Synchronization in Antiferromagnetic Oscillator Systems

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A bipartite group of oscillators in which all oscillators within the same group are positively coupled, while any pair between groups is negatively coupled, is known as a antiferromagnetic system. Synchronization of antiferromagnetic oscillators is achieved only when the coupling is greater than a threshold value: in this case, all oscillators within one group are in-phase with one another, and exactly out of phase with those of the other group. In the presence of an external driving force, the phase difference during synchronization is perturbed from the noiseless synchronization phase difference of π. We also show that this change in the phase difference is proportional to the external coupling strength. This work may contribute to the understanding and application of collection enhancement of precision in physical implementations of engineered antiferromagnetic systems.

Global synchronization is a ubiquitous problem arising in fields as disparate as developmental biology, distributed computation, geological magnetodating and antiferromagnetism in certain metals at low temperatures. Antiferromagnetism is a physical property of materials characterized by localized negative interactions. In oscillator networks, antiferromagnetic implies that two groups of oscillators are negatively coupled and possess distinct phases. In positively coupled oscillator systems (ferromagnetic), synchronization is achieved when all oscillators possess roughly equal phase for a prolonged period of time (unimodal distribution). In an antiferromagnetic network, synchronization is defined to occur when the phases of all oscillators fall into a narrow bimodal distribution.

INTRODUCTION

Diverse biological and physical phenomena, from the division of a cell to the cycle of an engine piston, display regular periodic behavior, and can be modelled as oscillators [2, 6]. When many instances of such systems are connected, or located in close proximity to each other, complex global behavior and patterns can emerge from strictly local communication and feedback. This has been the basis of research on cellular automata and neural networks[8]. One simple example of global coordination is synchronization, which can be subdivided broadly into two main classes: ferromagnetic and antiferromagnetic, wherein local coupling is uniformly positive or alternating positive and negative, respectively. Synchronization of oscillation leads to *collective enhancement of precision*, improving the group rhythm to far above the precision of any individual component [3, 7]; thus a group of faulty oscillators may act as a single reliable one.

The exact parameters and details of oscillator coupling can vary greatly. One well-studied example is the Kuramoto model [4, 5]. In this system, each of the N oscillators is coupled to all the other oscillators with the same coupling strength K. The ith oscillator has intrinsic frequency ω_i, drawn from a distribution $g(\omega)$. The dynamics of their phases θ_i are described by:

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N} H(\theta_j - \theta_i), \quad i = 1, \ldots, N. \tag{1}$$

The function $H(\cdot)$ is the phase coupling function, and is often taken to be the sine function. Each oscillator can be expressed as a phasor $z_i = r_k\, e^{i\theta_i}$, where r_i denotes the amplitude and θ_i denotes the phase. The overall synchronization state of the system can be described by a complex-valued *order parameter* χ, defined to be the weighted arithmetic mean of the phasors of all oscillators. However, in our case, all weights are equal.

$$\chi = R\, e^{i\Theta} = \frac{1}{N}\sum_{k=1}^{N} z_k = \frac{1}{N}\sum_{k=1}^{N} r_k e^{i\theta_k}, \tag{2}$$

The magnitude $R = |\chi|$ is defined as the amplitude of the average state. When $R = 1$, it means that the system is perfectly synchronized. In the Kuramato model (1), the amplitudes $r_k = 1$ for all k.

The group frequency can be modulated via an external driving force with phase ϕ and amplitude d. The phase dynamics are then given by:

$$\dot{\theta}_i = \omega_i + d\sin(\phi - \theta_i) + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad i = 1, \ldots, N \tag{3}$$

When the system is synchronized, the phase ϕ is similar to the group rhythm $\Omega = \frac{1}{N}\sum_{k=1}^{N}\theta_k$ studied by Needleman, Tiesinga, and Sejnowski [3]. When the oscillators are synchronized, the group rhythm describes the collective rhythm of the system. The group rhythm Ω follows a Gaussian distribution when a Gaussian noise perturbation is introduced in the system.

