ for the partition $\{1, 3/2, 5/2, 3\}$ with $c_1 = 3/2, c_2 = 2, c_3 = 3$.

Solution. $f(3/2)/2 + f(2) + f(3)/2 = 6.5$.

(b) Sketch the region whose area is given by the integral

$$\int_1^3 |2x - 4| dx.$$

Solution. It is the two right triangles with vertices at $(1, 0), (2, 0), (1, 2)$ and $(2, 0), (3, 0), (3, 2)$.

(15) 8. (a) State the Mean Value Theorem.

Solution. See page 192 of the text.

(b) Prove the following: If $f(x)$ is differentiable on (a, b) and $f'(x) = 0$ on (a, b), then there is a constant C so that $f(x) = C$ on (a, b).

Solution. See page 193 of the text.

1. Given the function

$$f(x) = \frac{x}{x-2} \qquad (x \neq 2),$$

determine where the function is concave upward and where it is concave downward.

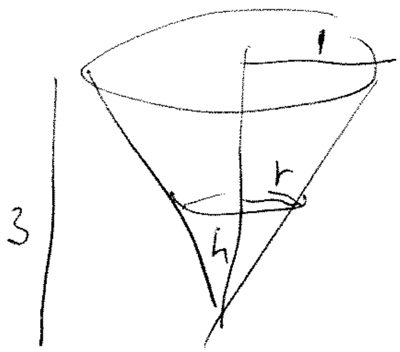
$$f'(x) = \frac{x-2-x}{(x-2)^2} = -2(x-2)^{-2}$$

$$f''(x) = 4(x-2)^{-3} = \frac{4}{(x-2)^3}$$

pos for $x > 2$ - concave up

neg for $x < 2$ - concave down

2. A paper cup in the shape of a cone is of height 3 in. and radius (at the top) of 1 in. Water is poured into the cup at a rate of $\frac{1}{3}$ in^3/s. How fast is the water level rising when the water is 2 in. deep? (Note: The volume of a cone of height h and radius r is $V = \frac{1}{3}\pi r^2 h$.)



$$\frac{r}{h} = \frac{1}{3} \qquad r = \frac{1}{3} h$$

$$V(h) = \frac{1}{3}\pi \left(\frac{1}{3}h\right)^2 h$$

$$= \frac{\pi}{27} h^3$$

$$\frac{1}{3} = \frac{dV}{dt} = \frac{\pi}{9} h^2 \frac{dh}{dt} \qquad \frac{dh}{dt} = \frac{3}{4\pi}$$

$$\frac{1}{3} = \frac{\pi}{9}(4)\frac{dh}{dt}$$

3. Find all the local maxima and local minima of

$$f(x) = (x^2 - 1)^{2/3}.$$

$$f'(x) = \frac{2}{3}(x^2-1)^{-1/3}(2x) = \frac{4}{3}\,\frac{x}{(x^2-1)^{1/3}}$$

Critical numbers: $x = 0$ and $+1, -1$
(no derivative)

Interval	x	$(x^2-1)^{1/3}$	$f'(x)$
$(-\infty, -1)$	neg	pos	neg
$(-1, 0)$	neg	neg	pos
$(0, 1)$	pos	neg	neg
$(1, \infty)$	pos	pos	pos

local min at $+1, -1$

local max at 0

4. A soft drink can in the shape of a right circular cylinder must contain 32 cubic inches of liquid. The metal used for the bottom of the can is the same as that used for the side but the metal used for the top costs 3 times as much as the metal used for the rest of the can. Find the dimensions (its height and radius of the base) of the can that will minimize the cost of the metal used in making it. (Note: You do not have to verify by means of a test that your answer is the minimum.)

$V = \pi r^2 h = 32 \qquad h = \dfrac{32}{\pi r^2}$

cost of side metal $= c$

$\text{Cost} = 2\pi r h c + \pi r^2 c + \pi r^2 (3c)$

$C(r) = 2\pi r\left(\dfrac{32}{\pi r^2}\right)c + 4c\pi r^2$

$C(r) = 64r^{-1}c + 4c\pi r^2$

$C'(r) = -64r^{-2}c + 8c\pi r = 0$

$-64 + 8\pi r^3 = 0$

$r^3 = \dfrac{8}{\pi} \qquad r = \dfrac{2}{\sqrt[3]{\pi}} \qquad h = \dfrac{32}{\pi\left(\dfrac{2}{\sqrt[3]{\pi}}\right)^2}$

