Course Announcement

Hardy Fields

Math 223M, Winter Quarter 2020

MWF 1–1:50 pm, MS 7608

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Office hours. by appointment; MS 5614.

Description. Hardy fields have their origin in the 19th century, with du Bois-Reymond's "orders of infinity". Later, G. H. Hardy made sense of du Bois-Reymond's original ideas, and focused on *logarithmic-exponential functions* (*LE-functions*, for short): these are the real-valued functions in one variable defined on neighborhoods of $+\infty$ that are obtained from constants and the identity function by algebraic operations, exponentiation and taking logarithms. The asymptotic behavior of non-oscillating real-valued solutions of algebraic differential equations can often be described in terms of LE-functions. For example, $x^{\sqrt{2}}$, $\exp(x^2)$, or $x/\log(1+x^2)$ all describe LE-functions. Hardy proved that the germs at $+\infty$ of LE-functions make up an ordered differential field: every LE-function, ultimately, has constant sign, is differentiable, and its derivative is again an LE-function. Bourbaki then took this result as the defining feature of a *Hardy field*: an ordered differential field of germs of real-valued differentiable functions defined on neighborhoods of $+\infty$ on the real line.

The modern theory of Hardy fields was mostly developed by Rosenlicht and Boshernitzan. Recently, Hardy fields have gained prominence in model theory and its applications to real analytic geometry and dynamical systems, via o-minimal structures on the real field. They have also found applications in ergodic theory and computer algebra.

In this course, after an introduction to the basics of Hardy fields, I plan to prove the classical extension theorems for Hardy fields, followed by a self-contained proof of Miller's growth dichotomy theorem for o-minimal structures. In the remainder of the quarter I will explore the elementary theory of maximal Hardy fields.

Prerequisites. Some basic knowledge of first-order logic, model theory, analysis, and abstract algebra should be sufficient. If in doubt about your background, ask me.

References. There will be no textbook. Here are some sources that I will rely on:

M. Aschenbrenner, L. van den Dries, *Asymptotic differential algebra,* in: O. Costin, M. D. Kruskal, A. Macintyre (eds.), *Analyzable Functions and Applications,* pp. 49–85, Contemp. Math., vol. 373, Amer. Math. Soc., Providence, RI, 2005.

L. van den Dries, *Tame Topology and O-Minimal Structures,* London Math. Soc. Lecture Note Series, vol. 248, Cambridge University Press, Cambridge (1998).

C. Miller, *Exponentiation is hard to avoid,* Proc. Am. Math. Soc. **122** (1994), 257–259.

______, *Basics of o-minimality and Hardy fields,* in: C. Miller et al. (eds.), *Lecture Notes on O-minimal Structures and Real Analytic Geometry,* pp. 43–69, Fields Institute Communications, vol. 62, Springer, New York, 2012.

M. Rosenlicht, *Hardy fields,* J. Math. Anal. Appl. **93** (1983), 297–311.