Pattern Formation in Vertically Vibrated Granular Media

Jeremy Corbett

gtg778a@mail.gatech.edu
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332

August 14, 2003

Abstract

This work concerns pattern formation in vertically driven granular media. We generalize the work of Shankar C. Vankataramani and Edward Ott to test notions about patterns found in vertically driven granular media. A generalization of this model is present along with some of the preliminary results.

1 Introduction

Discovered over 150 years ago, pattern formation in driven systems is a halmark of non-equilibrium physics.[2] Patterned states can be found in systems as diverse as visual hallucianations, chemical reactions, and granular media. Theoretical developments in mathematics have drastically affected understanding of pattern formation. Deterministic systems that exhibit pattern formation are typically described by nonlinear partial differential equations (PDEs) such as the Navier-Stokes equations for fluids, or reaction-diffusion equations for chemical systems.[4] However, not all systems with patterned behavior have known deterministic equations. One example of such a system is granular media. Which is consequently particularly interesting to study.

The motivation for Vankataramani and Ott's[8,9] work started with experiments by Francisco Melo, Paul B. Umbanhowar, and Harry L. Swinney,[6] who studied an apparatus consisting of two plates enclosing a layer of 0.15-0.18 mm diameter bronze spheres. This layer was then periodicly forced.

By varying the frequency and dimensionless acceleration, a wide variety of patterns were obtained–including stripes, squares, and hexagons. These patterns arose spontenously, not as a result of specific boundary conditions. In other words, the patterned behavior was in the intrinsic dynamics of the bronze spheres, where complete understanding is an important problem.

This situation motivated the work of Shankar C. Venkataramani and Edward Ott to develop a simple framework to describe the experiments.[8,9] Using a framework that they called continuum coupled maps (CCM) they created a simulation with striking qualitative agreement with the experiments of Melo et al.[6] This framework forms the basis of this project.

The only direct approach available to model granular media involve particle simulation. While phenomenological models (such as CCMs) can be used, their connection to the actual phenomena is not entirely clear.[1]

One can use a PDE model to provide a good description of the phenomenon which remain simple enough to allow for analytic investigation. Such methods often give insight that compliment numerical simulations.[3]

Vankataramani and Ott test Melo et. al[6] hypothesis that the patterned states they observed were a result temporal period doubling and a standing wave instability. Modeling the dynamics of sand in discrete time, as done in the CCM model, allows for period doubling to be include in a map the exhibits period doubling. Discrete time also makes the length of orbits a much more intuitive and testable phenomenon.

While there are apparant drawbacks with using phenomenological models and–in particular- models that discretize time or space, there are substantial computational gains offered of PDEs or simulations. Moreover, the spatial operator in the CCM model can efficently implemented using the fast Fourier transform techniques.

2 Continuum Coupled Map model

The CCM uses a continuous field while having discrete time. I will briefly present the model used for the simulations, which is presented more thoroughly by Vankataramani and Ott.[8,9]

Let $\mathbf{x} \in \mathbb{R}^2$. At time $n \in \mathbb{N}$, the height of the granular layer at position $\mathbf{x}$ is $\xi_n(\mathbf{x})$. To step $\xi_n(\mathbf{x})$ forward in time a one dimensional map M is applied to every $\mathbf{x}$ to obtain $\xi_n^{'}(x)$,

$$\xi_n^{'}(\mathbf{x}) = M[\xi_n(\mathbf{x}), r] \tag{1}$$

where r is a parameter of the map. A linear spatial coupling operator is then applied to the spatial field.

$$\xi_{n+1}(\mathbf{x}) = \mathcal{L}[\xi_n^{'}(\mathbf{x})] \tag{2}$$

Assuming translational invariance $\mathcal{L}$, is of the form

$$\mathcal{L}[\xi_n^{'}(\mathbf{x})] = f(\mathbf{x}) \otimes \xi_n^{'}(\mathbf{x}) \tag{3}$$

where $\otimes$ denotes convolution.

Let $\bar{\xi}_n{}'(\mathbf{k})$ and $\bar{f}(\mathbf{k})$ be the spatial Fourier transforms of $\xi_n^{'}(\mathbf{x})$ and $f(\mathbf{x})$ respectively. This yields

$$\bar{\xi}_{n+1}(\mathbf{k}) = \bar{f}(\mathbf{k})\bar{\xi}_n{}'(\mathbf{k}) \tag{4}$$

The framework is then completely determined by the choices of M and $\bar{f}(\mathbf{k})$:

$$M(\xi, r) = r\exp[-(\xi-1)^2/2] \tag{5}$$

This map is similar to the logistic map, but does not have any orbits that escape to negative infinity.[8,9]