5. Use the Mean Value Theorem to prove that, if $b > 0$, then

$$\sqrt{1+b} < 1 + \frac{b}{2}.$$

$f(x) = \sqrt{1+x}$ $\qquad f'(x) = \frac{1}{2}(1+x)^{-1/2}$

$= (1+x)^{1/2}$ $\qquad = \frac{1}{2\sqrt{1+x}}$

Mean Value Theorem $(a=0)$ $\quad 0 < c < b$

$$\sqrt{1+b} - \sqrt{1+0} = \frac{1}{2\sqrt{1+c}}(b-0)$$

$$\sqrt{1+b} - 1 = \frac{b}{2\sqrt{1+c}} < \frac{b}{2}$$

$$\sqrt{1+b} < 1 + \frac{b}{2}$$

MATH 31A 2ND PRACTICE MIDTERM

Problem 1. For each of the following functions, compute its derivative and then use linear approximation around the given point a to estimate the value of the function.
(a) $f(x) = \sin x$, $a = \pi$, $f(\frac{5\pi}{4})$;
(b) $f(x) = \sqrt{1 + \sqrt{1 + x}}$, $a = 0$, $f(0.1)$;

SOLUTION. (a) $f'(x) = \cos x$. Since $f(\pi) = 0$ and $f'(\pi) = \cos(\pi) = -1$, the linear approximation at $a = \pi$ is given by $L(x) = -1(x - \pi)$. Therefore, $f(5\pi/4) \simeq L(5\pi/4) = -\pi/4$;
(b) We have

$$f'(x) = \frac{1}{2} \cdot \frac{1}{\sqrt{1+\sqrt{1+x}}} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{1+x}} = \frac{1}{4} \cdot \frac{1}{\sqrt{1+\sqrt{1+x}} \cdot \sqrt{1+x}};$$

Since $f(0) = \sqrt{2}$ and $f'(0) = \frac{1}{4} \cdot \frac{1}{\sqrt{2}} \cdot 1 = \frac{1}{4\sqrt{2}}$, the linear approximation at $a = 0$ is given by $L(x) = \sqrt{2} + \frac{1}{4\sqrt{2}}x$. Therefore, $f(0.1) \simeq L(0.1) = \sqrt{2} + \frac{0.1}{4\sqrt{2}} = \frac{81}{80}\sqrt{2}$;

Problem 2. True or False. For each of the following statements, indicate if it is true or false (you may assume that f is everywhere differentiable for all of the questions below):

1. If $f'(x) > 0$ for all x, then $f(x)$ is increasing	True
2. If $f'(a) = 0$, then f attains either a maximum or a minimum at a	False
3. If f is concave up, then f' is increasing	True
4. The function $\sqrt{x^2+1}$ is concave down	False
5. The function $\sin x$ has an inflection point at π.	True

Comments:

1. The statement is true by the first derivative test.
2. If $f'(a) = 0$, $f(x)$ can have an inflection point at a, which is neither a local minimum nor a local maximum. For example, take $f(x) = x^3$ and $a = 0$.
3. If f is concave up (on some interval), then $f'' > 0$ on this interval. But $f''(x) = (f'(x))'$. Therefore, $(f'(x))' > 0$, therefore, $f'(x)$ is increasing (by the first derivative test applied to $f'(x)$).

4. Compute the derivatives: $f'(x) = \frac{1}{2}\frac{1}{\sqrt{x^2+1}} \cdot 2x = \frac{x}{\sqrt{x^2+1}}$;

$$f''(x) = \frac{\sqrt{x^2+1} - \frac{1}{2}\frac{2x}{\sqrt{x^2+1}}x}{x^2+1} = \frac{(x^2+1) - x^2}{(x^2+1)^{3/2}} = \frac{1}{(x^2+1)^{3/2}} > 0 \qquad \text{for all } x$$

Therefore, the function is concave up.

5. Either recall the graph of $\sin(x)$, or compute the second derivative: $(\sin(x))' = \cos x$, $(\sin x)'' = -\sin x$. Since $\sin \pi = 0$, π is an inflection point.

Problem 3. Let $f(x) = x^3 - 3x + 7$. Find the minimum and maximum values of f in the interval $[0, 2]$.

SOLUTION: Compute the derivative: $f'(x) = 3x^2 - 3$. $f'(x) = 0$ for $x = -1$ and $x = 1$. Only $x = 1$ is on the interval $[0, 2]$. Since at $x = 1$ the derivative changes sign from negative to positive, $x = 1$ is a local minimum. The value of the function at 1 is $f(1) = 5$.

Compute the values of $f(x)$ at the end points of $[0, 2]$. We have $f(0) = 7$ and $f(2) = 9$.

Comparing all of the values we computed, we obtain that on the interval $[0, 2]$, $f(1) = 5$ is the minimum of $f(x)$ and $f(2) = 9$ is its maximum.

Problem 4. Let $f(x) = \frac{1-2x^2}{1-x^2}$. Sketch the graph of f, indicating all properties of the function, such as asymptotes, extreme points, minima, maxima, convexity, points of inflection and intersepts.

SOLUTION: The domain of $f(x)$ is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

1). The intercept points:

- $f(0) = 1$;
- $f(x) = 0$ when $x = \pm\frac{1}{\sqrt{2}}$.

