# Bounded-confidence opinion models with random-time interactions

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In models of opinion dynamics, agents interact with each other and can change their opinions as a result of those interactions. One type of opinion model is a bounded-confidence model (BCM), in which opinions take continuous values and interacting agents compromise their opinions with each other if their opinions are sufficiently similar. In studies of BCMs, researchers typically assume that interactions between agents occur at deterministic times. This assumption neglects an inherent element of randomness in social interactions, and it is desirable to account for it. In this paper, we study BCMs on networks and allow agents to interact at random times. To incorporate random-time interactions, we use renewal processes to determine social-interaction event times, which can follow arbitrary interevent-time distributions (ITDs). We establish connections between these random-time-interaction BCMs and deterministic-time-interaction BCMs. We analyze the quantitative impact of ITDs on the transient dynamics of BCMs and derive approximate governing equations for the time-dependent expectations of the BCM dynamics. We find that BCMs with Markovian ITDs have consistent statistical properties (in particular, they have the same expected time-dependent opinions) when the ITDs have the same mean but that the statistical properties of BCMs with non-Markovian ITDs depend on the type of ITD even when the ITDs have the same mean. We numerically examine the transient and steady-state dynamics of our BCMs with various ITDs on different networks, and we compare their expected order-parameter values and expected convergence times.

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## I. INTRODUCTION

On social-media platforms, individuals engage in regular and frequent exchanges of opinions, and people's views and how those views change play a pivotal role in shaping societal discourse [1]. The study of opinion dynamics—which involves the intersection of the social and behavioral sciences, mathematics, complex systems, and other areas—has emerged as a vibrant research area that aims to determine the mechanisms that govern the formation, evolution, and dissemination of opinions in human (and animal) societies [2–8]. At its core, the study of opinion dynamics concerns how beliefs, attitudes, and perceptions evolve with time through agreement, compromise, persuasion, imitation, and conflict. Studying such dynamics is crucial to understanding both (1) the emergence of consensus, polarization, and fragmentation and (2) the resilience of diverse opinions in societies, especially in the modern ecosystem of increasingly interconnected and digital communication environments [9–11].

Researchers have studied many types of opinion models [2–4]. In opinion models, agents adjust their opinions based on their interactions with other agents. Their opinions can update either in discrete time or in continuous time. In opinion models with discrete-time updates, time progresses through a sequence of discrete steps. Examples of discrete-time opinion models include voter models [12], DeGroot consensus models [13], and bounded-confidence models (BCMs) [14,15]. Opinion models with discrete-time updates are straightforward to implement in code for numerical simulations, and one can readily incorporate various features (such as parameter adaptivity [16]) into such models. In opinion models with continuous-time interactions, agents continuously adjust their opinions at rates that are influenced by factors such as whether they have friendly or hostile relationships with their neighbors [17] and the difference between their opinions and the opinions of their neighbors [18,19]. Another prominent type of model with continuous-time interactions is density-based opinion models [20], which consider the collective evolution of opinions in a large population and often are described by integro-differential equations.

Several researchers have highlighted the importance of incorporating randomness into opinion models to accurately capture the probabilistic nature of human interactions [21,22]. One can incorporate randomness in the structure of social and communication ties between agents by using random networks, such as configuration models, stochastic block models (SBMs), and their generalizations [23]. Additionally, one can use tie-decay networks [24] (which distinguish between communication interactions and underlying social ties) and activity-driven networks [25] (which also incorporate randomness in the interactions between agents) to incorporate randomness in communication. Another way to include randomness in a model is to incorporate noise and employ a stochastic differential equation to describe opinion evolution

 

[26,27]. One can also incorporate probabilistic components into the decision-making process of agents during opinion updates [14,28,29] either by choosing a random pair of agents to interact at each time step [14] or by allowing agents to choose probabilistically between multiple opinion-update rules [29]. See Ref. [30] for a quantitative study of how randomness in the structure of specific network models [including Erdős–Rényi (ER) graphs and Barabási–Albert graphs] influences steady-state features, phase transitions, consensus formation, and finite-size effects in the Hegselmann–Krause (HK) BCM.

Temporal stochasticity is another form of randomness that is relevant to opinion models, but it is often overlooked. Existing opinion models typically treat time as deterministic and neglect the temporal stochasticity that is inherent in social interactions. In the present paper, we model social interactions using renewal processes [31]. A renewal process consists of a sequence of random events, and the time between consecutive events follows a desired interevent-time distribution (ITD). By employing renewal processes, we are able to study non-Markovian dynamical processes, which arise frequently in human dynamics, including in financial markets [32], the spread of infectious diseases [33], e-mail traffic [34], and opinion dynamics [35]. For concreteness, we frame our discussion in the context of BCMs [36,37]. We consider both HK models (in which agents update their opinions synchronously) and Deffuant–Weisbuch (DW) models (in which agents update their opinions asynchronously). BCMs have been studied extensively by physicists, mathematicians, and others. For results about consensus formation, convergence, and opinion clustering in BCMs, see Refs. [38,39] for HK models and [40–42] for DW models. We discuss two approaches to integrate temporal stochasticity into BCMs, and we investigate the effects of such stochasticity on the convergence of opinions, the formation of opinion clusters, and the transient dynamics of opinions. We establish connections between our opinion models and traditional BCMs, and we approximate the expected dynamics in non-Markovian settings using BCMs with interactions at deterministic times.

Our paper proceeds as follows. In Sec. II, we discuss single-process BCMs, in which a single renewal process dictates the interaction times of all agents. We explore these BCMs with both synchronous and asynchronous update rules by examining properties such as expected dynamics and convergence for different ITDs. In Sec. III, we discuss multiple-process BCMs, where independent renewal processes govern the interaction times between each pair of agents. We derive the expected dynamics of Markovian BCMs in this framework, and we use a Gillespie algorithm to efficiently simulate event times in non-Markovian BCMs. In Sec. IV, we conclude and discuss future directions. Our code is available at Ref. [43].

## II. SINGLE-PROCESS BCMs

### A. Random-time interactions

Consider an unweighted and directed network (i.e., graph) $G = (V, E)$, where $V = \{1, 2, \dots, N\}$ is the set of nodes (i.e., agents) and $E = \{e_{ij}\}$ is the set of edges (i.e., social ties between agents). The directed edge $e_{ij}$ starts at agent $j$ and ends at agent $i$. Each agent $i$ has a scalar continuous-valued opinion $x_i(t)$. When $e_{ij} = 1$, agent $j$ can potentially influence agent $i$'s opinion. In a traditional BCM [14,15,37], time is deterministic and takes discrete values, with social interactions and opinion updates occurring in intervals of duration $\Delta t$. For convenience, researchers often set $\Delta t = 1$.

Let $R(t)$ be a renewal process, which is a stochastic process that models a sequence of events that occur randomly in time [31]. Let $\mathcal{T} = \{t_0, t_1, t_2, \dots\}$ be the sequence of event times in the renewal process $R(t)$. We set $t_0 = 0$ as the starting time of the renewal process. The time increments (i.e., *interevent times*) $t_{k+1} - t_k$ constitute a sequence of independent and identically distributed (IID) random variables with finite expected values. Because $t_{k+1} > t_k$ for all $k$, the time increments are positive. Let $\psi(t)$ denote the probability density function (PDF) of the IID random variables $t_{k+1} - t_k$. It is common to refer to this PDF as an *interevent-time distribution* (ITD) [44,45]. In this section, we suppose that a single renewal process determines the interaction times between the agents in a network.