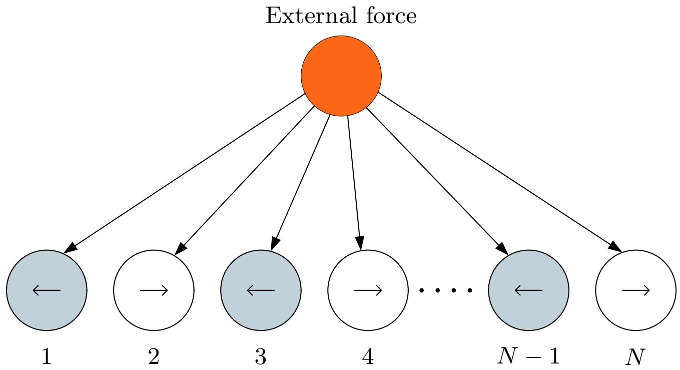

FIG. 1: [Color Online] Model of antiferromagnetic oscillator network. Oscillators are represented by circles, while the circled arrows represent the final phase of the oscillator. All blue (grey) oscillators are positively coupled to each other, and negatively coupled to white oscillators. All white oscillators are also positively coupled to each other.

THE ANTIFERROMAGNETIC SYSTEM

We used a variation of the Kuramoto model to study antiferromagnetic synchronization in a system of oscillators that includes negative coupling between groups of oscillators (Figure 1). The negative oscillator couplings form a bipartite set: odd-numbered oscillators (blue/grey) and even-numbered ones (white). Oscillators in the same odd/even group are positively coupled with each other.

The phases θ_i $(i = 1, \ldots, N)$ of the oscillators are described by:

$$\dot{\theta}_i = \omega_i + d\sin(\phi - \theta_i) + \sum_{j=1}^{N} K_{ij}\sin(\theta_j - \theta_i), \qquad (4)$$

$$K_{ij} = \begin{cases} K_+ & \text{if } (i+j) \text{ is even,} \\ K_- & \text{if } (i+j) \text{ is odd.} \end{cases} \qquad (5)$$

The coupling strengths $K_+ > 0$ and $K_- < 0$ respectively denote the coupling constant between cells of same and different parities. In analysis and simulations presented here, K_+ is abbreviated as K, and K_- is taken to be $-K$. Under these circumstances, all oscillators with odd indices synchronize to a phase different from that of all even indices. Due to positive coupling within the two groups, oscillators in the same group tend to be synchronized with each other, but are out of phase with those in the opposite group. This is known as *antiferromagnetic synchronization.*

SYNCHRONIZATION IN ABSENCE OF EXTERNAL DRIVING FORCE.

Analytically, the simplest case is when the external drive d is equal to zero. To avoid asymmetric boundary conditions, we use a fully connected network of oscillators. In the case of ferromagnetic oscillatory networks, a threshold value existed for K, above which the network eventually synchronizes (order parameter χ approaches 1)[5]. An analogous critical value K_c is expected to exist for the antiferrmagnetic network.

To calculate the value of K_c, I define the antiferromagnetic order parameter to be:

$$re^{i\psi} = \frac{1}{N}\sum_j (-1)^j e^{i\theta_j}. \qquad (6)$$

Algebraic rearrangement and derivation [1] leads to:

$$K_c = \frac{2}{\pi g(0)}. \qquad (7)$$

This result is confirmed by numerical simulations: Systems with coupling constants lower than K_c failed to synchronize, and the order parameter was near zero by the end of the simulations. In contrast, all systems exhibiting coupling constants larger than K_c quickly synchronized. The coupling constant of the external driving force, d, affects the behavior of the antiferromagnetic oscillator network by applying equal force to both groups of oscillators. This partially counters the groups' negative coupling constant, and the minimum energy state of the network has the two groups of oscillators separated in phase by less than π.

When $K > K_c$, one obtains global synchronization; that is, only two distinct phase states exist in the set of all oscillators. We represent these by θ_1 and θ_2. Thus, the dynamics reduces to:

$$\begin{cases} \dot{\theta}_1 = \omega + d\sin(\phi - \theta_1) - K\sin(\theta_2 - \theta_1), \\ \dot{\theta}_2 = \omega + d\sin(\phi - \theta_2) - K\sin(\theta_1 - \theta_2). \end{cases} \qquad (8)$$

By adopting a rotating frame of reference at rotation ϕ, we can set $\phi = 0$.

Zero Drive

We first analyze the degenerate case in which $\omega = 0$ and $d = 0$. Here the external drive is absent. From previous analyses, we expect the resulting phase difference $\theta_1 - \theta_2$ is π. We approach the problem both numerically and analytically.