2). First, notice that

$$f(x) = \frac{1-2x^2}{1-x^2} = \frac{1-x^2-x^2}{1-x^2} = 1 - \frac{x^2}{1-x^2};$$

Compute the derivative:

$$f'(x) = -\frac{2x(1-x^2) - (-2x)x^2}{(1-x^2)^2} = -\frac{2x}{(1-x^2)^2};$$

Since $f'(x) > 0$ for $x < 0$, on this interval the function is increasing. Since $f'(x) < 0$ for $x > 0$, on this interval the function is decreasing. Since $f'(0) = 0$ and the derivative changes sign from positive to negative, $f(x)$ has a local maximum at $x = 0$. The value at local maximum is 1.

3). Compute the second derivative:

$$\begin{aligned} f''(x) &= -2\frac{(1-x^2)^2 - 2(1-x^2)\cdot(-2x)\cdot x}{(1-x^2)^4} \\ &= -2\frac{(1-x^2)(1-x^2+4x^2)}{(1-x^2)^4} = -2\frac{(1+3x^2)}{(1-x^2)^3} \end{aligned}$$

For $x < -1$ and $x > 1$ $f''(x) < 0$ and the graph of the function is concave up. For $x \in (-1, 1)$ $f''(x) < 0$ and the graph of the function is concave down. There are no points of inflection.

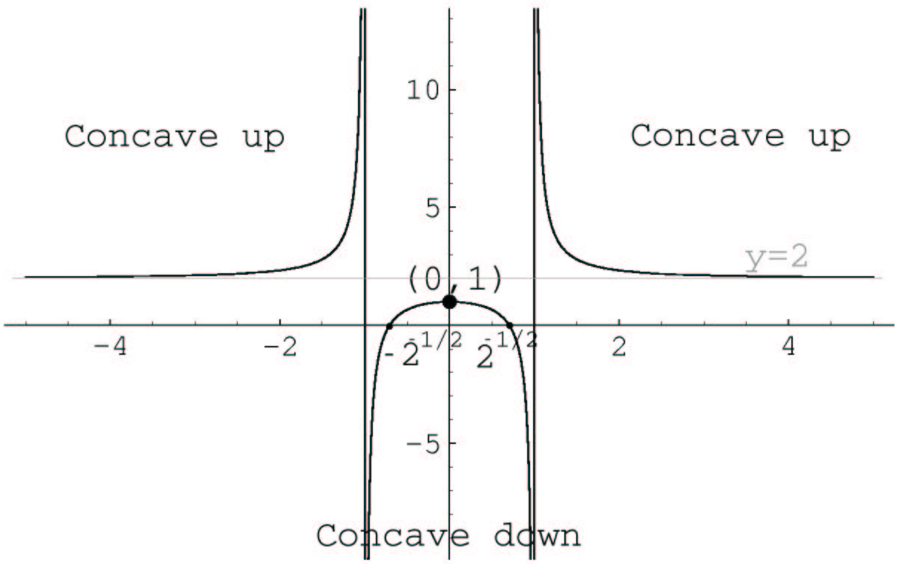
4). Since $\lim_{x\to 1^-} f(x) = -\infty$ and $\lim_{x\to 1^+} f(x) = \infty$, $x = 1$ is a vertical asymptote. Since $\lim_{x\to 1^-} f(x) = +\infty$ and $\lim_{x\to -1^+} f(x) = -\infty$, $x = -1$ is a vertical asymptote.

5). To find a horizontal asymptote, compute the limit

$$\lim_{x\to\infty}\frac{1-2x^2}{1-x^2}=\lim_{x\to\infty}\frac{\frac{1}{x^2}-2}{\frac{1}{x^2}-1}=1$$

Therefore, $y=1$ is the horizontal asymptote.

6). We can now draw the graph of $f(x)$:



Problem 5. Let $y=\sin 2x-2\sin x$. Sketch the graph of this function, indicating all properties of the function.

SOLUTION: 1). Since the period of the function $\sin 2x$ is π and the period of the function $\sin x$ is 2π, the function $y=\sin 2x-2\sin x$ is periodic with period 2π. It is also convenient to notice that $y=2\sin 2x-2\sin x=2\sin x\cos x-2\sin x=2\sin x(\cos x-1)$.

2). The intercept points:

- $y(0)=0$.
- $y(x)=0$ is equivalent to $\sin x(\cos x-1)=0$, i.e., either $\sin x=0$ or $\cos x=1$. The first equation has the solutions $x=\pi n$ for any integer n, and the second has the solution $x=2\pi n$ for all n. Therefore, $y(x)=0$ at the points $y=\pi n$ for all n.