### B. Synchronous and asynchronous opinion-update rules

The HK model [15,46] is a discrete-time BCM with a synchronous opinion-update rule. That is, all agents update their opinions simultaneously. Let[1]

$$\mathcal{N}_i(t) = \{i\} \cup \{j : e_{ij} \in E \;\text{ and }\; |x_i(t) - x_j(t)| < c\} \tag{1}$$

be the set of neighbors of agent $i$ (including $i$ itself) with which it interacts at time $t$. The parameter $c$ is the confidence bound. At each discrete time step, the opinion of each agent $i$ updates via the rule

$$x_i(t + \Delta t) = \frac{\sum_{j \in \mathcal{N}_i(t)} x_j(t)}{|\mathcal{N}_i(t)|}, \quad t \in \{0, \Delta t, 2\Delta t, \dots\}. \tag{2}$$

We extend the HK BCM to allow interactions at random times on directed graphs. Agents update their opinions synchronously when an event occurs in the renewal process $R(t)$. The opinion-update rule is thus

$$x_i(t) = \frac{\sum_{j \in \mathcal{N}_i(t_-)} x_j(t_-)}{|\mathcal{N}_i(t_-)|}, \quad t \in \mathcal{T} = \{t_1, t_2, \dots\}, \tag{3}$$

where $t_- = \lim_{\epsilon \to 0}[t - \epsilon]$ (with $\epsilon > 0$) denotes the time that is instantaneously before time $t$. Therefore, $x_j(t_-)$ is the opinion of agent $j$ right before it updates its opinion at time $t$. Unless an event occurs at time $t \in \mathcal{T}$, the opinions of all agents stay the same. The BCM with opinion-update rule (3) is a *single-process BCM* with synchronous updates. When the ITD $\psi$ is the Dirac delta distribution [i.e., $\psi(t) = \delta(t - \Delta t)$], the update rule (2) reduces to the update rule (2) in the classical HK BCM [15].

Deffuant *et al.* [14] introduced a discrete-time BCM with an asynchronous opinion-update rule. At each discrete time step, one selects a pair of agents uniformly at random and

[1]In Refs. [15,46], the set of neighbors of each agent $i$ is $\mathcal{N}_i(t) = \{i\} \cup \{j : e_{ij} \in E \;\text{ and }\; |x_i(t) - x_j(t)| \leqslant c\}$. To be consistent with the strict inequality in the classical DW BCM [14], we instead use a strict inequality.

updates their opinions to the mean of their opinions (or, more generally, to opinions that are closer to the mean) if their opinion difference is smaller than a confidence bound $c$. This model, which is the DW BCM, was proposed in the context of undirected graphs. We extend the DW BCM to a directed DW BCM. In this directed DW model, at time step $t$, one selects an edge $e_{ij}$ uniformly at random and updates the opinion of agent $i$ with the rule[2]

$$x_i(t+\triangle t) = \begin{cases} \frac{1}{2}[x_j(t) + x_i(t)] & \text{if } |x_i(t) - x_j(t)| < c \\ x_i(t) & \text{otherwise}. \end{cases} \quad (4)$$

The opinions of all other agents stay the same. The time $t$ takes values from the set $\{0, \triangle t, 2\triangle t, \ldots\}$. Unlike in the traditional DW model [14], we update the opinion of only agent $i$ rather than updating the opinions of both agent $i$ and agent $j$. Unidirectional interactions and influence occur in many social and biological systems [48,49]. For example, individuals may update their opinions by reading other individuals' social-media posts without commenting or otherwise signaling their engagement with those posts.

We generalize the directed DW BCM (4) by allowing interactions at random times. At time $t \in \mathcal{T}$, where $\mathcal{T}$ is the set of event times of a renewal process $R(t)$, we select an edge $e_{ij}$ uniformly at random and update the opinion of agent $i$ with the rule

$$x_i(t) = \begin{cases} \frac{1}{2}[x_j(t_-) + x_i(t_-)] & \text{if } |x_i(t_-) - x_j(t_-)| < c \\ x_i(t_-) & \text{otherwise}. \end{cases} \quad (5)$$

The opinions of all other agents stay the same. Additionally, unless an event occurs at time $t \in \mathcal{T}$, the opinions of all agents stay the same. The BCM with opinion-update rule (5) is a single-process BCM with asynchronous updates. When the ITD $\psi$ is the Dirac delta distribution, the update rule (5) reduces to the update rule (4) in the directed DW BCM.

In the random-time BCMs with synchronous (3) and asynchronous (5) opinion updates, the agent opinions converge almost surely (i.e., with probability 1) to isolated opinion clusters (i.e., maximal sets of agents with the same opinion value) that differ by at least the confidence bound $c$. This is a direct consequence of Lorenz's stability theorem [40].

### C. Exact and approximate dynamics of the expected opinions

Let $\boldsymbol{x}(t) = (x_1(t), \ldots, x_N(t)) \in \mathbb{R}^N$ be the time-dependent opinion vector in the opinion-update rule, which is given by (3) or (5). The randomness in $\boldsymbol{x}(t)$ arises from the interaction times, the selection of edges in the asynchronous-update model (5), and potentially random initial opinions. These three sources of randomness are independent of each other. In the rest of this section, we fix the initial opinion vector $\boldsymbol{x}_0$ and investigate how the other two sources of randomness influence the dynamics of the expected opinions. We also examine how the ITD influences the dynamics of the expected opinions in single-process BCMs with the synchronous update rule (3) and the asynchronous update rule (5).

Let $u_k(t)$ denote the probability that the renewal process $R(t)$ has $k$ events in the time interval $[0, t]$. With the ITD $\psi$, we have $u_0(t) = 1 - \int_0^t \psi(\tau)\, \mathrm{d}\tau$. If $k+1$ events occur in the time interval $[0, t]$, then $k$ events occur in the time interval $[0, t-\tau]$ and 1 event occurs in the time interval $(t-\tau, t]$ for some $\tau \in [0, t]$. Therefore, the probability $u_k(t)$ satisfies

$$u_{k+1}(t) = \int_0^t u_k(t-\tau)\psi(\tau)\, \mathrm{d}\tau\,, \quad k \geqslant 0\,. \quad (6)$$

For any function $f : \mathbb{R}^N \longrightarrow \mathbb{R}$, let $\mathbb{E}[f]$ denote the expectation of $f(\boldsymbol{x})$. Armed with this notation, we write

$$\mathbb{E}[f](t) = \mathbb{E}[f(\boldsymbol{x}(t))]\,. \quad (7)$$

We take this expectation with respect to all sources of randomness except for the initial opinions. Let $\boldsymbol{x}[k]$ denote the opinion vector after $k$ updates, and let $\mathbb{E}_k[f]$ be the expected value of $f(\boldsymbol{x}[k])$. The event times are independent of opinion updates, so

$$\mathbb{E}[f](t) = \sum_{k=0}^{\infty} \mathbb{E}_k[f] u_k(t)\,. \quad (8)$$

The probability $u_k(t)$ is determined solely by the ITD $\psi$; it is independent of the update rules (3) and (5). The expectation $\mathbb{E}_k[f]$ is independent of both the ITD $\psi$ and the renewal process $R(t)$; it is determined solely by the update rules (3) and (5). Using the expression (8), we disassociate the expected opinion dynamics from the temporal stochasticity that arises from random-time interactions. By introducing a cutoff for $k$ in Eq. (8), we obtain an approximate formula to compute the expected dynamics of our BCMs with random-time interactions.