We randomly selected the initial phase states of the two oscillators from a Gaussian distribution with mean 0 and standard deviation 1. After stabilization, we find that $\theta_1 - \theta_2 \approx 3.1416 \approx \pi$. See Fig. 2 for an example.

We analytically solve the differential equation presented earlier for the $d = 0$ case and compare with the simulation results. To do this, we define two new variables S and D:

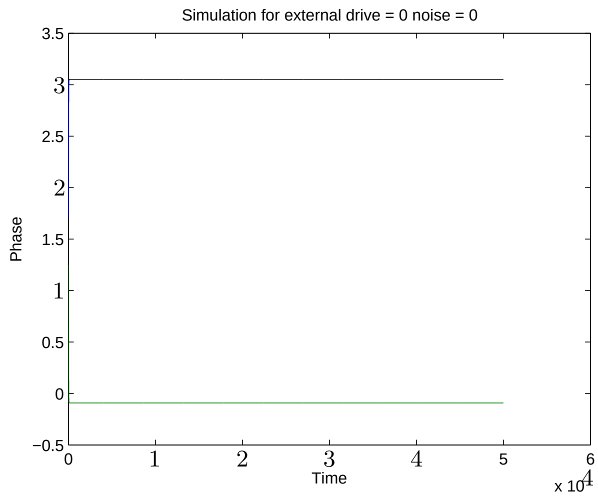

FIG. 2: Two oscillators with no external drive. We selected initial phases randomly from a standard normal Gaussian distribution. Initial frequencies are zero. Coupling constant K was set to be 0.5. The phases of the two oscillator groups stabilized very rapidly, with $\theta_1 \approx -0.0868$ and $\theta_2 \approx 3.0548$, $\theta_1 - \theta_2 \approx \pi$

$$S = \theta_1 + \theta_2$$

$$D = \theta_1 - \theta_2$$

For $\phi = 0$ this gives:

$$\begin{cases} \dot{S} = -d(\sin\theta_1 + \sin\theta_2), \\ \dot{D} = d(-\sin\theta_1 + d\sin\theta_2) + 2K\sin(D). \end{cases} \tag{9}$$

For $d = 0$, we get

$$\begin{cases} \dot{S} = 0, \\ \dot{D} = 2K\sin(D). \end{cases} \tag{10}$$

Rearranging and integrating the above gives

$$e^{-2Kt} = |\frac{1}{\sin D}(1 + \cos D)|.$$

When time goes to infinity, $e^{-2Kt} \longrightarrow 0$ so $|\frac{1}{\sin D}(1 + \cos D)|$ does as well. This gives

$$1 + \cos D = 0,$$

so that

$$\theta_1 - \theta_2 = \pi. \tag{11}$$

Thus, our analysis agrees with the simulation results.

Small Amplitude Drive

When $0 < d \ll 1$, we would expect the results to be very similar to the $d = 0$ case. Define a new variable ξ to denote the deviation of the phase of each group of oscillators from the $d = 0$ case. Because the phases of the two groups of oscillators deviate in opposite ways in the presence of external drive, we write $\theta_2 = \theta_1 - \pi + 2\xi$. In this section, we derive an explicit relation between ξ and d/K.

A typical result of this case is shown in Fig. 3.

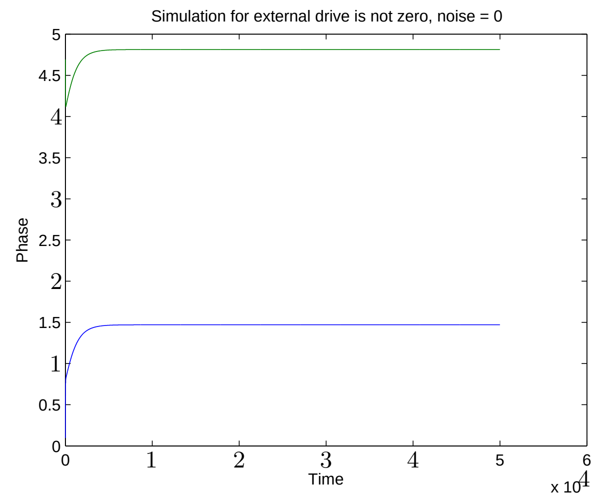

FIG. 3: Phase as a function of time for two oscillators with non-zero external drive. $d = 0.05$, $K = 0.5$. We selected initial phases randomly from a standard normal Gaussian distribution. We let initial frequencies set to zero. At steady state, $\theta_1 \approx 4.8126$ and $\theta_2 \approx 1.4706$. The phase difference $\theta_1 - \theta_2 \approx 3.3420 = \pi + 2\xi$. This gives $\xi \approx 0.1002$

The relation between ξ and d/K is further characterized with repeated runs for various values of d/K. We show our result in Fig. 4.

