Recurrences in multiple-particle Billiard systems

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September 23, 2010

1 Abstract

We investigate recurrence statistics of two-particle circular billiards. Dynamics of this billiard is generated by free-motion of two particles (with non-zero radius) inside a closed surface with piecewise smooth circular boundary in $2-D$ Euclidean space. Particles collide elastically against the boundary and each other, with angle of incidence equal to angle of reflection. We are interested in recurrences of time interval needed for both particles to come back to certain area. We used time series, recurrence plots and Poincaré sections in order to investigate this time recurrences. We compare our results with ones obtained from [3, 5, 13]. Although more investigation and statistics (as well as simulation running time) is needed, we were still able to make conclusions about correlation decay of certain variables in our system, as well as of frequencies and periodicity of appearance of integrable regions in our phase space. Correlation decay for position and velocity obeys a power low with coefficient between -0.2 and -0.8, whilst decay of probability of first return time obeys power low with coefficient approximately -1. We make conjecture about dependence of this coefficient on rate of inner and outer circle radius.

2 Modelling circular billiard system

In a billiard, a point particle of mass m and momentum $p = mv$ moves freely until it encounters boundary or the other ball. Without loss of generality, one can take $m = |v| = 1$. If the ball hits the boundary normal component of it's momentum becomes

$$p'_n = p_n - 2\frac{\langle p, \vec{n}\rangle}{\vec{n}} \tag{1}$$

where $\vec{n}$ is unit normal to the boundary ∂Q at the collision point, and billiard surface $Q \in R^2$. If balls hit each other the change in momentum of one ball obeys same formula, but with $\vec{n}$ being normal to the boundary line of the other ball. Two-particle billiard is Hamiltonian system with a $4D$- dimensional phase space $x = (q1, p1, q2, p2)$ and potential $V(q) = 0$ for $q \in Q$, and $V(q) = \infty$ for $q \in \partial Q$. To define a billiard flow we use Poincaré section defined by Birkhoff coordinates s_n and p_n. Here s_n is arc length along the boundary of n^{th} ball-wall collision (any point can be the starting point because of the symmetry of the system), and $p_n = |p| \cos \phi_n$, where ϕ_n is angle between the normal to boundary and outgoing direction [11]. As we have assumed $|p_1| = |p_2| = 1$ hence eliminated two coordinates, and using Poincaré section we eliminate another one, our Poincaré return map is (4D-3)-dimensional. The configuration space for two hard balls of radii R_1 in a circular container ∂Q of radius R is Q_2, the direct product of 2 copies of the circle of radius $R - R_1$ reduced from ∂Q, from which one should remove cylinder which corresponds to the interactions of the particles. Even though we can explain topology of the configuration space quite simply, mixing and ergodic properties of this system are still not completely understood [7].

Main tool we used was a simulation of two particle circular billiard in Matlab programming language. Running this simulation for different initial positions and velocities of particles as well as different radius, we obtained time series for position, angle of collision and recurrence time probability, which we studied using correlation plots and statistics techniques.

2.1 Billiard types, chaos and Lyapunov exponent

Lyapunov exponent measures exponential decay in chaotic systems, and can be described as the mean rate of separation of trajectories of the system. At time t it is defined as parameter λ, where $|\delta x(t)| = e^{\lambda t}|\delta x(0)|$ and $\delta x(t)$ is the separation of two paths starting on a distance $\delta x(0)$. Dynamic of the system is only predictable up to Lyapunov time, which is defined as time until two neighboring trajectories diverge by e. Positive Lyapunov exponent by itself isn't enough for appearance of chaos, and we also need mixing of trajectories. Hence after long enough period of time paths can get arbitrarily close to each other. Usually chaotic properties are investigated when the Lyapunov time gets small enough in comparison with observational time of the system.

We calculated Lyapunov exponent for our system by doing small perturbation of initial conditions for both balls and calculating distances between two paths obtained for these two starting positions and velocities. We have calculated Lyapunov exponent for position coordinates as well as for velocity coordinate v_x (we didn't do calculation for v_y as the magnitude of velocity is constant). For Lyapunov exponent at time t we used following formula, as stated in [14]:

$$\lambda = \frac{\log(\parallel \delta Z(t) \parallel / \parallel \delta Z(0) \parallel)}{t} \tag{2}$$

where t is time in discrete time steps of 0.2s, and $Z(t)$ is system parameter (position or velocity) at time t (time step 0.2 has been chosen as big enough for simulation efficiency, and small enough to get precise results).
To get Lyapunov exponent for the system we averaged above values for different time of our system transience, and took the biggest exponent of the ones obtained for different variables in our system - position and velocity coordinates.

Billiards can have chaotic, regular or mixed dynamics. Circular one-point billiard is completely defined by radius of the circle. Circular and elliptical one-point billiards have regular dynamics and are integrable. This means that number of constants of motion equals number of degrees of freedom of the system. Chaotic behaviour appears if the system is deterministic, locally unstable (has positive Lyapunov exponent) and globally mixing (has positive *entropy* -growth rate of number of topologically distinct trajectories).

Poincaré vision of chaos is the interplay of local instability (unstable periodic orbits) and global mixing (intertwining of their stable and unstable manifolds). In a chaotic system any open ball of initial conditions, no matter how small, will in finite time overlap with any other finite region and in this sense spread over the extent of the entire asymptotically accessible state space [11].

A dynamical system is *ergodic* if for some smooth well-behaving function f describing it, space average equals time average, except for trajectories on set of measure zero. We can also say that a system is *ergodic* if any function that is constant on orbits is constant on the whole space.

Mushroom billiards exhibit mixed dynamics, hence if a ball is confined only to the cap of the mushroom the behaviour of the system is regular, but if the ball gets in the stem it becomes chaotic. Another example of chaotic behaviour in billiards is Sinai billiard, which represents rectangular billiard with a circle taken out from it's center. Diamond billiard (table bounded by 4 circles curved inwards) and stadium billiard are also well investigated types of both chaotic and ergodic billiard system [2, 13, 6].

Billiard is integrable if divergence and convergence of neighboring orbits are balanced (Elliptic), defocusing if divergence of neighboring orbits (in average) prevails over convergence (Stadium) and dispersing if neighboring orbits diverge (Sinai blliards).