We compute the probability $u_k(t)$ either directly using (6) or by employing the Laplace transform of $u_k(t)$ to circumvent calculating a convolution. See Refs. [35,44] for how to derive the Laplace transform of the probability $u_k(t)$. The synchronous single-process BCM has a deterministic update rule (3). Therefore, $\mathbb{E}_k[f] = f(x[k])$ and we obtain $\mathbb{E}_k[f]$ in (8) with a single simulation of the deterministic-time HK BCM (2). That is, we simulate "one realization" of the deterministic-time HK BCM (2). For the asynchronous single-process BCM (5), it is often challenging to evaluate $\mathbb{E}_k[f]$ due to the randomness in selecting node pairs for potential opinion updates. This randomness can yield different opinion trajectories for any ITD (even for the Dirac delta ITD). Therefore, we need to simulate multiple realizations of the deterministic-time directed DW BCM (4) to approximate the expectation $\mathbb{E}_k[f]$.

To quantify the amount of consensus in a simulation of a single-process BCM, we calculate the order parameter

$$Q(\boldsymbol{x}) = \frac{1}{|E|} \sum_{e_{ij} \in E} \mathbb{1}_{x_i = x_j}\,, \quad (9)$$

where $\mathbb{1}_S$ denotes the indicator function on the set $S$. In practice, we relax the condition $\mathbb{1}_{x_i = x_j}$ by instead using $\mathbb{1}_{|x_i - x_j| < \mathtt{tol}}$ (where `tol` is a tolerance parameter) to hasten the convergence of simulations. For a complete graph, when $Q = 1$, all

[2] In the traditional DW model [14], one chooses a random edge and potentially updates the opinions of its two attached nodes. By contrast, in the present paper, we choose a random edge and then potentially update the opinion of only one node. A third option, which was employed in Ref. [47], is to first choose a random node, then randomly choose one of its neighboring nodes to interact with it, and then potentially update the opinions of both nodes.

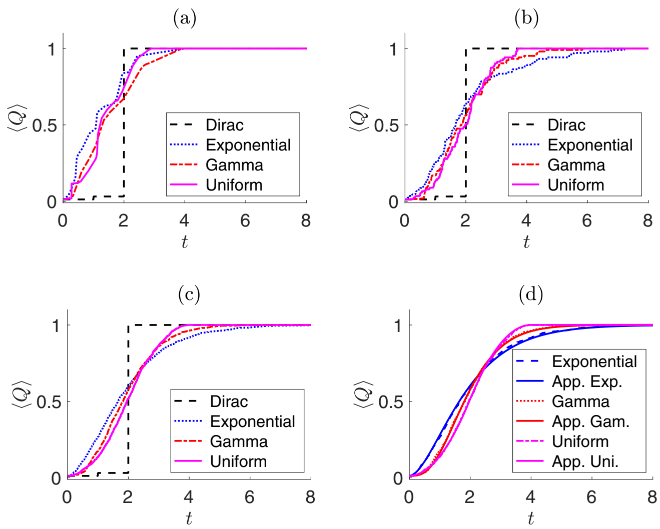


FIG. 1. In (a)–(c), we show empirical means of the order parameter $Q$ [see (9)] from (a) 10, (b) 100, and (c) 1000 simulations of the synchronous single-process BCM (3) on a 100-node complete graph. For each simulation, we draw the initial opinions from the uniform distribution on [0,1]. The confidence bound is $c = 0.5$, and the tolerance parameter is $\texttt{tol} = 10^{-2}$. In (d), we plot the empirical means of $Q(\boldsymbol{x})$ from (c) for different ITDs and their approximations using (8) with the cutoff $k = 15$.

agents have the same opinion and the system is in its most ordered state; conversely, when $Q = 1/N$ (where $N$ is the number of agents), each agent has a different opinion, so the system is in its least ordered state.

In Fig. 1, we compute the mean of the order parameter $Q(\boldsymbol{x})$ for the synchronous single-process BCM (3) for different ITDs and approximate the associated expected order parameters using (8) with the cutoff $k = 15$. We distinguish between empirical means $\langle \cdot \rangle$ and expectations $\mathbb{E}[\cdot]$ of quantities. We consider renewal processes $R(t)$ with ITDs

$$\psi_{\text{Dirac}}(t) = \delta(t - \mu)\,, \tag{10a}$$

$$\psi_{\text{exponential}}(t) = \frac{1}{\mu}\exp(-t/\mu)\,, \tag{10b}$$

$$\psi_{\text{gamma}}(t) = \frac{4t}{\mu^2}\exp(-2t/\mu)\,, \tag{10c}$$

$$\psi_{\text{uniform}}(t) = \mathbb{1}_{[0,2\mu]}(t)\,. \tag{10d}$$

All ITDs have the same mean value $\mu$. As we increase the number of simulations, we observe that the time-dependent order parameters become smoother for the continuous ITDs (i.e., the exponential, Gamma, and uniform ITDs) and that the trajectories of the approximate expected order parameters closely match the trajectories of the empirical means of the order parameters for all ITDs.

## III. MULTIPLE-PROCESS BCMs

The single-process BCMs in Sec. II assume that a single renewal process governs the times of the interactions between agents. In reality, however, individuals exchange opinions at various times, so one cannot expect the interactions between agents to be governed by a single renewal process. Therefore, we consider multiple independent renewal processes $R_{ij}$ for $i, j \in \{1, \dots, N\}$. The events in $R_{ij}$ trigger potential opinion updates of agent $i$ and determine when it interacts with each agent $j$.

### A. Interactions that are induced by multiple renewal processes

Let $R_{ij}$ be a renewal process that generates a sequence $\mathcal{T}_{ij} = \{t_0, t_1, t_2, \dots\}$ of event times with initial time $t_0 = 0$. We suppose that all renewal processes $R_{ij}$ are independent of each other and have the same ITD $\psi$. The renewal process $R_{ij}$ determines the times that agent $i$ interacts with agent $j$ and is potentially influenced by agent $j$. At time $t \in \mathcal{T}_{ij}$, we update the opinion $x_i$ of node $i$ using the update rule[3]

$$x_i(t) = \begin{cases} \frac{1}{2}[x_i(t_-) + x_j(t_-)] & \text{if } |x_i(t_-) - x_j(t_-)| < c \\ x_i(t_-) & \text{otherwise}\,. \end{cases} \tag{11}$$

The opinions of all other agents stay the same. If multiple events that involve the same agent $i$ occur simultaneously at time $t$, then we update its opinion $x_i$ to

$$x_i(t) = \frac{\sum_{j \in \tilde{\mathcal{N}}_i(t_-)} x_j(t_-)}{|\tilde{\mathcal{N}}_i(t_-)|}\,, \tag{12}$$

where

$$\tilde{\mathcal{N}}_i(t) = \{i\} \cup \{j \in \mathcal{N}_i(t) : t \in \mathcal{T}_{ij}\} \tag{13}$$

is a restricted neighbor set [which differs from the neighbor set (1)] that includes all neighboring nodes of $i$ that (1) interact with node $i$ at time $t$ and (2) have an opinion that differs from the opinion $x_i$ by less than the confidence bound $c$. The model (11) is a *multiple-process BCM*. When the ITD $\psi$ is continuous, the events of two renewal processes occur simultaneously with 0 probability, so opinion updates in (11) are asynchronous almost surely (i.e., with probability 1). When the ITD is $\psi(t) = \delta(t - \Delta t)$ (i.e., the Dirac delta distribution), the events of different processes occur simultaneously at times $t \in \{\Delta t, 2\Delta t, \dots\}$, and we obtain the synchronous single-process BCM (3) in Sec. II B. We can extend the multiple-process BCM (11) to a heterogeneous scenario in which each renewal process $R_{ij}$ has a different ITD $\psi_{ij}$. In such a model, opinion updates can occur as a hybrid of synchronous and asynchronous updates.