3). $y'=2\cos 2x-2\cos x$. We have $y'=0$ iff $\cos 2x=\cos x$. Since $\cos 2x=2\cos^2 x-1$, $\cos 2x=\cos x$ is equivalent to $4\cos^2 x-2\cos x-2=0$. This is a quadratic equation with respect to $\cos x$. Solving it, we obtain $\cos x=\frac{1\pm\sqrt{9}}{4}$, that is $\cos x=-\frac{1}{2}$ or $\cos x=1$. The first of these has the following solutions:

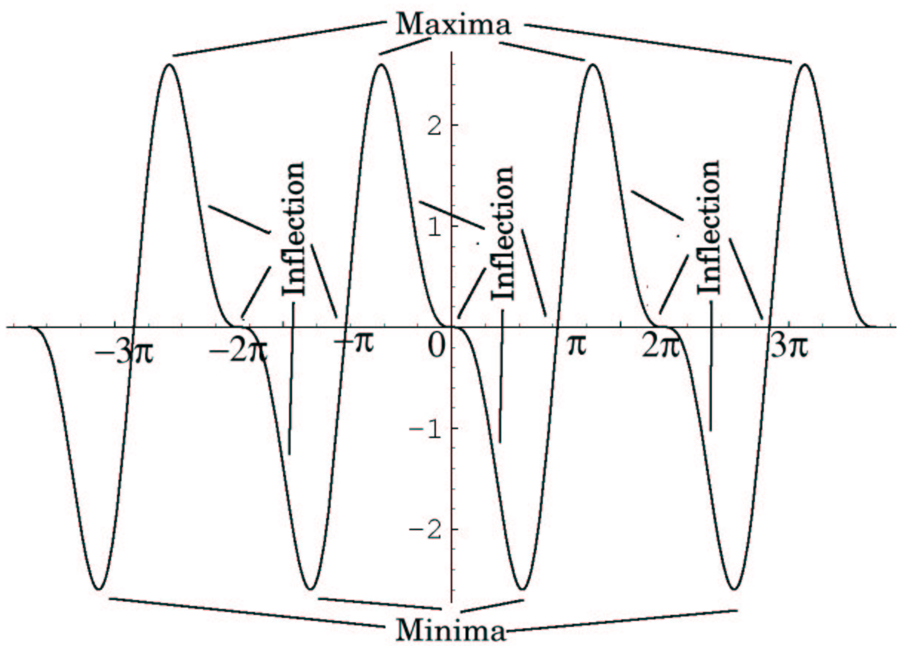
$x_1 = 2\pi/3 + 2\pi n$ and $x_2 = 4\pi/3 + 2\pi n$, where n is any integer. The second has the solutions: $x_3 = 2\pi n$ where n is any integer.

Since at $x_1 = 2\pi/3 + 2\pi n$ the derivative changes sign from $-$ to $+$, these are the points of local minima. At $x_2 = 4\pi/3 + 2\pi n$ the derivative changes sign from $+$ to -; therefore, x_2 are the points of local maxima. The values of the function at these points are $y(x_1) = -\frac{3\sqrt{3}}{2}$ and $y(x_2) = \frac{3\sqrt{3}}{2}$.

Near $x_3 = 2\pi n$ the derivative is negative. Since it does not change sign, these are not extremal points.

4). $y'' = -4\sin 2x + 2\sin x = -8\sin x\cos x + 2\sin x = 2\sin x(-4\cos x + 1)$. We have $y'' = 0$ iff $\sin x = 0$ or $\cos x = \frac{1}{4}$. The first equation has the solutions $x_5 = \pi n$ and the second one has the solutions $x_{6,7} = \pm\arccos(1/4) + 2\pi n$ for all integers n. These are the points of inflection. Since at $(x_5)_1 = 2\pi n$ the second derivative changes sign from $+$ to $-$, the concavity changes from upward to downward. Since at $(x_5)_2 = \pi + 2\pi n$ the second derivative changes the sign from $+$ to $-$, the concavity changes from upward to downward. At $x_6 = \arccos(1/4) + 2\pi n$ the second derivative changes the sign from $-$ to $+$, therefore, the concavity changes from down to up. At $x_7 = -\arccos(1/4) + 2\pi n$ the second derivative changes sign from $-$ to $+$, which implies that the function changes concavity from down to up.

5). We can now draw the graph of this function.



Problem 6. Let f be a differentiable function. Prove that if the equation $f(x) = x$ has more than one solution, then there must be a point c at which $f'(c) = 1$.

Proof: By assumption, the equation $f(x) = x$ has at least two distinct solutions. Suppose that a and b are two solutions, $a \neq b$. Then $f(a) = a$ and $f(b) = b$. By the Mean Value Theorem applied on the interval $[a, b]$, there exists a point

$c \in (a, b)$ such that

$$f'(c) = \frac{f(a) - f(b)}{a - b} = \frac{a - b}{a - b} = 1.$$

Problem 7. Compute the limits:

1. $\lim_{x \to \infty} \frac{x^4 + 9x^3 + \pi x^2 - 17x + 106}{3x^4 - 16x^3 - 149} = \frac{1}{3}$ by dividing both the numerator and denominator by x^4.
2. $\lim_{x \to \infty} \frac{5x^3 + 55x^2 + 555x + 5555}{x^8 - .003} = 0$ since the degree of the numerator is less then the degree of the denominator.
3. $\lim_{x \to \infty} \sqrt{x^4 + 2x + 4} - \sqrt{x^4 - 2x - 4}$:

$$\lim_{x \to \infty} \sqrt{x^4 + 2x + 4} - \sqrt{x^4 - 2x - 4} =$$

$$\lim_{x \to \infty} \frac{(\sqrt{x^4 + 2x + 4} - \sqrt{x^4 - 2x - 4})(\sqrt{x^4 + 2x + 4} + \sqrt{x^4 - 2x - 4})}{\sqrt{x^4 + 2x + 4} + \sqrt{x^4 - 2x - 4}} =$$

$$\lim_{x \to \infty} \frac{x^4 + 2x + 4 - x^4 + 2x + 4}{\sqrt{x^4 + 2x + 4} + \sqrt{x^4 - 2x - 4}} = \lim_{x \to \infty} \frac{4(x + 2)}{\sqrt{x^4 + 2x + 4} + \sqrt{x^4 - 2x - 4}} =$$

$$\lim_{x \to \infty} \frac{4(\frac{1}{x} + \frac{2}{x^2})}{\sqrt{1 + \frac{2}{x^3} + \frac{4}{x^4}} + \sqrt{1 - \frac{2}{x^3} - \frac{4}{x^4}}} = 0$$

4. $\lim_{x \to \infty} \frac{\sin^2 x}{x^2}$: First, notice that $0 \leq \sin^2 x \leq 1$. Therefore, $0 \leq \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$. Since $\lim_{x \to \infty} \frac{1}{x^2}$, by the Squeeze Thereom, it follows that $\lim_{x \to \infty} \frac{\sin x^2}{x^2} = 0$.

Problem 1. Approximate $\sqrt{\dfrac{5}{2}}$ using the linear approximation of the function $f(x) = \sqrt{1+3x}$ at $a = 0$ and $a = 1$. Given that $\sqrt{\dfrac{5}{2}} \approx 1.58$, which of the two approximations is better?

SOLUTION: First, notice that $\sqrt{\frac{5}{2}} = f(\frac{1}{2})$. Compute the derivative:

$$f'(x) = \frac{3}{2}\frac{1}{\sqrt{1+3x}}$$

Since $f(0) = 1$, $f'(0) = \frac{3}{2}$, the linear approximation at 0 is

$$L_0(x) = 1 + \frac{3}{2}x,$$

which for $x = \frac{1}{2}$ gives

$$L_0(\frac{1}{2}) = 1\frac{3}{4} = 1.75.$$

Similarly, since $f(1) = 2$, $f'(1) = \frac{3}{4}$, we have

$$L_1(x) = 2 + \frac{3}{4}(x-1),$$

which for $x = \frac{1}{2}$ gives

$$L_1(\frac{1}{2}) = 1\frac{5}{8} = 1.625$$

Since $L_0(1/2) - 1.58 = 0.17$ is bigger then $L_1(1/2) - 1.58$, the approximation at 1 gives better result.

Problem 2. True or False. For each of the following statements, indicate if it is true or false. This problem will be graded as follows: you will receive 4 points for a correct answer, 0 points if there is no answer, and -4 points if the answer is wrong.

1. If $f(x)$ is continuous on $[a, b]$, it is differentiable on (a, b)	False
2. If $f''(a) = 0$, then $f(x)$ can not have a local minimum at a	False
3. The limit $\lim\limits_{x\to\infty} \dfrac{\sin^2 x + 3}{\sqrt{x} + 3}$ does not exist	False
4. The graph of $y = \tan x$ is concave down on $(\pi/2, \pi)$	True
5. If $f'(x)$ is increasing, then $f(x)$ is concave up	True

Problem 3. Find the minimal and maximal values of the function $f(x) = \cos 2x - 2\cos x$ on the interval $[\pi/4, 3\pi/4]$.

SOLUTION: Since $f(x)$ is continuous, it attains its maximum and minimum on the given interval.

$$f'(x) = -2\sin 2x + 2\sin x = -2 \cdot 2\sin x \cos x + 2\sin x = -2\sin x(2\cos x - 1)$$

$f'(x) = 0 \Leftrightarrow \quad \sin x = 0 \,\text{or}\, \cos x = 1/2 \quad \Rightarrow \quad$ on $[\pi/4, 3\pi/4$, the only solution is $x = \pi/3$.