By putting two particles inside a circle container we change the dynamics of circular billiard, and balls show chaotic behaviour [13]. By Poincaré recurrence theorem, every system will after long enough transience come back approximately close to its initial state (to the inner circle of radius R_{in} around initial point). It is interesting to study the statistics for this time depending on how close to initial state we want our system to return to. We start with circle of radius $R_{in} = 0.5$ around initial point, hence outer circle has two times bigger radius then inner one, increasing this rate(R/R_{in}) and investigating impact of it on properties of our system.

3 Recurrence plot

Recurrence plot (RP) was introduced in 1987 by Eckmann as a tool to visualize the recurrence of states x_i in a phase space. Higher dimensional phase spaces can only be visualized by projection to $2-D$ or $3-D$ space, but RP technique enables us to analyze this systems with help of $N \times N$ matrix, where N denotes time of our system evolution [9]. We can represent an RP mathematically as

$$R_{i,j} = \Theta(\varepsilon_i - \|x_i - x_j\|), x_i \in \mathbb{R}^m, i, j = 1, ..., N \tag{3}$$

where ε_i is a threshold distance, $\|.\|$ a norm and Θ is the Heaviside function

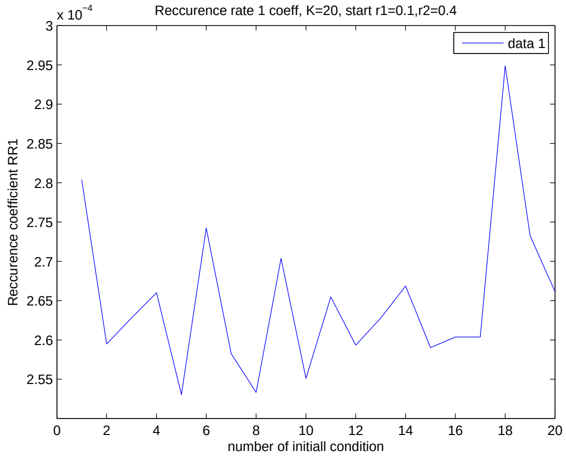

Figure 1: Recurrence rate for ball 1 for 20 different uniformly random initial conditions, balls radius 0.02, outer circle of radius 0.9

[9]. We have made RP of evolution of position of particles in time, hence our x_i was distance of particle from the center, and we were interested in how close and often will particle come back to points of its trajectory. Following the suggestions of [9] as we have circular symmetry in our system, we have taken 0.01 (one percent of circle radius) for a threshold distance ε , and L_2-norm for vector distances. Then using recurrence quantification analysis (RQA) we measure complexity of our billiard system. One of the measures of RQA is Recurrence rate coefficient (RR) which measures the density of recurrence points in the RP and is defined as

$$RR(\varepsilon) = \frac{\sum_{i=1}^{N}(R_{i,j})}{N^2} \tag{4}$$

As $N \to \infty$ RR represent the probability for a state to recur to its ε neighborhood in the phase space [9].
On figures 1 and 2 we show RR coefficient for 20 uniformly random initial conditions, when balls radius is 0.02, and outer circle radius is 0.9, from which we can see that starting position influences this coefficient, as the consequence of chaotic properties of our system.

4 Autocorrelation and Statistics

Autocorrelation is the tendency for observations made at adjacent time points to be related to one another. It refers to the correlation of a time series with its own past and future values. *The correlation coefficient* between series of observations x and y is given by

$$r_l = \frac{\sum_1^n (y_i - \langle y \rangle)(x_i - \langle x \rangle)}{\sqrt{(\sum_1^n (y_i - \langle y \rangle)^2)(\sum_1^n (x_i - \langle x \rangle)^2)}} \tag{5}$$

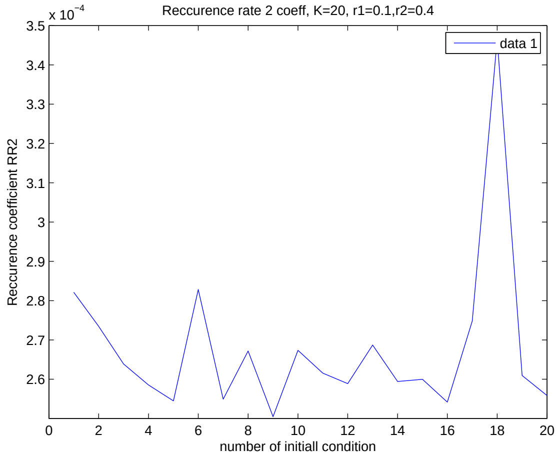

Figure 2: Recurrence rate for ball 2 for 20 different uniformly random initial conditions, balls radius 0.02, outer circle of radius 0.9

where the summations are over the n observations, and $\langle x \rangle$, $\langle y \rangle$ are mean values of x and y coordinates, respectively. A similar idea can be applied to time series for which successive observations are correlated. Instead of two different time series, the correlation is computed between one time series and the same series lagged by one or more time units. Hence *autocorrelation coefficient* is calculated as

$$R_l = \frac{\sum_1^{n-l}(x_i - \langle x \rangle)(x_{i+l} - \langle x \rangle)}{\sum_1^n (x_i - \langle x \rangle)^2} \tag{6}$$

Negative autocorrelation shows that the direction of the influence is changing in successive time intervals, hence if at time t_1 values were above average, at time t_2 we would have them below the average. Positive autocorrelation might be considered a specific form of "persistence", a tendency for a system to remain in the same state from one observation to the next.

Autocorrelation plots (correologrems) are a commonly-used tool for checking randomness in a data set. This randomness is ascertained by computing autocorrelations for data values at varying time lags. For random data, such autocorrelations should be near zero for any and all time-lag separations, and if a time series has a regular pattern, then a value of the series should be a function of previous values.