For the single-process BCMs (3) and (5), the steady-state behaviors are statistically the same as in the baseline BCMs (2) and (4), respectively, as the random interevent times do not affect the order of agent interactions. By contrast, in multiple-process BCMs, multiple renewal processes determine both the event times and the order of agent interactions. Therefore, the steady-state behaviors are now statistically different from those in the baseline BCMs.

We illustrate the statistical difference between the steady states from different ITDs using a three-node network. Consider a network with node set $V = \{1, 2, 3\}$ and edge set $E = \{e_{21}, e_{23}\}$. Suppose that the initial agent opinions are $x_1(0) = 0$, $x_2(0) = 0.5$, and $x_3(0) = 1$ and that the confidence bound is $c = 0.6$. Due to our unidirectional setting, only agent 2 can update its opinions from its interactions. If agent

[3]This is the same update rule as (5), which we repeat for clarity.

2 interacts first with agent 1, the opinion vector becomes $(0, 0.25, 1)$ after one interaction and we eventually obtain the steady state $(x_1^*, x_2^*, x_3^*) = (0, 0, 1)$. If agent 2 interacts first with agent 3, the opinion vector becomes $(0, 0.75, 1)$ after one interaction and we eventually obtain the steady state $(x_1^*, x_2^*, x_3^*) = (0, 1, 1)$. Therefore, the system's steady state is determined entirely by whether agent 2 interacts first with agent 1 or agent 3.

We consider two different ITDs for the edges $e_{21}$ and $e_{23}$. In the first scenario, both edges follow the same ITD, which has a mean of 1. Therefore, agent 2 interacts with agent 1 or agent 3 with equal probability. We thus obtain the steady state

$$(x_1^*, x_2^*, x_3^*) = \begin{cases} (0, 0, 1) & \text{with probability } 0.5 \\ (0, 1, 1) & \text{with probability } 0.5\,. \end{cases} \tag{14}$$

In the second scenario, the edge $e_{21}$ follows the uniform ITD on $[0.5, 1.5]$ and the edge $e_{23}$ follows the ITD

$$\psi_{23}(t) = \tfrac{3}{5}\delta\big(t - \tfrac{1}{3}\big) + \tfrac{2}{5}\delta(t - 2)\,, \tag{15}$$

which is a sum of two Dirac delta distributions. Therefore, when the interevent time of $e_{23}$ is $1/3$, agent 2 interacts with agent 3 before it interacts with agent 1 because the interevent time of $e_{21}$ is at least $1/2$, which is larger than $1/3$. With probability $3/5$, agent 2 interacts with agent 3 first and thereby yields the steady state $x^* = (0, 1, 1)$. When the interevent time of $e_{23}$ is 2, agent 2 interacts with agent 3 after it interacts with agent 1 because the interevent time of $e_{21}$ is at most $3/2$, which is smaller than 2. With probability $2/5$, agent 2 interacts with agent 1 first and thereby yields the steady state $x^* = (0, 0, 1)$. We thus obtain the steady state

$$(x_1^*, x_2^*, x_3^*) = \begin{cases} (0, 0, 1) & \text{with probability } 2/5 \\ (0, 1, 1) & \text{with probability } 3/5\,. \end{cases} \tag{16}$$

In both scenarios, the ITDs of all edges have the same mean (which is equal to 1). However, the two scenarios yield different steady-state statistics. This discrepancy arises because the order of edge events (and hence the sequence of pairwise interactions) determines the overall dynamics and is not statistically equivalent for different ITDs even when they have identical means. Accordingly, in multiple-process BCMs (and unlike in single-process BCMs), the randomness in interaction times affects steady-state behavior. In Secs. III B and III C, we examine how the ITD type and parameters affect the dynamics of multiple-process BCMs on various networks.

### B. Dynamics of Markovian multiple-process BCMs

We now discuss the dynamics of some Markovian multiple-process BCMs.

When the ITD is a Dirac delta distribution, both the single-process BCM (3) and the multiple-process BCM (11) become discrete-time Markovian processes. In this situation, both models yield the traditional HK BCM (2). In the rest of this subsection, we consider Markovian BCMs, which have exponential ITDs. To help highlight their dynamics, we also compare them to BCMs with a Dirac delta ITD.

When the ITD $\psi(t)$ is exponential, the renewal processes $R_{ij}$ are Poisson point processes. We write $\psi(t) = \lambda e^{-\lambda t}$, where $\lambda$ is the rate parameter of the process. The sum (i.e., "superposition") of $|E|$ Poisson point processes is a Poisson point process $P(t)$ with rate parameter $\Lambda = \lambda|E|$. In this case, the multiple-process BCM is the same as the asynchronous single-process BCM (5) with an exponential ITD with rate parameter $\Lambda$. We show that the opinion model that is induced by the exponential ITD is Markovian, and we relate the dynamics of the expected opinions to a continuous-time HK BCM [18].

Let $P(t)$ be the superposition of all $|E|$ Poisson point processes $R_{ij}$ (where $E$ is the set of edges of a network), and let $Z$ denote the total number of events in the time interval $[t, t + \tau)$ for $P(t)$. We have

$$Z = \begin{cases} 0 & \text{with probability} \quad e^{-\Lambda\tau} \\ 1 & \text{with probability} \quad \Lambda\tau\, e^{-\Lambda\tau} \\ \geqslant 2 & \text{with probability} \quad \sum_{k=2}^{\infty} \frac{(\Lambda\tau)^k}{k!} e^{-\Lambda\tau}\,. \end{cases} \tag{17}$$

When $Z = 0$, no events occur in the time interval $[t, t + \tau)$, so this situation does not contribute to opinion updates. When $Z = 1$, one event occurs in the time interval $[t, t + \tau)$. This event is generated by the process $R_{ij}$ with probability $1/|E|$. In this event, agent $i$ adjusts its opinion to $x_i(t) + \Delta_{i,j}(t)$, where

$$\Delta_{i,j}(t) = \tfrac{1}{2}\mathbb{1}_{|x_i(t)-x_j(t)|<c}[x_j(t) - x_i(t)]\,. \tag{18}$$

Let $y_i(t) = \mathbb{E}[x_i(t)]$ be the expectation of $x_i(t)$ with respect to the point-process superposition $P(t)$. The expected opinion $y_i(t + \tau)$ satisfies

$$y_i(t + \tau) = y_i(t) + \sum_{\{j: e_{ij} \in E\}} \frac{\Lambda\tau}{|E|} e^{-\Lambda\tau}\mathbb{E}[\Delta_{i,j}(t)] + \mathcal{O}(\tau^2)\,, \tag{19}$$

where the $\mathcal{O}(\tau^2)$ correction arises from the contribution for $Z \geqslant 2$. We use the relation $\Lambda = \lambda|E|$ and take the limit $\tau \to 0$ to obtain

$$\dot{y}_i(t) = \frac{\lambda}{2} \sum_{\{j: e_{ij} \in E\}} \mathbb{E}\{\mathbb{1}_{|x_i(t)-x_j(t)|<c}[x_j(t) - x_i(t)]\}\,, \tag{20}$$

where $i \in \{1, \ldots, N\}$. The system (20) is not closed. We make the bold approximation

$$\mathbb{E}\{\mathbb{1}_{|x_i(t)-x_j(t)|<c}[x_j(t) - x_i(t)]\} \approx \mathbb{1}_{|y_i(t)-y_j(t)|<c}[y_j(t) - y_i(t)] \tag{21}$$

and insert (21) into (20) to obtain a closed set of equations. We thereby obtain

$$\dot{y}_i(t) = \frac{\lambda}{2} \sum_{\{j: e_{ij} \in E\}} \mathbb{1}_{|y_i(t)-y_j(t)|<c}[y_j(t) - y_i(t)]\,, \tag{22}$$