Since at $x = \pi/3$ the derivative changes sign from negative to positive, $\pi/3$ is a local minimum.

$$f(\pi/3) = \cos 2\pi/3 - 2\cos \pi/3 = -\frac{3}{2}$$

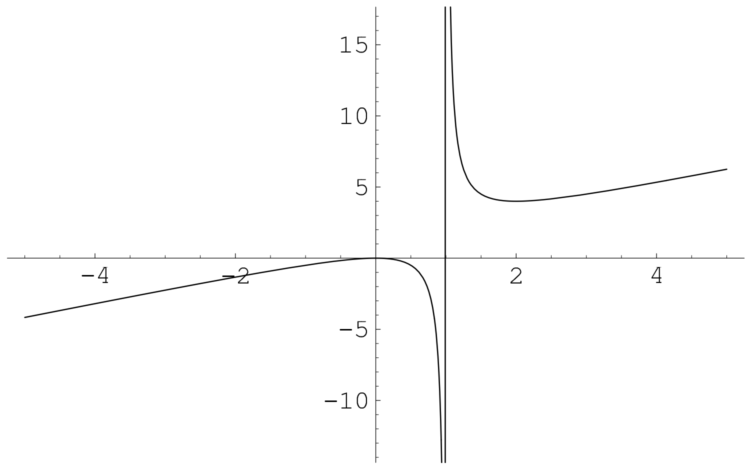
Compute the values at the end points of the interval:

$$f(\pi/4) = \cos \pi/2 - 2\cos \pi/4 = 0 - 2 \cdot \sqrt{2}/2 = -\sqrt{2}$$
$$f(3\pi/4) = \cos 6\pi/4 - 2\cos 3\pi/4 = 0 - 2 \cdot (-\sqrt{2}/2) = \sqrt{2}$$

Comparing the values at $\pi/3$, $\pi/4$ and $3\pi/4$, we obtain that the minimal value is $f(\pi/3) = -3/2$ and the maximal value is $f(3\pi/4) = \sqrt{2}$.

Problem 4. Let $f(x) = \dfrac{x^2}{x-1}$. Compute the first and second derivatives of f and sketch the graph of f, indicating all properties of the function, such as asymptotes, minima, maxima, convexity, points of inflection and intercepts.

SOLUTION:



$f(x)$ is defined on the following domain: $(-\infty, 1) \cup (1, \infty)$. Since $\lim_{x\to 1^+} f(x) = +\infty$ and $\lim_{x\to 1^-} = -\infty$, the line $x = 1$ is a vertical asymptote. Since

$$f(x) = \frac{x^2 - 1 + 1}{x - 1} = (x - 1) + \frac{1}{x - 1},$$

and, therefore, the limit $\lim_{x\to\pm\infty}(f(x) - (x - 1)) = \lim_{x\to\pm\infty} \frac{1}{x-1}$ is zero, the line $y = x - 1$ is a slant asymptote.

$$f'(x) = 1 - \frac{1}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}$$

$$f'(x) = 0 \quad \Leftrightarrow \quad x = 0, x = 2$$

Since at $x = 0$ the derivative changes sign from $-$ to $+$, $x(0) = 0$ is a local maximum. Since at $x = 2$ the derivative changes sign from $+$ to $-$. $x(2) = 4$ is a local minimum.

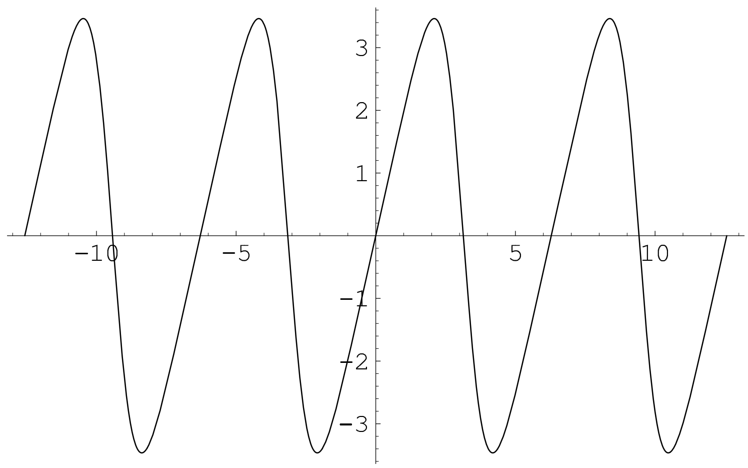
$$f''(x) = \frac{2}{(x-1)^3}$$

Since on $(-\infty, 1)$ $f''(x) < 0$, on this interval the curve $f(x)$ is concave down. Since on $(1, \infty)$ $f''(x) > 0$, on this interval the curve $f(x)$ is concave up.

We can now sketch the graph of $f(x)$:

Problem 5. Let $f(x) = \dfrac{6 \sin x}{2 + \cos x}$. Sketch the graph of this function, indicating all properties of the function, such as asymptotes, minima, maxima, convexity, points of inflection and intercepts.

SOLUTION:



Since $\sin x$ and $\cos x$ are periodic functions with a period 2π, and 2 is a constant, $f(x)$ is periodic with a period 2π. We will consider $f(x)$ on the interval $[-\pi, \pi]$ and sketch its graph on this interval, and then extend it periodically.