5 Results

We are interested in investigating autocorrelation decay for x and y coordinate of balls position and velocity, as well as the angle of collision with the wall. When we have long sequence of wall hits without balls hitting each other in

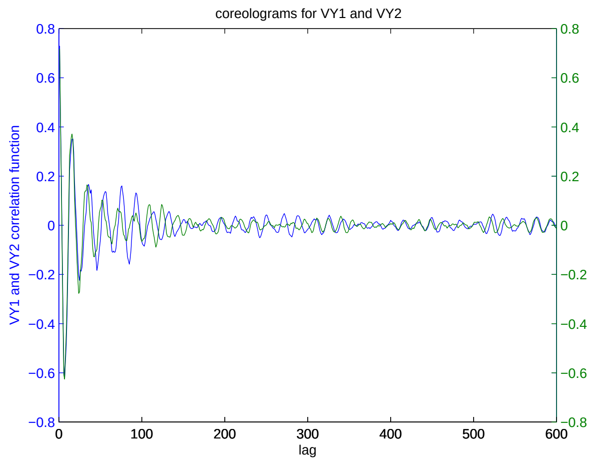

Figure 3: Autocorrelation plots for v_{y1} (blue) and v_{y2} (green)

between, the system is predictable and stable, and time series for both balls show periodic behaviour, whilst on Poincaré section we find integrable regions. Also, when balls hit each other, the system is no longer predictable, and positions and velocities of both balls show chaotic behaviour in time. We proceed on investigating correlation plots for just one particle, because they will both have peaks at same places as shown on Figure 3.

Figures 4 and 5 show us change in rate of ball vs wall hits for 20 different initial conditions, uniformly spread around the circle. From Figure 4 we see that rate between ball and wall hits, which we denote by hitBW, becomes bigger as radius of balls gets bigger, which is as expected as there are more ball collisions for bigger ball radius. Mean value of number of ball versus wall hits is 0.1059 with minimum 0.0131 and maximum 0.3493 (hence standard deviation is very big, 0.1402).

Phase space of dependence of incident angle against arc-length of collision on boundary (Figure 6) show us the chaotic behaviour of our billiard system, as well as appearance of some stable areas. Horizontal lines that appear are integrable regions in which balls were not hitting each other but moving as they are alone in the circle - hitting the wall with same angle. When the ball collision happens behaviour becomes chaotic again.

By plotting angle of wall collision (Figures 7 and 8) we again see integrable regions where straight lines appear on these plots (hence ball was only colliding with the wall). If we annotate with $n_1, n_2, n_3, ...$ number of collisions with the wall for each of these integrable regions (hence n_i is for region i) it's an open question to investigate the pattern appearing among this sequence n_i. We are particularly interested if we can get statistics of this sequence for different initial conditions by considering one random initial condition and letting our simulation run for long enough period.

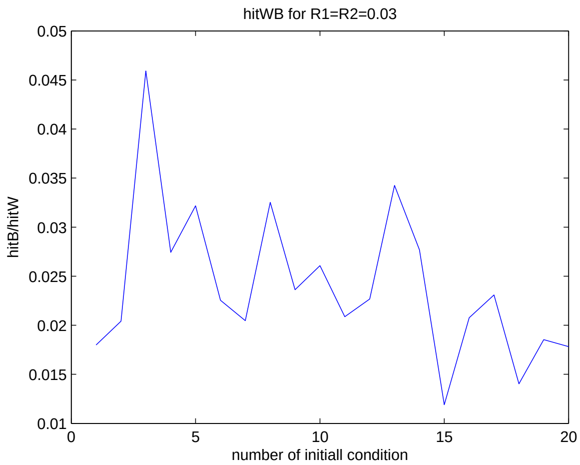

Figure 4: rate of number of ball and wall collision for 20 uniformly randomly picked initial conditions and radius of balls 0.03, circle R=1

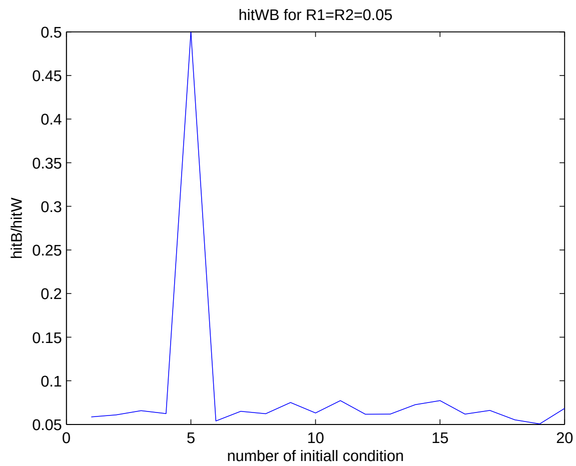

Figure 5: rate of number of ball and wall collision for 20 uniformly randomly picked initial conditions and radius of balls 0.05, circle R=1

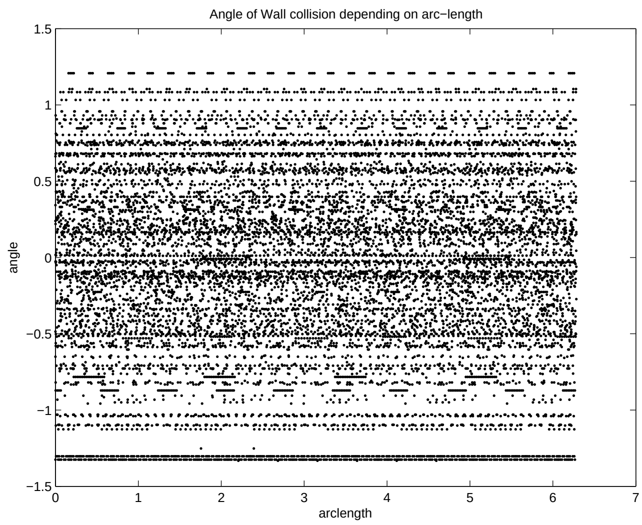

Figure 6: Angle of ball to wall collision depending on arc length of collision point

5.1 Recurrence plots and Recurrence rate coefficient

In Table 1 we show the mean, minimum and maximum values of RR (Recurrence rate) coefficient for different ball radius over 20 initial conditions (where starting positions of balls were $r_1 = 0.1, r_2 = 0.3$ and uniformly distributed around circle). If we look at the mean of this coefficient for both balls we notice it is bigger as ball radius becomes bigger (except in case of ball radius 0.035). This is maybe opposite result from expected, because the system should show more chaotic properties as the ball radius becomes bigger, because then we have more collisions among balls. Hence we would expect it will come back close to initial position more rarely. On the other hand, when radius is bigger it's greater possibility that it will cross particular area in the future. To investigate this more we need to go to limit of very big and very small ball radius. In case of radius 0.2 and random initial condition we get RR1 = 0.00027041, and when radius is 0.22 RR1=0.00024625. Hence at the moment we can't make conclusion that RR is bigger for bigger ball radius, but we leave this question for future work. Differences in RR for different ball radius are of order 10^{-5}, while the average value of coefficient is 0.00021569 hence this differences can't be a secure parameter showing the dependence of chaos appearance on ball radius.
When we run our simulation again for balls of radius 0.02 over 20 uniformly distributed initial conditions (so velocity and starting angle of ball2 are distributed around the circle, whilst initial velocity and position of 1st ball are fixed), but with starting positions at $r_1 = 0.2$, $r_2 = 0.4$, we get slightly bigger recurrence rate coefficients, of order 10^{-5}.
In Table 2 Recurrence coefficients for two different initial positions of balls of radius 0.02 are presented.