The assumption of isotropy implies f is function of $k = |\mathbf{k}|$. The Fourier transform of this function is given by

$$\bar{f}(k) = \phi(k)\exp[\gamma(k)] \tag{6}$$

where ϕ and γ satisfy

$$\begin{aligned} \gamma(k) &= \frac{1}{2}\left(\frac{k}{k_0}\right)^2\left[1 - \frac{1}{2}\left(\frac{k}{k_0}\right)^2\right], \\ \phi(k) &= \mathrm{sgn}(k_c^2 - k^2), \end{aligned} \tag{7}$$

where $\mathrm{sgn}(y) = 1$ for $y \geq 0$ and $\mathrm{sgn}(y) = -1$ for $y < 0$. The function γ has the property that $f(0) = 1$ wit maximum at $k = k_0^{-1}$, where k_0^{-1} is the characteristic length of the model. The function $\bar{f}(k)$ decays quickly for $k > k_0^{-1}$ to discourage interactions that are longer than the characteristic length.

The system has two parameters: r and $(k_c/k_0)^2$. The former is roughly analogous to the dimensionless acceleration, whereas the latter is analogous to the forcing frequency. The patterns and bifurcation diagrams are similar qualitatively to patterns produced by in the granular media experiments.[6]

The first stage of the work this summer invovled duplicating the patterns given in Vankataramani and Ott.[8,9] The following four figures are sampling of the patterns that I produced, along with what figure they correspond to in Venkataramani and Ott.[9]

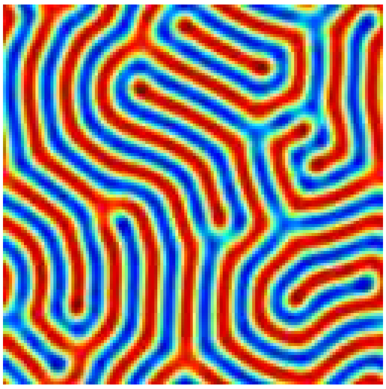

Figure 1: Period-2 stripes obtained with $r = 1.9$ and $(k_c/k_0)^2 = 5$. (Fig 3a)[9]

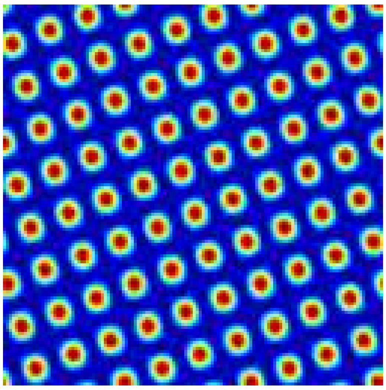

Figure 2: Period-2 squares obtained with $r = 1.9$ and $(k_c/k_0)^2 = 1.5$. (Fig 3b)[9]

3 Two-frequency Forcing

Amoung the most facinating patterns are quasi-patterns (patterns that are not spatially periodic but have rotational symmetry, for an example see

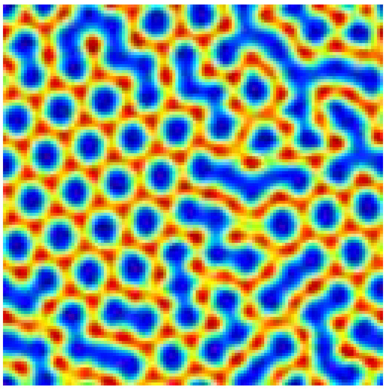

Figure 3: Period-2 hexagons obtained with $r = 2.05$ and $(k_c/k_0)^2 = 2.6$. (Fig 3c)[9]

Gollub and Langer[4]), which can result from multiple frequency forcing,[5] I decided to adapt two-frequency forcing into the model.

To introduce these changes as simply as possible, it ws natural to make $\bar{f}(k)$ a linear combination of two functions that could have different frequencies.

Let $\gamma_0(k) := \gamma(k)$ and $\phi_0(k) := \phi(k)$ so that $\bar{f}_0(k) := \bar{f}(k)$. Define

$$\begin{aligned}\gamma_1(k) &= \frac{1}{2}\left(\frac{k}{k_1}\right)^2\left[1-\frac{1}{2}\left(\frac{k}{k_1}\right)^2\right]\\ \phi_1(k) &= \operatorname{sgn}(k_{c_1}^2-k^2)\\ \bar{f}_1(k) &= \phi_1(k)\exp[\gamma_1(k)].\end{aligned} \tag{8}$$

We then define $\bar{f}_c(k)$ to be

$$\bar{f}_c(k) = \cos^2\theta\bar{f}_0(k)+\sin^2\theta\bar{f}_1(k) \tag{9}$$

where θ represents the relative contribution of each frequency.