$f(0) = 0$, $f(-\pi) = 0 = f(\pi)$.

$$f'(x) = \frac{6 \cos x (2 + \cos x) + \sin x \cdot 6 \sin x}{(2 + \cos x)^2} = \frac{12 \cos x + 6}{(2 + \cos x)^2} = 6 \frac{2 \cos x + 1}{(2 + \cos x)^2}$$

$$f'(x) = 0 \;\Leftrightarrow\; \cos x = -\frac{1}{2} \;\Leftrightarrow\; x = 2\pi/3 \text{ and } x = -2\pi/3$$

Since at $x = 2\pi/3$ the derivative changes the sign from positive to negative, this is a point of local maximum, $f(2\pi/3) = \frac{6 \cdot \sqrt{3}/2}{2 - 1/2} = 2\sqrt{3}$. Since at $x = -2\pi/3$ the sign of the derivative changes from negative to positive, this is a point of local minimum, $f(-2\pi/3) = -2\sqrt{3}$.

$$f''(x) = 6 \frac{-2 \sin x (2 + \cos x)^2 + 2(2 + \cos x) \sin x (2 \cos x + 1)}{(2 + \cos x)2^4} = -\frac{12 \sin x (1 - \cos x)}{(2 + \cos x)^3}$$

$$f''(x) = 0 \;\Leftrightarrow\; \sin x = 0 \text{ or } \cos x = 1 \;\Leftrightarrow\; x = -\pi,\ 0,\ \pi$$

Since the second derivative changes sign from positive to negative at $x = 0$, $f(x)$ is concave up on $(-\pi, 0)$ and concave down on $(0, \pi)$.

We can now sketch the graph of $f(x)$ on $[-\pi, \pi]$ and then periodically extend it:

Problem 6. Prove that the equation $x^5 + 20x + \cos x = 0$ has exactly one solution.

SOLUTION: Let $f(x) = x^5 + 20x + \cos x$. This function is continuous and differentiable everywhere.

Consider $f(x)$ on the interval $[-\pi/2, 0]$. Since $f(-\frac{\pi}{2}) = -\frac{\pi^5}{64} - 10\pi + 0 = -\pi \cdot (\frac{\pi^4}{64} + 10) < 0$ and $f(0) = 1 > 0$, by Intermediate Value Theorem, there is a point $a \in [-\pi/2, 0]$ such that $f(a) = 0$. Therefore, there is at least one root.

Suppose that there are two roots, a and b. By the Mean Value Theorem applied to the interval $[a, b]$, there is a point c on the interval (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a} = 0$. However,

$$f'(x) = 5x^4 + 20 - \sin x$$

and since $5x^4 > 0$ and $20 - \sin x > 0$ (since $|\sin x| \leq 1$), $f'(x) > 0$. We obtained a contradiction. Therefore, there can't be two roots of this equation.

Sample Midterm Math 31A

Student ID : ____________________

First Name: ____________________

Last Name: ____________________

There are a total of 5 problems. SHOW YOUR WORK ON ALL PROBLEMS. Please write clearly.

1. Find all the local maxima and local minima of the function
$$f(x) = x^{\frac{2}{3}}(1-x)^2.$$
Justify your answer.

2. Consider the function
$$f(x) = \frac{x^2}{\sqrt{x^4-1}}.$$
   (i) Find the domain of f.
   (ii) Find the horizontal and vertical asymptotes of $f(x)$, if there are any.

   Justify your answer.

3. Prove that
$$\frac{1}{3} \leq \int_4^6 \frac{1}{x}\,dx \leq \frac{1}{2}.$$
Justify your answer.

4. Consider the point $P = (5,3)$ in the plane.

   (i) Find the slope of the line through P such that the triangle bounded by this line and the axes in the first quadrant has minimal area.

   (ii) Show that P is the midpoint of the hypothenuse of this triangle.

   Justify your answer.

5. Consider the function $f(x) = x^2(x-4)$. Determine the intervals on which the function is concave up or down and find the points of inflection.
   Justify your answer.

1. Given the function

$$f(x) = x^4 - 2x^3 - 12x^2 + 5$$

determine where the function is concave upward and concave downward, and find the points of inflection, if there are any.

$$f'(x) = 4x^3 - 6x^2 - 24x$$

$$f''(x) = 12x^2 - 12x - 24$$

$$= 12(x^2 - x - 2)$$

$$= 12(x-2)(x+1)$$

Interval	$x-2$	$x+1$	$f''(x)$	Concave
$(-\infty, -1)$	neg	neg	pos	upward
$(-1, 2)$	neg	pos	neg	downward
$(2, \infty)$	pos	pos	pos	upward

points of inflection: $x = -1, 2$

2. Consider the function

$$f(x) = \frac{\sqrt{2x^2 - 1}}{x}.$$

(a) Determine the domain of $f(x)$.

(b) Find the horizontal asymptotes of $f(x)$, if there are any.