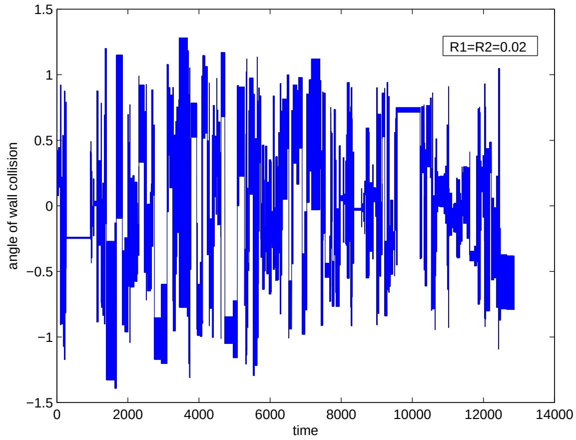

Figure 7: Time series of angle of ball-wall collision

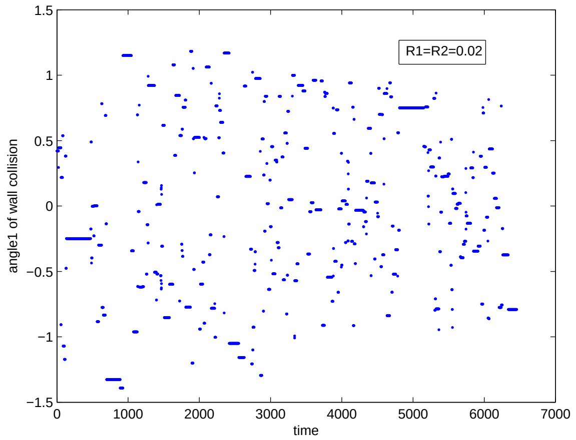

Figure 8: Plot of angle of ball-wall collision for $R_1 = R_2 = 0.02$

RR—statistics	mean	median	min	max	STD
R=0.02	0.0002129	0.0002139	0.0002036	0.0002196	0.000003939
R=0.03	0.00021375	0.00021425	0.000206	0.00022443	0.0000048255
R=0.035	0.0002052	0.0002098	0.0002052	0.0002203	0.0000038
R=0.04	0.0002181	0.0002103	0.0002047	0.0003408	0.00002929
R=0.05	0.0002167	0.0002153	0.0002074	0.000245	0.000007858

Table 1: statistics of RR coefficient for different ball radius

RR2—statistics	min	max	mean	median	std
$r_1 = 0.1, r_2 = 0.3$	0.0002036	0.0002196	0.0002129	0.0002139	0.000003939
$r_1 = 0.1, r_2 = 0.4$	0.0002505	0.0003466	0.0002678	0.0002613	0.00002064

Table 2: Recurrence rate of ball 2 for two initial radial positions of balls

5.2 Probability for balls to come back to certain area

Probability $P(\tau) = P(\tau > t)$, where τ is Poinacaré recurrence time (here first time of balls getting back to the inner circle) decays with a power law $48.82 \times x^{-0.9628}$ in case of averaging over two initial conditions, and as $66.63 \times x^{-0.9038}$ in case of averaging over 20 initial conditions. This is for balls of radius 0.02 in circle of radius 0.9, with radius of inner circle 0.4. We have run the simulation for 6500 iterations. Power regression is in paper by [8] explained to be due to the trapping of chaotic trajectories in the hierarchically structured vicinity of islands of regular motion. In paper by [1] is shown that in Hamiltonian systems with KAM islands of stability, probability density has power regression $t^{-\alpha}$, where $1.5 < \alpha < 2.5$, whilst for probability itself parameter α should be between 2.5 and 3.5. [7] In most Billiard system papers [13],[3],[5],[10] the topics discussed were mushroom, diamond or stadium one-particle billiards. Here we deal with more simple boundary shape (which 1-particle counterpart is completely investigated integrable circular billiard), but the effect of ball collisions brings in chaotic component whose behaviour is still not well understood. By numerical methods we show that recurrence time of correlation decay has smaller absolute value then in above mentioned investigated cases. This shows a difference between one-particle and two-particle billiards, but also leave an open question of influence of boundary shape on correlation decay.

Conjecture: Power regression of probability that τ (mean return time to inner circle of radius R_{in} over certain time interval) is greater then time t, $P(\tau > t)$ depends on the relation between radius of outer and inner circle, R/R_{in}, as well as on R_{in}/R_1, where R_1 is radius of balls. We show this by our numerical results for different R, R_{in}, R_1 and time periods. On figures below are plotted probability function depending on t, for several different conditions. Table below gives some of numerical results we calculated.

K=2, it=3000
$R = 0.9, R_{in} = 0.3, R_1 = 0.02 : y = 45x^{-1}(exclude\, x < 100)$
$R = 0.9, R_{in} = 0.3, R_1 = 0.01, K = 30, it = 5000 : y = 143.2x^{-0.9957}(excludex < 180, x > 1000), y = 245.3x^{-0.9985}(exclude\, x < 1105)$
$R = 1.2, R_{in} = 0.4, R_1 = 0.02$
$R = 1.8, R_{in} = 0.6, R_1 = 0.04 : y = 179x^{-1}(exclude\, x < 210)$
$R = 0.6, R_{in} = 0.2, R_1 = 0.01$
$R = 1.5, R_{in} = 0.5, R_1 = 0.02 : y = 332.1x^{-0.9825}$
$R = 0.6, R_{in} = 0.2, R_1 = 0.0133(exclude\, x < 220)$
$R = 1.2, R_{in} = 0.4, R_1 = 0.026 : y = 114.7x^{-0.975}(exclude\, x < 210)$
$R = 1.5, R_{in} = 0.5, R_1 = 0.033 : y = 120.8x^{-0.9839}$
$R = 1.8, R_{in} = 0.6, R_1 = 0.04 : y = 179x^{-1}(exclude\, x < 210)$
K=20, it=3000
Exact fits when $R/R_{in} = 3, R_{in}/R_1 = 15$
$R = 0.45, R_{in} = 0.15, R_1 = 0.01 : y = 45x^{-1}(exclude\, x < 100, x > 990)$
$R = 0.9, R_{in} = 0.3, R_1 = 0.02 : y = 45x^{-1}(exclude\, x < 80, x > 1000)$
$R = 1.35, R_{in} = 0.45, R_1 = 0.03 : y = 20x^{-1}(exclude\, x < 60, x > 990)$
$R = 1.8, R_{in} = 0.6, R_1 = 0.04 : y = 179x^{-1}(exclude\, x < 210)$