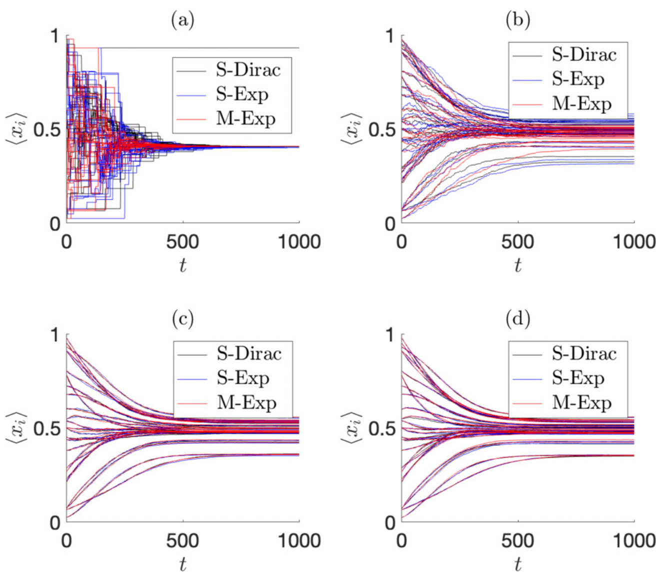


FIG. 2. Empirical means of the time-dependent opinions $x_i(t)$ in asynchronous single-process BCMs (5) with Dirac delta (S-Dirac) and exponential (S-Exp) ITDs and the multiple-process BCM (11) with an exponential (M-Exp) ITD for (a) 1, (b) 100, (c) 1000, and (d) 2000 simulations. All simulations have the same initial opinions, which we draw uniformly at random from [0,1]. We generate one directed 25-node $G(N, p)$ ER graph with connection probability $p = 0.5$, and we run all simulations on this ER graph. The confidence bound is $c = 0.4$.

which is a continuous-time HK BCM [18] with $\lambda = 2$ and $c = 1$. The approximation (21) is the special form of the approximation $\mathbb{E}[g(r)] \approx g(\mathbb{E}[r])$ when $g(r) = \mathbb{1}_{|r|<c} r$ and $r = x_j - x_i$. For a general random variable $r$, the expectation $\mathbb{E}[g(r)]$ does not equal $g(\mathbb{E}[r])$. These two quantities are equal in two noteworthy special cases: (1) when $g$ is linear; and (2) when $r$ follows a Dirac delta distribution. In our numerical simulations, we observe discrepancies between (20) and (22).

The expected dynamics (20) is related to the asynchronous single-process BCM (5) when the ITD is the Dirac delta distribution $\psi(t) = \delta(t - \Delta t)$ and $\Delta t = 1/(\lambda|E|)$. In this case, agents interact with each other and potentially update their opinions at times $t \in \{\Delta t, 2\Delta t, \ldots\}$. We employ a linear interpolation of the opinions $x_i(t)$ on each time interval $[k\Delta t, (k+1)\Delta t)$ using the opinions at the two interval end points. The expected opinions satisfy

$$\dot{y}_i(t) = \frac{\lambda}{2} \sum_{\{j: e_{ij} \in E\}} \mathbb{E}\{\mathbb{1}_{|r_{ji}(k\Delta t)|<c} r_{ji}(k\Delta t)\} \tag{23}$$

for times $t \in [k\Delta t, (k+1)\Delta t)$, where $r_{ji}(k\Delta t) = x_j(k\Delta t) - x_i(k\Delta t)$. Equation (23) gives the expected dynamics of the single-process BCM (5) with a Dirac delta ITD, and it is also a discrete-time version of the expected dynamics (20) for single-process (5) and multiple-process (11) BCMs with exponential ITDs.[4] We observe numerically that (20) and (23) yield the same dynamics when we approximate the expectations using the empirical means of the time-dependent opinions.

[4]See Ref. [50] for a discussion of continuous-time and discrete-time approximations of Markovian dynamics.

ALGORITHM 1. Gillespie algorithm to simulate $m$ independent renewal processes.

1: Initialize $t_\alpha = 0$ for all $\alpha \in \{1, \ldots, m\}$, where $t_\alpha$ records the time that has elapsed since the most recent event in the $\alpha$th renewal process.[a]
2: Draw a value $u$ uniformly at random from [0,1] and determine the time increment $\Delta t$ by solving

$$\Phi(\Delta t \mid \{t_\alpha\}) = \prod_\alpha \frac{\psi_\alpha(t_\alpha + \Delta t)}{\Psi(t_\alpha)} = u\,, \tag{24}$$

where $\psi_\alpha$ is the ITD of the $\alpha$th renewal process and $\Psi_\alpha(t) = \int_t^\infty \psi_\alpha(\tau)\, d\tau$ is the survival function.
3: Randomly select a process $\beta$ that generates an event with probability

$$\Pi_\beta = \frac{\lambda_\beta(t_\beta + \Delta t)}{\sum_{\alpha=1}^m \lambda_\alpha(t_\alpha + \Delta t)}\,, \tag{25}$$

where

$$\lambda_\alpha(t) = \frac{\psi_\alpha(t)}{\Psi_\alpha(t)} \tag{26}$$

is the instantaneous rate of the $\alpha$th process.
4: Set $t_\beta = 0$ and update $t_\alpha$ to $t_\alpha + \Delta t$ for $\alpha \neq \beta$.
5: Repeat steps 2–4 (or terminate the algorithm if a stopping criterion is satisfied).

[a]This meaning of $t_\alpha$ is different from our prior use of the notation $t_k$, which we used to denote the $k$th event time of a process.

In Fig. 2, we show mean time-dependent opinions of three Markovian models: the asynchronous single-process BCM (5) with the Dirac delta ITD $\psi(t) = \delta(t - 1/(\lambda|E|))$ (which we denote by "S-Dirac"), the asynchronous single-process BCM (5) with the exponential ITD $\psi(t) = \lambda|E| \exp(-\lambda|E|t)$ (which we denote by "S-Exp"), and the multiple-process BCM (11) with the exponential ITD $\psi(t) = \lambda \exp(-\lambda t)$ (which we denote by "M-Exp"). Each realization of these processes can have distinct opinion-update times, so we use a piecewise-linear interpolation of the opinions at discrete update times for each realization and then compute the mean of the interpolated opinion trajectories on the entire time domain. We then compute the mean opinion dynamics by averaging the interpolated dynamics across multiple simulations of the same model.

In our simulations, we observe the same expected dynamics in all of our Markovian BCMs, which have either a single-process Dirac delta ITD, a single-process exponential ITD, or a multiple-process ITD in which all edges have the same ITD (i.e., a homogeneous multiple-process ITD). Importantly, our observation that these three Markovian ITDs lead to the same expected opinion dynamics is independent of the network size $N$ and the confidence bound $c$. For our simulations in Fig. 2, we use a small network (which has 25 nodes) due to the slow convergence of the dynamics to their expected values and the significant increase in computation time as we increase network size.