(a) $x \neq 0 \quad 2x^2 - 1 \geq 0 \quad x^2 \geq \frac{1}{2}$

so $x \geq \sqrt{\frac{1}{2}}$ or $x \leq -\sqrt{\frac{1}{2}}$

Domain: $(-\infty, -\sqrt{\frac{1}{2}}]$ and $[\sqrt{\frac{1}{2}}, \infty)$

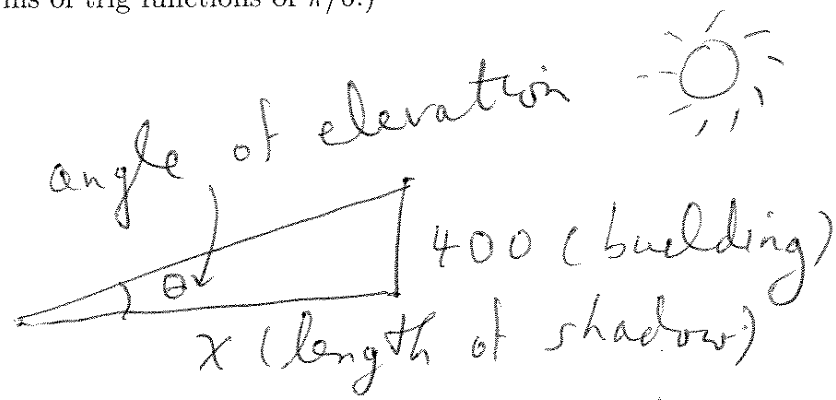
(b) $\displaystyle\lim_{x \to \infty} \frac{\sqrt{2x^2-1}}{x} = \lim_{x \to \infty} \frac{\sqrt{2x^2-1}}{\sqrt{x^2}} = \lim_{x \to \infty} \sqrt{\frac{2x^2-1}{x^2}}$

$\displaystyle = \lim_{x \to \infty} \sqrt{2 - \frac{1}{x^2}} = \sqrt{2}$

$\displaystyle\lim_{x \to -\infty} \frac{\sqrt{2x^2-1}}{x} = \lim_{x \to \infty} \frac{\sqrt{2(-x)^2-1}}{-x}$

$\displaystyle = -\lim_{x \to \infty} \frac{\sqrt{2x^2-1}}{x} = -\sqrt{2}$

3. As the Sun sets, the angle of elevation of the Sun above the horizon is decreasing at the rate of $\frac{1}{4}$ radian/hr. How fast is the shadow cast by a 400-foot-tall building increasing when the angle of elevation of the Sun is $\pi/6$ radians. (You can leave your answer in terms of trig functions of $\pi/6$.)



Note x and θ depend on time (as sun moves)

$$\cot\theta = \frac{x}{400} \qquad \frac{d\theta}{dt} = -\frac{1}{4}$$

$$x = 400\cot\theta$$

$$\frac{dx}{dt} = 400(-\csc^2\theta)\frac{d\theta}{dt}$$

$$= 400\left(-\csc^2\left(\frac{\pi}{6}\right)\right)\left(-\frac{1}{4}\right)$$

4. Find all the local maxima and local minima of the function

$$f(x) = x^{2/3}(1-x)^2.$$

$$f'(x) = \frac{2}{3}x^{-1/3}(1-x)^2 + x^{2/3}\,2(1-x)(-1)$$

$$= 2x^{-1/3}(1-x)\left(\frac{1}{3}(1-x) - x\right)$$

$$= \frac{2}{3}x^{-1/3}(1-x)(1-x-3x)$$

$$= \frac{2}{3}x^{-1/3}(1-x)(1-4x)$$

critical numbers : $x = 0$ (no derivative), 1, $1/4$

Interval	$x^{-1/3}$	$1-x$	$1-4x$	$f'(x)$
$(-\infty, 0)$	neg	pos	pos	neg
$(0, 1/4)$	pos	pos	pos	pos
$(1/4, 1)$	pos	pos	neg	neg
$(1, \infty)$	pos	neg	neg	pos

local min at $x = 0, 1$

local max at $x = 1/4$

5. Suppose that a function $f(x)$ has second derivative $f''(x)$ for all real numbers x.

(a) Prove that there exists a point c in $(0,1)$ such that

$$f(1) - f(0) = f'(c).$$

(b) Prove that if $|f''(x)| \leq 1$ for all x in $(0,1)$, then

$$|f(1) - f(0) - f'(0)| < 1.$$

(a) By the Mean Value Theorem

$$f(1) - f(0) = f'(c)(1-0) = f'(c)$$

(b) By part (a)

$$f(1) - f(0) - f'(0) = f'(c) - f'(0)$$

By the Mean Value Theorem, there is d in $(0,c)$ so that

$$f'(c) - f'(0) = f''(d)(c-0)$$

$$|f(1) - f(0) - f'(0)|$$
$$= |f'(c) - f'(0)| = |f''(d)(c)|$$
$$= |f''(d)|\,|c| < 1$$

because $|f''(d)| \leq 1$ and $c < 1$.