In case of running our system for 5000 seconds averaged over two initial conditions, was noticed that between probability iterations 1000 and 1200 we have

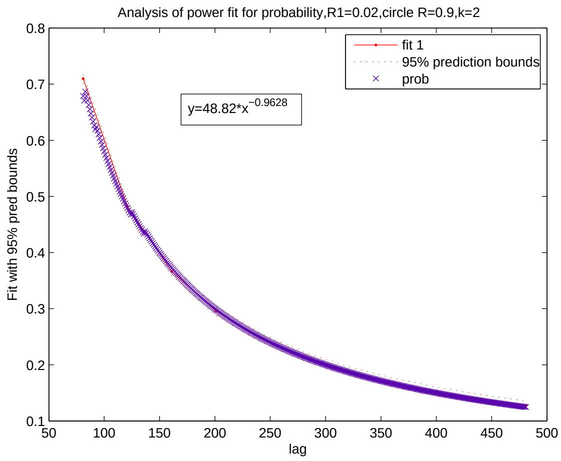

Figure 9: K=2,balls radius 0.02, circle 0.9, inner circle 0.4, 20 initial conditions, time=2000

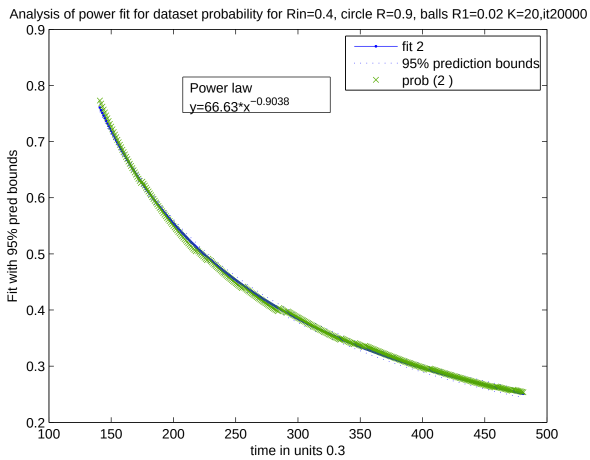

Figure 10: K=20,balls radius 0.02, circle 0.9, inner circle 0.4, 20 initial conditions, time=2000

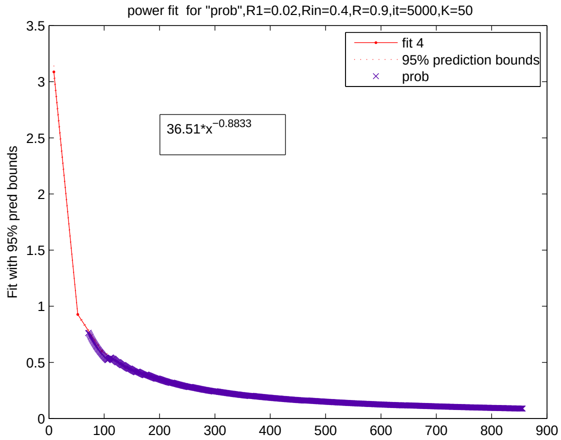

Figure 11: Probability of return time for $R = 0.9, R_{in} = 0.4, R_1 = 0.02, K = 20$ random, uniformly picked initial conditions, time=5000

non-predictable behaviour of probability function, whilst before and after it comes back to power regression with coefficient near -1. This starting behaviour is probably due to the existence of non-integrable regions in which system behaves unpredictable. But after long enough transience it's probability function stabilizes again.We get exact coefficient -1 as on Figure 12, when $R/R_{in} = 3$. On Figures 9, 10 and 11 we represent probability function for several other values of outer and inner radius. We have again excluded some starting points due to the transience needed for system to 'stabilize', start behaving deterministic as well as chaotic. This was also done in paper by [5], and we applied same reasoning when determining power and exponential regression for coordinates of position and velocity. We need to exclude more points as radius of balls and circles becomes bigger, as then system is more chaotic and need longer time to stabilize. We would expect that as the coefficient R/R_{in} becomes bigger, power regression has smaller modulus of coefficient. This is because balls will get into inner circle less frequent, hence time τ will get bigger and there will be more t for which it is satisfied that $t < \tau$.

5.3 Autocorrelation plots (Correolograms)

We have fit exponential curve on angle correologram, which can be obtained by fitting linear function to log-log plot of wall collision angle. This is shown in Figures 13 and 14, for ball radius 0.05, and outer circle R=1, where the exponential regression is approximately $0.69 \times 0.99593^x = 0.69 \times e^{-0.0041x}$. We can also see non-exponential initial decay, which was found for diamond billiards in work by [12]. Possible for same reason in paper by [5], they excluded first two units measured in appropriate units of Lyapunov exponent from their results, due to transient phenomena that are apparent in short time period. On Figure 22 we also show correologram for angle of ball 1 but made with bigger time step.

