Problem Set 3
Solutions

Mathematical Logic

Math 114L, Spring Quarter 2008

1. (a) We proceed by induction on n to show that given a set Σ consisting of n wffs there exists an independent equivalent subset Σ_0 of Σ. If $n = 0$, then there is nothing to show, since Σ is then automatically independent. Suppose $n > 0$. If Σ is already independent, we are done. If not, let $\alpha \in \Sigma$ with $\Sigma' := \Sigma \setminus \{\alpha\} \models \alpha$. Then clearly Σ and Σ' are equivalent: if $\Sigma' \models \beta$ then $\Sigma \models \beta$ since $\Sigma' \subseteq \Sigma$; and if $\Sigma \models \beta$, and v is a truth assignment satisfying Σ', then $\bar{v}(\alpha) = T$ since $\Sigma' \models \alpha$, hence v satisfies $\Sigma = \Sigma' \cup \{\alpha\}$ and thus also β, so $\Sigma' \models \beta$. Since Σ' has $n - 1$ elements, by inductive hypothesis there exists an equivalent independent subset Σ'_0 of Σ'. Then Σ and Σ'_0 are also equivalent. (So we may take $\Sigma_0 := \Sigma'_0$.)
   (b) Consider $\Sigma = \{A_1, A_1 \wedge A_2, A_1 \wedge A_2 \wedge A_3, \dots, A_1 \wedge \cdots \wedge A_n, \dots\}$.
   (c) The equivalent independent subsets are $\big\{\alpha\wedge\beta, \beta\wedge\gamma\big\}$ and $\big\{\alpha\wedge\beta\wedge\gamma\big\}$.
2. Take $\alpha = (A_1 \wedge A_1)$, $\beta = A_1$. Then $(\alpha \wedge \beta) = (\gamma \wedge \delta)$ where $\gamma = (A_1$ and $\delta = A_1) \wedge A_1$, with $\alpha \neq \gamma$.
3. Let v be the truth assignment with $v(A_n) = T$ for all n. We claim that $\bar{v}(\alpha) = T$ for every positive wff α. We show this by using the induction principle. If $\alpha = A_n$ is a sentence symbol, then the claim holds trivially: $\bar{v}(\alpha) = v(A_n) = T$. Otherwise $\alpha = (\beta\Box\gamma)$ where β, γ are positive wffs and $\Box \in \{\wedge, \vee\}$. By inductive hypothesis we have $\bar{v}(\beta) = \bar{v}(\gamma) = T$; hence also $\bar{v}(\alpha) = T$.
4. Can be done in a similar way as the Example on p. 50 of the textbook.
5. Consider the set Σ_n consisting of all wffs of the form $\Box_1 A_1 \vee \cdots \vee \Box_n A_n$ where each $\Box_i$ is either empty or equals $\neg$. So we have

   $$\Sigma_1 = \{A_1, \neg A_1\},\ \Sigma_2 = \{A_1 \vee A_2, A_1 \vee \neg A_2, \neg A_1 \vee A_2, \neg A_1 \vee \neg A_2\}, \text{etc.}$$

   Then every subset of size at most n of Σ_n is satisfiable; we prove this by induction on n, the case $n = 1$ being trivial. Suppose Σ is a subset of Σ_n of size at most n, where $n > 1$. If every wff in Σ has the form $\cdots \vee A_n$ or every wff in Σ has the form $\cdots \vee \neg A_n$ then we are done: any truth assignment v with $v(A_n) = T$ (resp. $v(A_n) = F$) satisfies Σ. So suppose otherwise; so there exists a wff $\cdots \vee A_n$ and a wff $\cdots \vee \neg A_n$ in Σ. Let Σ' be the set of all wffs α such that $\alpha \vee \neg A_n \in \Sigma$. Then Σ' is a subset of Σ_{n-1} of size at most $n - 1$, so by inductive hypothesis there is a truth assignment v' satisfying Σ'. Then v defined by $v(A_i) = v'(A_i)$ for $i \neq n$ and $v(A_n) = T$ satisfies Σ.