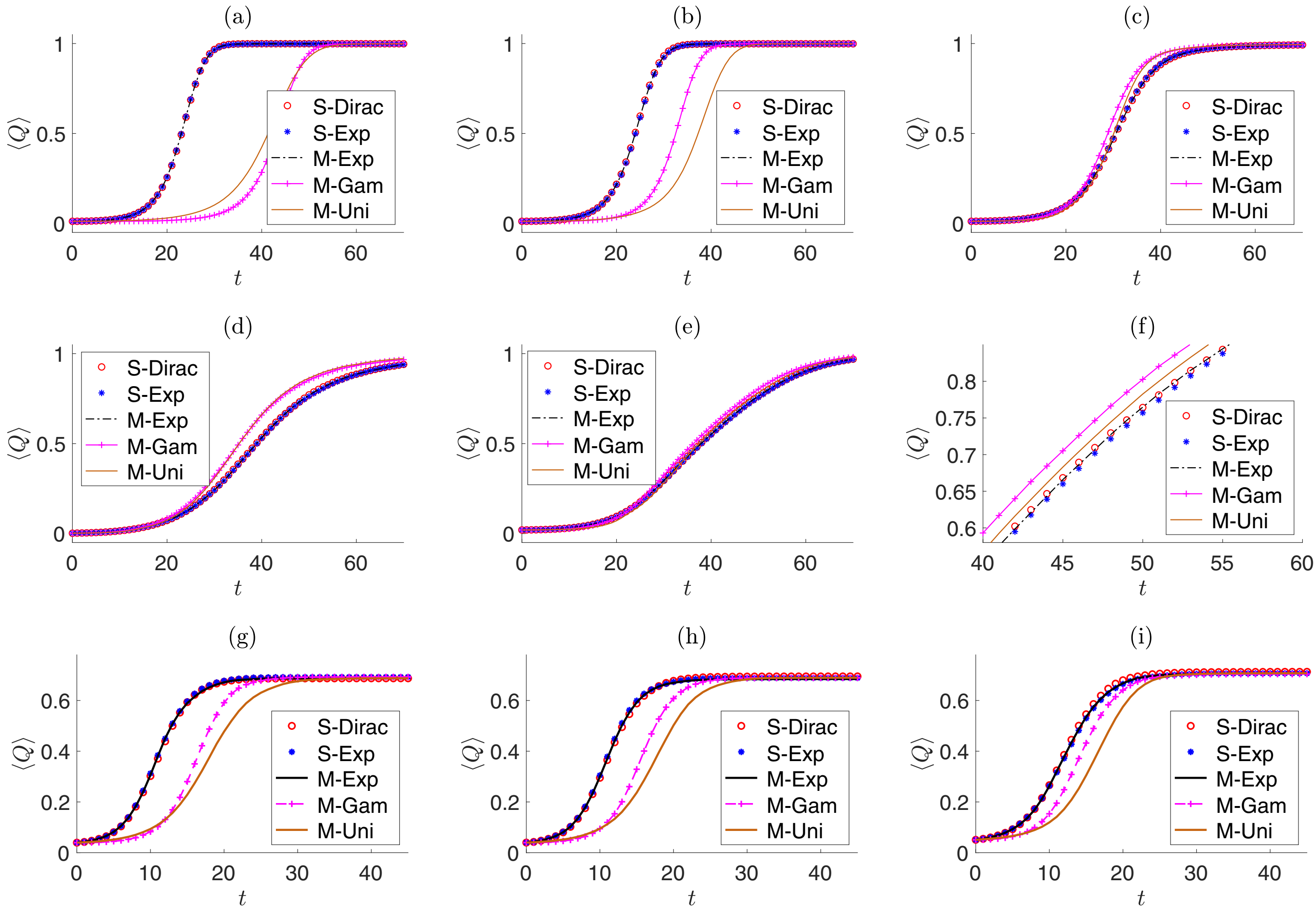


FIG. 3. Empirical means of the order parameter $Q$ [see (9)] versus time for single-process BCMs (5) with Dirac delta (S-Dirac) and exponential (S-Exp) ITDs and for multiple-process BCMs (11) with exponential (M-Exp), gamma (M-Gam), and uniform (M-Uni) ITDs on (a, g) a complete graph, (b, h) directed $G(N, p)$ ER graphs with $p = 0.4$, (c, i) directed $G(N, p)$ ER graphs with $p = 0.1$, (d) symmetric and directed Chung–Lu graphs, and (e) directed SBM graphs with two communities. In (f), we show a magnification of (e). All ITDs have mean $\mu = 0.01$, and all graphs have $N = 50$ nodes. The confidence bound is $c = 0.5$ in panels (a)–(f) and is $c = 0.3$ in panels (g)–(i). We compute the mean of $Q$ using 3000 BCM simulations. We draw the initial opinions uniformly at random from [0,1] for each simulation, and we generate a new random graph for each simulation that involves a random-graph model.

### C. Gillespie algorithm for non-Markovian multiple-process BCMs

It is computationally challenging to simulate a large number of processes in a multiple-process BCM (11) with $|E|$ independent and concurrent renewal processes. It is prohibitively complex to simulate these processes separately, organize their events chronologically, and execute opinion updates. To mitigate this computational burden, we use a Gillespie algorithm [51], which allows us to generate independent stochastic processes efficiently and statistically correctly.

The traditional Gillespie algorithm [52] is for independent Poisson processes, whose ITDs are exponential. Boguñá *et al.* [53] extended the Gillespie algorithm to simulate the events of multiple independent renewal processes. Their non-Markovian Gillespie algorithm draws a time increment $\triangle t$ for the time to the next event from the superposition of $m$ renewal processes and determines the process that produces that event with a probability that depends on the elapsed time (i.e., the current interevent time) of each renewal process. This non-Markovian Gillespie algorithm, which we state in Algorithm 1, generates a statistically correct sequence of event times. One can terminate the algorithm after a specified number of events or when the time reaches a specified value. When all renewal processes are Poisson processes, the instantaneous rate $\lambda_\alpha(t)$ [see (26) in Algorithm 1] reduces to a constant $\lambda_\alpha$, which is the rate of the $\alpha$th Poisson point process. That is, in this situation, this non-Markovian Gillespie algorithm reduces to the traditional Gillespie algorithm.

We use the non-Markovian Gillespie algorithm in Algorithm 1 to simulate the multiple-process BCM (11) on four distinct types of graphs (with $N = 50$ nodes): (1) a complete graph; (2) directed analogues of $G(N, p)$ Erdős–Rényi (ER) random graphs in which each edge exists with independent and homogeneous probability $p$; (3) symmetric and directed Chung–Lu graphs (in which we start with undirected graphs and treat each undirected edge as two directed edges) [54], which are similar to configuration-model networks and are parametrized by sequences of expected degrees [55], which we choose to be $\{10/\ln(k)\}_{k=2,\dots,N+1}$; and (4) directed stochastic-block-model (SBM) graphs with two communities, intracommunity probabilities $p_{AA} = p_{BB} = 0.2$, and inter-