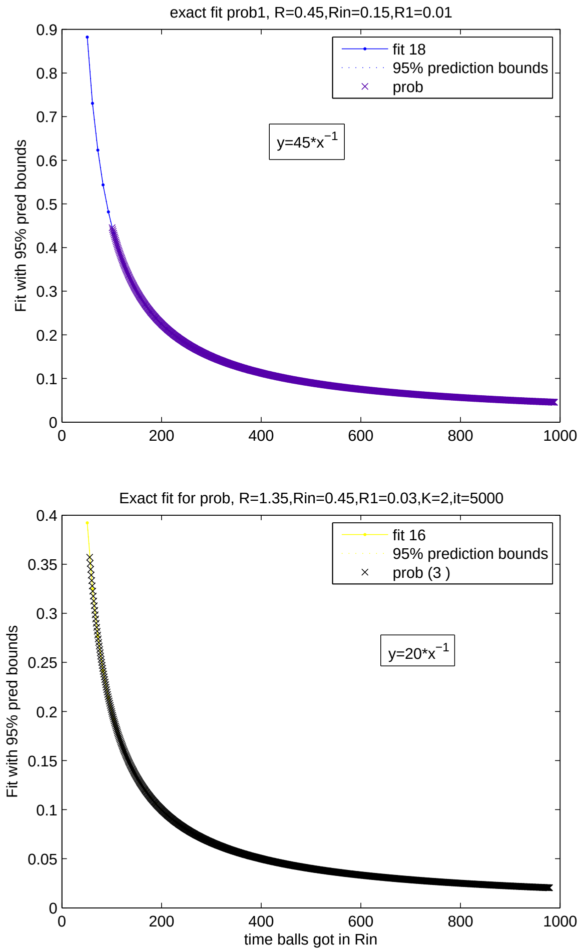

Figure 12: Probability when $R/R_{in} = 3$, power fit with coefficient -1

In paper [4] it was shown that the appearance of long tails in correlation decay is due to integrable regions. Results we get for angle correlation coincide with this, because we make time series of angle with wall collision, and this happens at start of each integrable region(when ball collides only with wall). For a future research is left to determine some rules of behaviour for lengths of this integrable regions by using symbolic dynamics and similar tools.

From autocorrelation plots of x and y coordinate for one ball, we can see the periodic behaviour of autocorrelation coefficient. We average correlation coefficients over initial conditions for particle 2 uniformly distributed around circle (we fix particle 1 position at $r_1 = 0.1, \theta_1 = \pi/3, v_{x1} = 1, v_{y1} = 0$). Then fitting exponential and power regression to it, we find that the power coefficient has modulus less then 1 (average around 0.5).

Like stated in [5], in absence of rigorous asymptotic, we can only check if our numerical results for decay of correlation functions are compatible with some a priori function describing the decay. Hence we test if our time series for correlation functions of position and velocity coordinates can be fitted with exponential function of form $a \times e^{-b \times t}$, where t is time expressed in units of 0.3. Using this method from [5] we try to fit function of the form $e^{-at} f(wt)$, for some periodic function f. In this case, we should be able to fit successive maxima of logarithmic plot for correlation function with several parallel lines. (as f would be free parameter of linear fit, and as it is periodic would only take several different values, hence parallel lines).

On the Figure 15 we show the power fit for x coordinate of ball 1 just for maximum points is $1.2878 \times t^{-0.26843}$ whilst for all points it is $1.304 \times t^{-0.4116}$. But we can see that all maximum points don't fit well in mentioned power law, and following the example of [5] we try to fit several parallel lines to maximum points on log-log plot, in order to get power regression in form $t^{-a} f(wt)$, where f is some periodic function of time. Coefficient a is then as for fit for all points, $a = -0.4116$ as shown on Figure 15. Comparing this result with ones obtained in [3], where in case of mushroom billiards the correlation decays as a power function with modulus of exponent greater then 1, we see that fact of two balls interacting in a simple shape billiard makes decay more more slowly.

Calculation of Lyapunov exponent gave us the mean value over radius 0.03,0.05 of 0.44584 for position, and 10.4162 for Cartesian velocity coordinates, which is the maximum Lyapunov exponent for our system.

Results for exponential decay (Figure 17 and 18) are however similar to those obtained in unpublished part of paper by S.Lansel and M.Porter where for X coordinate they get exponential regression approximately 0.077×0.99025^x. This was in the case of one-particle mushroom billiard.

When comparing period of autocorrelation functions for x coordinate of ball1 for different ball radius, we see that period becomes smaller as radius becomes bigger - this is because of islands of stability being bigger in case of smaller balls, hence correlation coefficient changes less frequent. By reading from graphs of correlation function, we get that the periods for balls of radius 0.02, 0.03, 0.04,

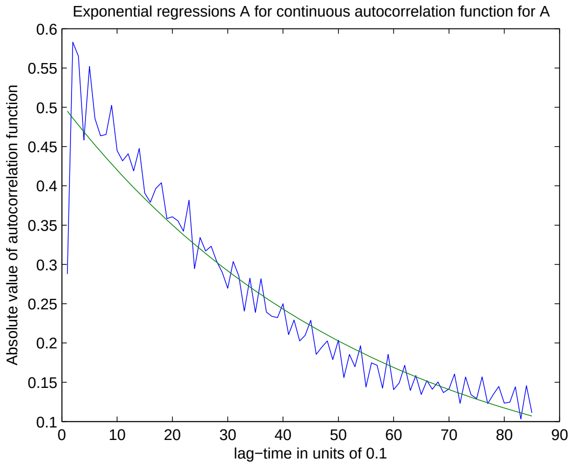

Figure 13: Correlation function and exponential fit for angle of wall-ball1 collision, for radius of balls 0.05

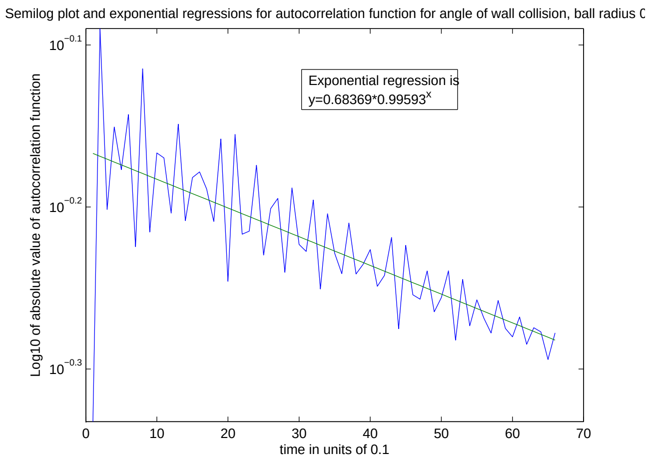

Figure 14: Semilog plot for autocorrelation coefficient of angle of wall-ball1 collision, ball radius 0.05