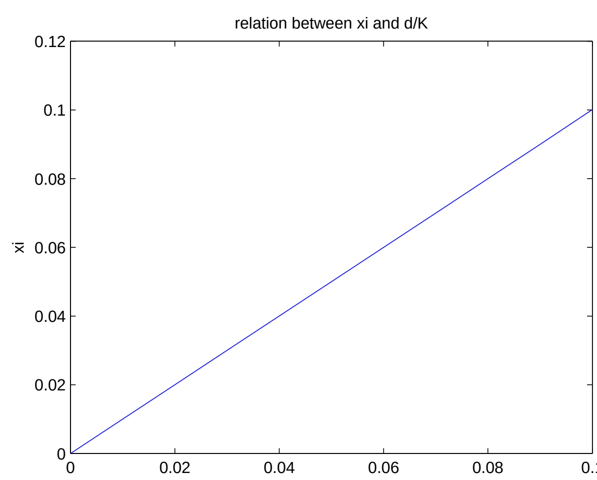

FIG. 4: Relation between ξ and $\frac{d}{K}$

From the figure we can see that ξ is growing proportionately with $\frac{d}{K}$, when $\frac{d}{K}$ is small.

For convenience, we define $\tilde{d} = \frac{d}{K}$. The differential equations describing the trajectory of the phases of the oscillators (9) can be rearranged to yield:

$$\begin{cases} \dot{\tilde{S}} = -\tilde{d}(\sin\theta_1 + \sin\theta_2) \text{ and} \\ \dot{\tilde{D}} = 2\sin(\theta_1 - \theta_2) - \tilde{d}(\sin\theta_{-}\sin\theta_2). \end{cases}$$

(in the equations above $\dot{\tilde{S}}, \dot{\tilde{D}}$ represent $\frac{\dot{S}}{K}$ and $\frac{\dot{D}}{K}$.)

This gives:

$$\begin{cases} \dot{\tilde{S}} = -\tilde{d}(\sin(\frac{S+D}{2}) + \sin(\frac{S-D}{2})) \text{ and} \\ \dot{\tilde{D}} = 2\sin(D) - \tilde{d}(\sin(\frac{S+D}{2}) - \sin(\frac{S-D}{2})). \end{cases}$$

Because we assumed that $\theta_2 \approx \theta_1 - \pi + 2\xi$, we obtain:

$$D = \pi - 2\xi = \pi + \zeta(\zeta = -2\xi).$$

Hence,

$$\dot{\tilde{D}} = \dot{\zeta} = -2\sin\zeta + \cos(S/2)\cos(\zeta/2)2\tilde{d} \qquad (12)$$

We Taylor expand $\tilde{D}$ and $\tilde{S}$ to obtain:

$$\begin{cases} \dot{\tilde{D}} \approx -2\zeta + \cos(S/2)2\tilde{d} \text{ and} \\ \dot{\tilde{S}} \approx -2\tilde{d}\sin(S/2)\zeta/2. \end{cases} \qquad (13)$$

At steady state, set $\dot{S} = 0$ and $\dot{D} = 0$. From the equation $\dot{\tilde{S}} = 0$, we know either $S = 0$ or $\zeta = 0$. But because we are trying to figure out the relationship between ζ and $\tilde{d}$, we cannot set $\zeta = 0$. Plugging this result into (12) gives

$$\zeta = \tilde{d} = \frac{d}{K} \qquad (14)$$

This analysis comfirms the result with numerical simulation.

CONCLUSIONS

The synchronization properties of antiferromagnetic oscillator systems were studied in both the absence and presence of external drive. In order to effect global synchronization, the coupling strength between oscillators must be larger than a critical $K_c = \frac{2}{\pi g(0)}$. After synchronization, the phase difference between the two groups of oscillators tends towards $\pi - \frac{d}{K}$. Further work on noise-prone oscillators and non-uniform coupling constants K may lead to insight on network robustness and dynamic behavior of locally-coupled systems.

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