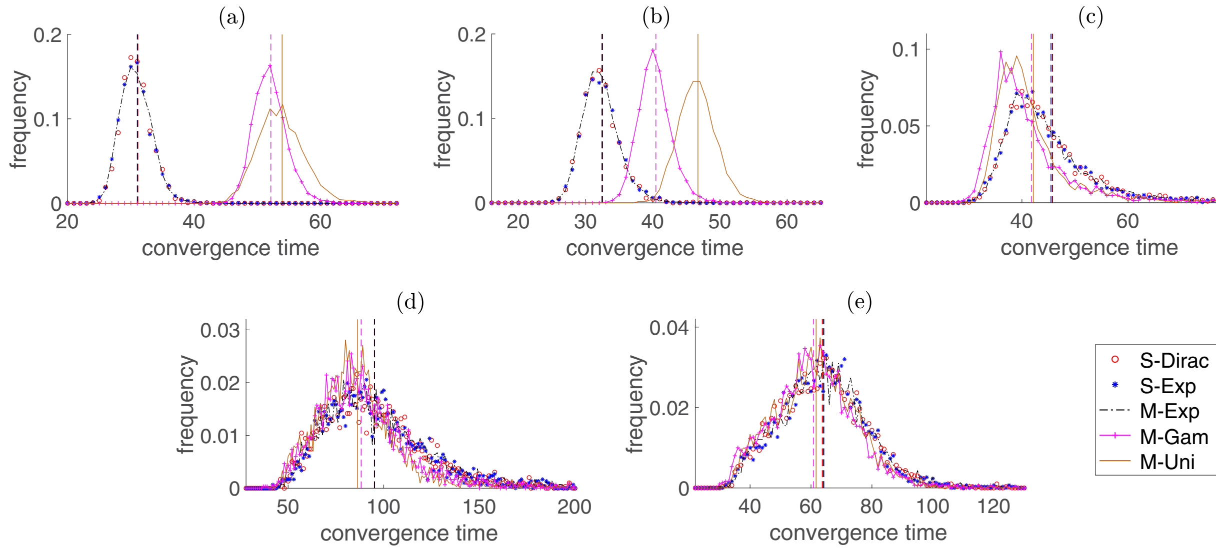


FIG. 4. Normalized histograms of the convergence times of several of the simulations in Fig. 3. The vertical lines indicate the mean convergence times of the BCMs on (a) a complete graph, (b) directed $G(N, p)$ ER graphs with $p = 0.4$, (c) directed $G(N, p)$ ER graphs with $p = 0.1$, (d) symmetric and directed Chung–Lu graphs, and (e) directed SBM graphs with two communities. In panels (d) and (e), which show results for graphs with prominently heterogeneous degree distributions, the variances of the convergence times are larger than those in panels (a)–(c).

community probabilities $p_{\mathrm{AB}} = p_{\mathrm{BA}} = 0.02$. We explore how randomness, which arises through the ITDs and specific structures in the random-graph models, influences the order parameter $Q$ [see (9)] and convergence time in both single-process and multiple-process BCMs. For the single-process BCMs, we use Dirac delta and exponential ITDs with mean $\mu|E|$. Because the mean is the same, all simulations have the same expected number of events. For the multiple-process BCMs (11), we consider exponential, gamma, and uniform ITDs with mean $\mu$.

In Fig. 3, we plot the mean of the order parameter $Q$ [see Eq. (9)] from 3000 simulations of each scenario. We generate a new random graph for each simulation that involves a random-graph model. The three Markovian models—the single-process BCM with the Dirac delta ITD, the single-process BCM with the exponential ITD, and the multiple-process BCM with the exponential ITD—yield almost identical mean time-dependent order parameters $Q$, which agrees with the results in Fig. 2. We use piecewise-linear interpolation to construct $\mathbb{E}[Q^{\mathrm{Dirac}}(\boldsymbol{x}(t))]$ in the model with the Dirac delta ITD. Because of this construction, $\mathbb{E}[Q^{\mathrm{Dirac}}(\boldsymbol{x}(t))]$ is a piecewise-linear approximation of the value $\mathbb{E}[Q^{\mathrm{exp}}(\boldsymbol{x}(t))]$ that we obtain from the model with the exponential ITD. As $\Delta t \to 0$, we anticipate that the expected dynamics of both models converge to the same dynamics. In our simulations, we use $\Delta t = 0.01$. Multiple-process BCMs with gamma and uniform ITDs yield different mean dynamics of the time-dependent order parameter $Q$, and the convergence rates depend both on the ITDs and on network structure.

Multiple-process BCMs with different non-Markovian ITDs can have distinct steady states, and their order parameters almost never converge to the same value (although they tend to yield similar values). Moreover, when the confidence bound is $c = 0.5$, our simulations (of models with either Markovian or non-Markovian ITDs) do not always converge to a consensus, so the order parameter $Q$ does not converge exactly to 1. Instead, it converges to values less than 1. When the confidence bound is $c = 0.3$, we observe polarization and fragmentation[5] in the steady states, with the order parameter again converging to a value less than 1.

In Fig. 4, we show normalized histograms of the convergence times of several of the simulations in Fig. 3. We treat a simulation as having converged if the opinion difference between each pair of adjacent nodes is either (1) at least the confidence bound $c$ or (2) smaller than $10^{-3}$. Based on our numerical observations, we conclude that both the ITD and network structure influence convergence time. Additionally, the convergence-time variance increases dramatically as we consider more heterogeneous degree distributions.

## IV. CONCLUSIONS AND DISCUSSION

Social systems include various elements of randomness, and it is important to account for such randomness in models of opinion dynamics. In this paper, we extended traditional bounded-confidence models (BCMs) of opinion dynamics by incorporating randomness into agent interaction times. In our BCMs, opinion updates occur randomly in time as events of renewal processes. The interevent times are random and follow non-Markovian interevent-time distributions (ITDs). Traditional Hegselmann–Krause (HK) and Deffuant–Weisbuch (DW) BCMs arise from specific choices of renewal processes and are thus special cases of our models. We investigated how ITDs affect the transient dynamics of our BCMs, and we derived approximate governing equations for the time-dependent expectations of opinions. We then numerically

[5] A "polarized" state has two major opinion clusters, and a "fragmented" state has three or more major opinion clusters.

simulated our BCMs on various types of networks to explore how different network structures impact their dynamics.

In the single-process BCMs (3) and (5), for which a single renewal process governs the interaction times between agents, the ITDs only influence the transient opinion dynamics. One obtains the same steady-state outcome for all ITDs. For a Dirac delta ITD, we highlighted that the BCM (3) reduces to a traditional HK BCM [15] and that the BCM (5) reduces to a directed variant of the classical DW BCM [14]. Additionally, we derived a relationship (8) between single-process BCMs [(3) and (5)] and the deterministic-time BCMs [(2) and (4), respectively] in terms of their expected dynamics. This relationship (8) yields an approximation method to efficiently compute the expected dynamics of the single-process BCMs. Using numerical simulations, we demonstrated that our approximation is accurate for exponential, Gamma, and uniform ITDs.

We also developed multiple-process BCMs (12), which use multiple independent renewal processes to determine the interaction times between agents. Multiple-process BCMs with Dirac delta and exponential ITDs yield Markovian models that are equivalent to single-process BCMs with appropriately chosen ITDs and parameters. We derived an approximate governing equation for the expected opinions in these two Markovian models, and we showed that one can interpret the expected dynamics of the BCMs with Dirac delta ITDs as discrete-time analogues of the expected dynamics of the BCMs with exponential ITDs. For specific parameter values, these two models reduce to the continuous-time BCM in Ref. [18]. To numerically simulate our multiple-process BCMs efficiently and statistically accurately, we employed a non-Markovian Gillespie algorithm [53]. In our numerical computations, we observed that both ITDs and network structure significantly influence the transient properties—including both the order parameter (9) and the convergence times—of the non-Markovian BCMs. We also observed that network structure has more influence than ITDs on convergence-time variance and that degree heterogeneities further amplify this variance.