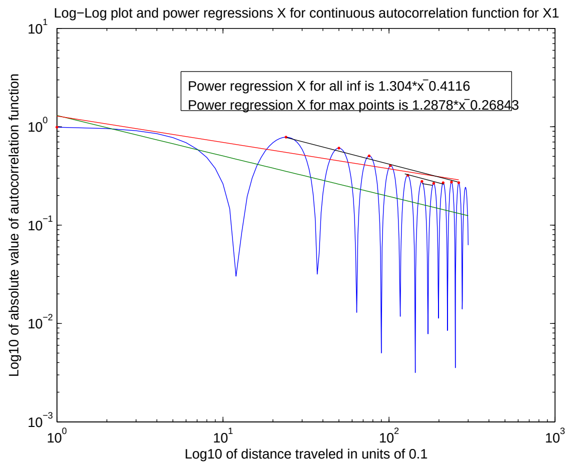

Figure 15: Correlation function of $x1$ coordinate over 15 random, uniformly picked initial conditions, for balls of radius 0.04, outer circle of radius 1 (red=fit for max points, green=fit for all points)

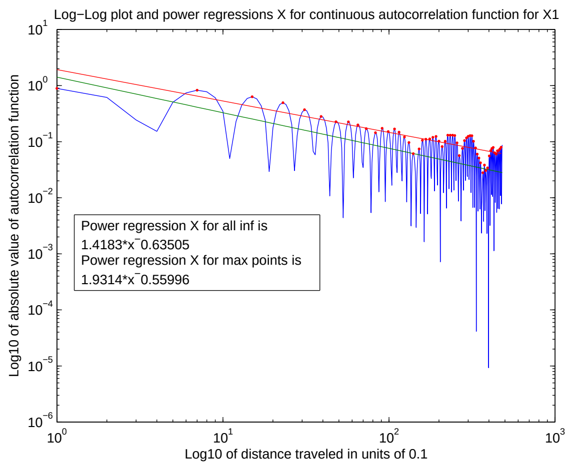

Figure 16: Log-log plot for x coordinate of ball1, balls radius 0.02, outer circle 0.9, inner circle 0.4, 20 random initial conditions and time=2000 (red=fit for max points, green=fit for all points)

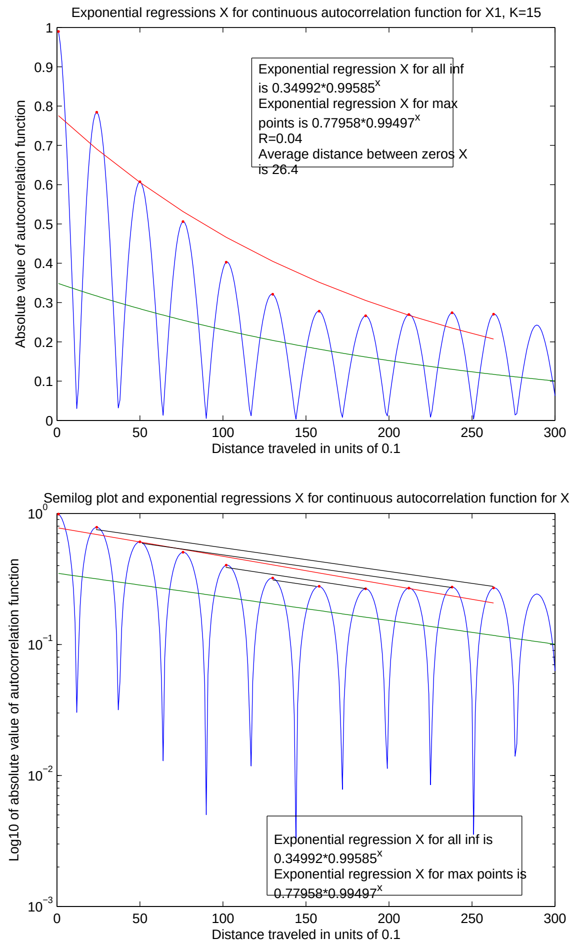

Figure 17: x coordinate correlation function for ball 1 of radius 0.04 averaged over 15 random uniform initial conditions (red=fit for max points, green=fit for all points)

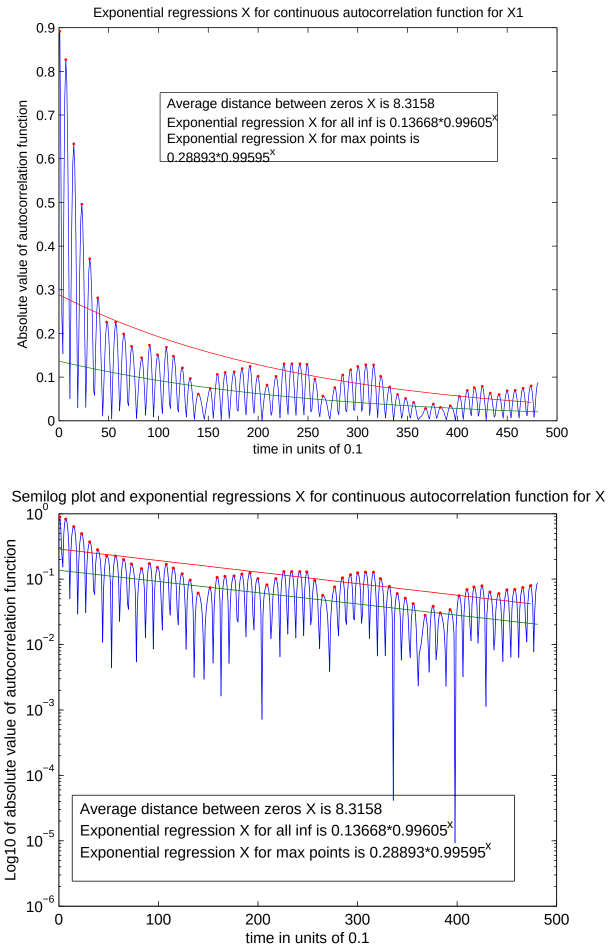

Figure 18: Corr plots for x coord of ball1, balls radius 0.02, circle 0.9, inner circle 0.4, 20 uniformly picked random initial conditions, time=2000 (red=fit for max points, green=fit for all points)

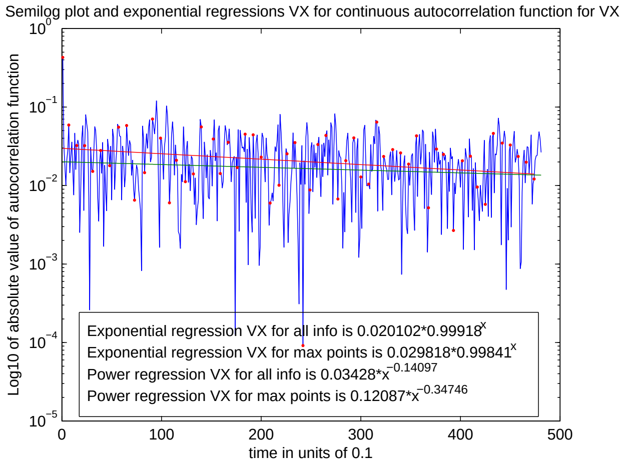

Figure 19: Correlation plot and fit for v_x coord of ball1, balls radius 0.02, circle 0.9, inner circle 0.4, 20 initial conditions,time=2000 (red=fit for max points, green=fit for all points)

0.05, 0.05 are as follows: 12.5, 8.72, 6.66, 4.76, 4.34.