The function $\bar{f}_c(k)$ can be used in place of $\bar{f}(k)$ 6 in the original CCM framework as it preserves the quality that $\bar{f}(0) = 1$

This modification (eq. 9) introduces a second forcing frequency.

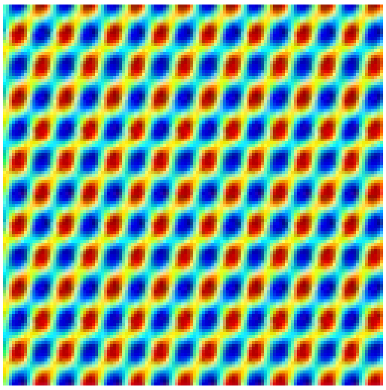

Figure 4: Period-2 "hills" in stripe pattern obtained with $r = 1.73$ and $(k_c/k_0)^2 = 2.5$ (Fig 8d)[9]

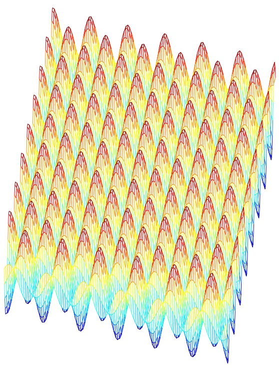

Figure 5: A three-dimensional look at "hills" pattern

4 Initial Findings

The most interesting pattern found so for in the two-frequency model are period-2 quadrilaterals.

Continuation of this work will begin by doing a stability analysis of this pattern. Unfortunately, quasi-patterns have yet to be found in this model.

5 Future Work

There are many directions to take this project from here. The first thing that I would like to accomplish is to find a quasi-pattern in the two fre-

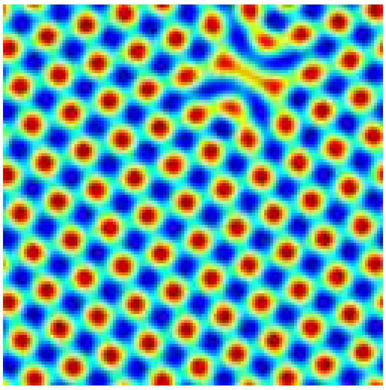

Figure 6: Period-2 quadrilaterals for at $r = 1.9$, $(k_{c_0}/k_0)^2 = 5$, $(k_{c_1}/k_1)^2 = 1.5$, and $\theta = \pi/4 + .1975$

quency model. Mason Porter would like to see all the patterns form the Venkataramani and Ott's papers[8,9]subject to two-frequency forcing model. Shankar Venkataramani has shown interest in the stability or instablity of hexagons in the two-frequency model.[7] After finding a few more patterns and showing their stablility, through linear stablity analysis, I would like to see if they can be reproduced experimentally.

6 Acknowledgements

This project was advised by Shui-Nee Chow and Mason Porter. Special thanks to Peter Mucha for help with numerical implementation of the CCM simulation. This project was funded by VIGRE grant through the math departments REU program.

References

[1] C. Bizon, M. D. Shattuck, J. B. Swift, W. D. McCormick, and Harry L. Swinney. Patterns in 3d vertically oscillated granular layers: Simulation and experiment. *Physical Review Letters*, 80(1):57–60, January 1998.

[2] Michael C. Cross and Pierre C. Hohenberg. Pattern formation outside equilibrium. *Reviews of Modern Physics*, 65(3):851–1112, July 1993.

[3] Jens Eggers and Hermann Riecke. Continuum description of vibrated sand. *Physical Review E*, 59(4):4476–4483, April 1999.

[4] J. P. Gollub and J. S. Langer. Pattern formation in nonequilibruim physics. *Reviews of Modern Physics*, 71(2):S396–S403, March 1999.

[5] Ron Lifshitz and Dean M. Petrich. Theoretical model for faraday waves with multiple-frequency forcing. *Physical Review Letters*, 79(7):1261–1264, August 1997.

[6] Francisco Melo, Paul B. Umbanhowar, and Harry L. Swinney. Hexagons, kinks, and disorder in oscillated granular layers. *Physical Review Letters*, 75(21):3838–3842, November 1995.

[7] Shankar C. Venkataramani. Private communication. July 2003.

[8] Shankar C. Venkataramani and Edward Ott. Spatiotemporal bifurcation phenomena with temporal period doubling: Patterns in vibrated sand. *Physical Review Letters*, 80(16):3495–3498, April 1998.

[9] Shankar C. Venkataramani and Edward Ott. Pattern selection in extended periodically forced systems: A continuum coupled map approach. *Physical Review E*, 63(046202):1–27, 2001.