Homework 1

Algorithms for Elementary Algebraic Geometry

Math 191, Fall Quarter 2007

Due Monday, October 8, 2007

1. Let \( \mathbb{F}_2 = \{0, 1\} \), and define two operations \(+\) and \(\cdot\) on \( \mathbb{F}_2 \) by

\[
0 + 0 = 1 + 1 = 0, \quad 0 + 1 = 1 + 0 = 1
\]

and

\[
0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0, \quad 1 \cdot 1 = 1.
\]

Verify that \( \mathbb{F}_2 = \{0, 1\} \), equipped with these operations, forms a field. (You may skip the verification of the associative and distributive properties.)

2. Let \( \mathbb{F}_2 \) be the field from Problem 1. Consider the polynomial

\[
g(x, y) = x^2y + y^2x \in \mathbb{F}_2[x, y].
\]

Show that \( g(x, y) = 0 \) for every \((x, y) \in \mathbb{F}_2^2\). Why does this not contradict what we proved in class?

3. Let \( k \) be a field. Prove that every single point \((a_1, \ldots, a_n) \in k^n\) is an affine variety. Use this to prove that every finite subset of \( k^n \) is an affine variety.

4. Show that the set

\[
X = \{(x, x) : x \in \mathbb{R}, x \neq 1\} \subseteq \mathbb{R}^2
\]

is not an affine variety. (This is Exercise 1.2.8 in the textbook, where you can find some hints.)

5. Consider the equations:

\[
x^2 + y^2 = 1
\]

\[
xy = 1
\]

which describe the intersection of a circle and a hyperbola.

(a) Use algebra to eliminate \( y \) from the above equation.

(b) Show that the polynomial you found in part (a) lies in the ideal \((x^2 + y^2 - 1, xy - 1)\).