On Figure 19 we see exponential and power fit for x-coordinate of velocity, with power coefficient -0.14 for all info and -0.34 just for max points. Hence velocity correlation is decaying more slowly then position coordinates, which is expected as velocity only changes when collision happens.

5.4 Time series analysis

We have made autocorrelation plots for time interval needed for both balls to get inside circle of radius R_{in} (series of difference between time both get in and previous time at least one of them was out-Figure 20 and 21), as well as of angle of wall collision (Figure 22) . As in case of random time series we expect autocorrelation plot to be around 0, we can see that in angle of wall hitting is very correlated, whilst the time interval series becomes more correlated as the radius becomes bigger. (as we could have expected, as then is the greater probability of both particles getting into smaller circle).The autocorrelation plot of recurrence time series (time intervals denoted by timeRec) averaged over many initial conditions, the correologram has linear regression (best fit is linear with residuals distributed randomly), but scattering of points around 0 show that there is no connection between how correlated series is for different lags we take. Also the mean value of the correlation is close to 0, and only about 10 percent of points get to 0.15 correlation coefficient.

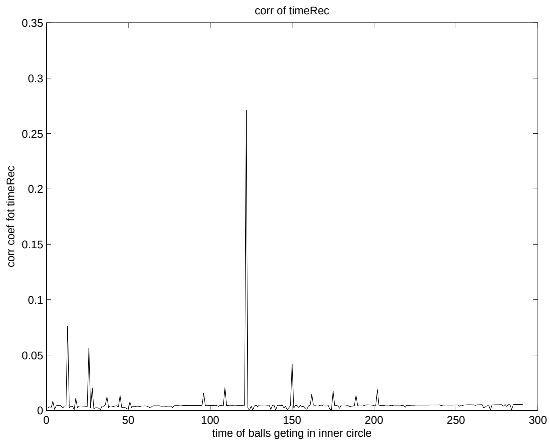

Figure 20: Time series of recurrence time

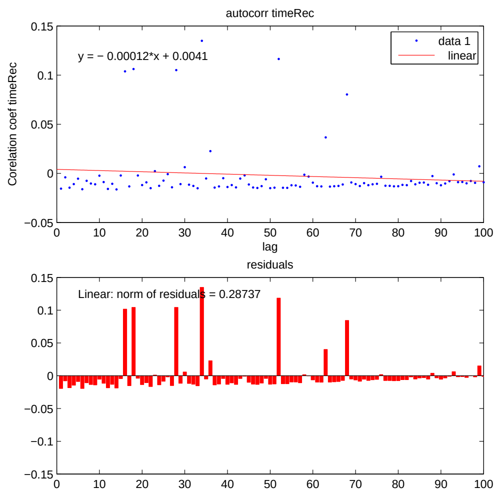

Figure 21: Mean values of correlation function of return time over 20 random initial conditions, and residuals plot

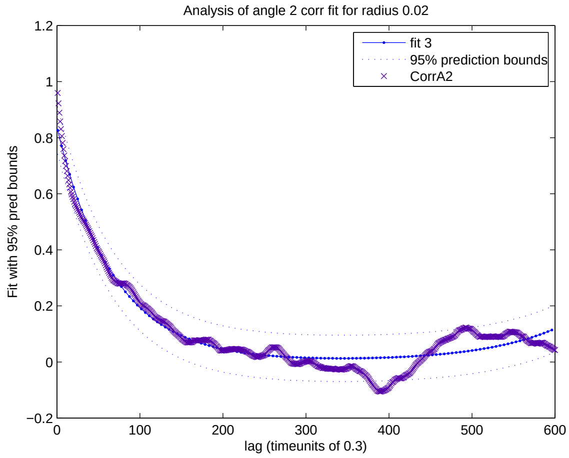

Figure 22: Correologram and fit for angle of wall hit of ball 2, for ball radius 0.02, outer circle radius 1

6 Summary

In the paper we have analyzed different recurrence times in two-particle circular billiard system. Starting from time for both balls to get back into a circle of radius half and quarter of outer radius, for different ball size, we can make conclusions that correlation in time series obtained depends on this coefficient significantly. Autocorrelation plots for position and velocity components have also been analyzed, and their exponential and power regressions found. Comparing these with results obtained in [3, 6, 5, 4] and others we have seen similarities in decay properties, but noticeable differences in coefficients of these regressions. Investigating connections between shape of boundary, number and size of balls, and similar system properties with correlation decay is topic for future research. Also, we have found interesting results in power regression of probability of mean recurrence time being greater then particular time interval. Power coefficient is between -0.95 and -1, and we get exact result -1 when outer radius is three times bigger then inner one. This is also topic for future investigation, where simulation should be run for longer time interval over several hundred initial conditions, so that we can numerically conjecture the dependence of probability on radius of circles and balls with more precision. In future, properties of lengths of integrable periods and dependence of it on initial conditions should be analyzed. Also coefficient of exponential fit for correlation decay of position and velocity coordinates is one of parameters we calculated and should be compared with the coefficients obtained in work by [7, 3, 5].

7 Acknowledgments

I would first like to thank my supervisor Mason A. Porter for invaluable help, guidance, time and patience during this research. To L.A. Bunimovich I am grateful for useful ideas and comments at the beginning and during the research, as well as to Steven Lansel for providing me with his past work on similar topics and help with coding. I also thank Nick Jones for advices with time series analysis, and St John's College for providing me with accommodation during the five weeks of summer research.

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