Problem Set 1
Solutions

Mathematical Logic

Math 114L, Spring Quarter 2008

1. (30 pt.) By using the Induction Principle for wffs we show that every wff has length 1, 4, 5, or length > 7. This clearly holds for sentence symbols (they have length 1). Suppose α, β are wffs whose length is 1, 4, 5, or > 7. Let a, b denote the length of α, β, respectively. Then $\alpha' = (\neg\alpha)$ has length $a' = a + 3$, so $a' = 4$, $a' = 7$, or $a' > 7$, and $\gamma = (\alpha\Box\beta)$ (where $\Box \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$) has length $g = a + b + 3$, so $g = 5$ or $g > 7$. This shows that there are no wffs of length 2, 3 or 6. To show that every other positive length is possible, we first verify the cases of length 1, 4, 5, 7, and 8 by hand: the wffs

$$A_1, \quad \alpha := (\neg A_1), \quad \beta := (A_1 \wedge A_1)$$

have lengths 1, 4 and 5, respectively. For $n = 7$ and $n = 8$ we consider $(\neg\alpha)$ and $(\neg\beta)$, where α, β are as above. It remains to prove that for every $n \geq 9$ there is a wff of length n, which we do by induction on n. The case $n = 9$ is witnessed by $((A_1 \wedge A_1) \wedge A_1)$. Suppose $n > 9$; then $n - 3 > 6$. If $n - 3 \geq 9$ then by induction hypothesis there is a wff γ of length $n - 3$, and if $n - 3 < 9$, then $n - 3 \in \{7, 8\}$, and as we've seen above, in both cases there is a wff γ of length $n - 3$. Applying the negation operation we get a formula $(\neg\gamma)$ of length n.

2. (30 pt.) Let S be the set of all wffs α for which $s(\alpha) = c(\alpha)+1$, where $s(\alpha)$, $c(\alpha)$ denotes the number of occurrences of sentence symbols respectively binary connective symbols in α. This set clearly contains every α of the form $\alpha = A_k$ for some sentence symbol A_k, since then we have $s(\alpha) = 1$, $c(\alpha) = 0$. Suppose $\alpha \in S$; then for $\alpha' = (\neg\alpha)$ we obtain the same values $s(\alpha') = s(\alpha)$ and $c(\alpha') = c(\alpha)$ as for α, hence $\alpha' \in S$. If $\alpha, \beta \in S$ and $\Box \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$, then for $\gamma = (\alpha\Box\beta)$ we compute

$$s(\gamma) = s(\alpha) + s(\beta) = c(\alpha) + 1 + c(\beta) + 1 = c(\gamma) + 1,$$

hence $\gamma \in S$. Thus S consists of all wffs, by the Induction Principle.

3. (20 pt.) An expression is a finite sequence of elements of a certain set of symbols, consisting of the finitely many logical symbols and the infinitely many sentence symbols $A_1, A_2, \ldots$. The disjoint union $S = F \cup A$ of a finite set F and a countable set A is countable: to see this, let $\Phi\colon A \to \mathbb{N}$ be one-to-one, and suppose $F = \{f_1, \ldots, f_n\}$ has n elements; then $\Psi(f_i) = i$ and $\Psi(a) = \Phi(a) + (n + 1)$ for $a \in A$ defines a one-to-one map $\Psi\colon S \to \mathbb{N}$. Therefore, the set of symbols is countable. Theorem 0B says that if S is

a countable set, then the set of all finite sequences of elements of S is also countable. Hence the set of expressions is countable.

4. (20 pt.) Suppose S is a countable set, and let $S' \subseteq S$. Let $\Phi\colon S \to \mathbb{N}$ be one-to-one. Then the restriction of Φ to S' is a one-to-one map $S' \to \mathbb{N}$, showing that S' is countable. By the previous problem, we know that the set of all expressions is countable. The set of wffs is a subset thereof, whence countable by the above.

5. (30 pt. extra credit.) The following sequence of applications of (P1)–(P4) produces $MUUIU$ from MI:

$$
\begin{aligned}
MI &\underset{\text{(P2)}}{\longrightarrow} MII \\
&\underset{\text{(P2)}}{\longrightarrow} MIIII \\
&\underset{\text{(P1)}}{\longrightarrow} MIIIIU \\
&\underset{\text{(P2)}}{\longrightarrow} MIIIIUIIIIU \\
&\underset{\text{(P3)}}{\longrightarrow} MIUUIIIIU \\
&\underset{\text{(P4)}}{\longrightarrow} MIIIIIU \\
&\underset{\text{(P3)}}{\longrightarrow} MUIIU \\
&\underset{\text{(P2)}}{\longrightarrow} MUIIUUIIU \\
&\underset{\text{(P4)}}{\longrightarrow} MUIIIIU \\
&\underset{\text{(P3)}}{\longrightarrow} MUUIU.
\end{aligned}
$$

I only sketch the solutions of the second part of the problem. We first define (similarly to what we did for wffs) a *construction* sequence to be a finite sequence $\langle s_1, \ldots, s_n \rangle$ of strings s_i consisting of the letters M, U, I with the property that each s_i either equals MIor is obtained from a string s_j with $j \in \{1, \ldots, i-1\}$ by applying one of the rules (P1)–(P4). So a string s is in P if and only if there is a construction sequence as above with $s_n = s$. Next one proves, by induction on n, that for every construction sequence $\langle s_1, \ldots, s_n \rangle$ the number of I's in any of the strings $s_1, \ldots, s_n$ is always congruent to 1 or 2 modulo 3, i.e., of the form $3k+1$ or $3k+2$ for some $k \in \mathbb{N}$. The number of I's in MU is 0, and not of this form. Hence $MU \notin P$.