In the present paper, we examined BCMs on unweighted graphs and assumed that the ITDs are homogeneous across all edges. It is worthwhile to incorporate both heterogeneous edge weights and heterogeneous ITDs. In such extensions, one can consider deterministic edge weights and use weighted averages in synchronous opinion updates or employ heterogeneous ITDs with parameters that are linked to edge weights. In most of our numerical simulations, we used confidence-bound values that typically lead to consensus. To thoroughly study transitions between consensus steady states and other outcomes (especially polarization and fragmentation), it is important to also systematically examine many values of the confidence bound. It will be particularly interesting to explore the impact of different ITDs on the transitions between different steady states (such as between consensus and polarization) and on their convergence times. Another interesting research avenue is to incorporate temporal stochasticity into density-based BCMs [20,56], which describe the collective behavior of large populations of agents and traditionally take the form of integro-differential equations. Naturally, it is also worth exploring the behavior of BCMs with random-time interactions on real social networks and with ITDs that one estimates from empirical data.

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## DATA AVAILABILITY

The only data in the present paper is simulation data, which is not available. The code that we used to produce our simulations is available at Ref. [43].


[1] J. B. Bak-Coleman, M. Alfano, W. Barfuss, C. T. Bergstrom, M. A. Centeno, I. D. Couzin, J. F. Donges, M. Galesic, A. S. Gersick, J. Jacquet, A. B. Kao, R. E. Moran, P. Romanczuk, D. I. Rubenstein, K. J. Tombak, J. J. Van Bavel, and E. U. Weber, Stewardship of global collective behavior, Proc. Natl. Acad. Sci. USA **118**, e2025764118 (2021).

[2] M. Starnini, F. Baumann, T. Galla, D. Garcia, G. Iñiguez, M. Karsai, J. Lorenz, and K. Sznajd-Weron, Opinion dynamics: Statistical physics and beyond, arXiv:2507.11521.

[3] G. Caldarelli, O. Artime, G. Fischetti, A. N. Stefano Guarino, F. Saracco, P. Holme, and M. de Domenico, The physics of news, rumors, and opinions, arXiv:2510.15053.

[4] H. Noorazar, K. R. Vixie, A. Talebanpour, and Y. Hu, From classical to modern opinion dynamics, Int. J. Mod. Phys. C **31**, 2050101 (2020).

[5] H. Olsson and M. Galesic, Analogies for modeling belief dynamics, Trends Cognit. Sci. **28**, 907 (2024).

[6] C. Castellano, S. Fortunato, and V. Loreto, Statistical physics of social dynamics, Rev. Mod. Phys. **81**, 591 (2009).

[7] P. Sen and B. K. Chakrabarti, *Sociophysics: An Introduction* (Oxford University Press, Oxford, UK, 2014).

[8] M. Jusup, P. Holme, K. Kanazawa, M. Takayasu, I. Romić, Z. Wang, S. Geček, T. Lipić, B. Podobnik, L. Wang, W. Luo, T. Klanjšček, J. Fan, S. Boccaletti, and M. Perc, Social physics, Phys. Rep. **948**, 1 (2022).

[9] V. Amelkin, F. Bullo, and A. K. Singh, Polar opinion dynamics in social networks, IEEE Trans. Autom. Control **62**, 5650 (2017).

[10] J.-D. Mathias, S. Huet, and G. Deffuant, An energy-like indicator to assess opinion resilience, Physica A **473**, 501 (2017).

[11] A. Grabowski, Opinion formation in a social network: The role of human activity, Physica A **388**, 961 (2009).

[12] V. Sood and S. Redner, Voter model on heterogeneous graphs, Phys. Rev. Lett. **94**, 178701 (2005).

[13] M. H. DeGroot, Reaching a consensus, J. Am. Stat. Assoc. **69**, 118 (1974).

[14] G. Deffuant, D. Neau, F. Amblard, and G. Weisbuch, Mixing beliefs among interacting agents, Adv. Complex Syst. **03**, 87 (2000).

[15] R. Hegselmann and U. Krause, Opinion dynamics and bounded confidence: Models, analysis, and simulation, J. Artif. Soc. Social Simul. **5**(3), 2 (2002).

[16] G. J. Li, J. Luo, and M. A. Porter, Bounded-confidence models of opinion dynamics with adaptive confidence bounds, SIAM J. Appl. Dyn. Syst. **24**, 994 (2025).

[17] C. Altafini, Dynamics of opinion forming in structurally balanced social networks, PLoS One **7**, e38135 (2012).

[18] V. D. Blondel, J. M. Hendrickx, and J. N. Tsitsiklis, Continuous-time average-preserving opinion dynamics with opinion-dependent communications, SIAM J. Control Optim. **48**, 5214 (2010).

[19] H. Z. Brooks, P. S. Chodrow, and M. A. Porter, Emergence of polarization in a sigmoidal bounded-confidence model of opinion dynamics, SIAM J. Appl. Dyn. Syst. **23**, 1442 (2024).

[20] E. Ben-Naim, P. L. Krapivsky, and S. Redner, Bifurcations and patterns in compromise processes, Physica D **183**, 190 (2003).

[21] A. Diekmann, Cooperation in an asymmetric volunteer's dilemma game theory and experimental evidence, Int. J. Game Theory **22**, 75 (1993).

[22] I. Ajzen, The theory of planned behavior: Frequently asked questions, Hum. Behav. Emerging Technol. **2**, 314 (2020).

[23] M. E. J. Newman, *Networks*, 2nd ed. (Oxford University Press, Oxford, UK, 2018).

[24] K. Sugishita, M. A. Porter, M. Beguerisse-Díaz, and N. Masuda, Opinion dynamics on tie-decay networks, Phys. Rev. Res. **3**, 023249 (2021).

[25] N. Perra, B. Gonçalves, R. Pastor-Satorras, and A. Vespignani, Activity driven modeling of time varying networks, Sci. Rep. **2**, 469 (2012).

[26] B. D. Goddard, B. Gooding, H. Short, and G. A. Pavliotis, Noisy bounded confidence models for opinion dynamics: The effect of boundary conditions on phase transitions, IMA J. Appl. Math. **87**, 80 (2022).

[27] M. Pineda, R. Toral, and E. Hernández-García, Noisy continuous-opinion dynamics, J. Stat. Mech. (2009) P08001.

[28] S. Redner, Reality-inspired voter models: A mini-review, C. R. Phys. **20**, 275 (2019).

[29] J. Fernández-Gracia, V. M. Eguíluz, and M. San Miguel, Update rules and interevent time distributions: Slow ordering versus no ordering in the voter model, Phys. Rev. E **84**, 015103(R) (2011).

[30] H. Schawe, S. Fontaine, and L. Hernández, When network bridges foster consensus. Bounded confidence models in networked societies, Phys. Rev. Res. **3**, 023208 (2021).

[31] W. Feller, *An Introduction to Probability Theory and Its Applications* (John Wiley & Sons, New York, NY, USA, 1971), Vol. 2, Chap. XI.

[32] E. Scalas, The application of continuous-time random walks in finance and economics, Physica A **362**, 225 (2006).

[33] N. Masuda and P. Holme, Predicting and controlling infectious disease epidemics using temporal networks, F1000Prime Rep. **5**, 6 (2013).

[34] A.-L. Barabási, The origin of bursts and heavy tails in human dynamics, Nature **435**, 207 (2005).

[35] W. Chu and M. A. Porter, Non-Markovian models of opinion dynamics on temporal networks, SIAM J. Appl. Dyn. Syst. **22**, 2624 (2023).

[36] J. Lorenz, Continuous opinion dynamics under bounded confidence: A survey, Int. J. Mod. Phys. C **18**, 1819 (2007).

[37] C. Bernardo, C. Altafini, A. Proskurnikov, and F. Vasca, Bounded confidence opinion dynamics: A survey, Automatica **159**, 111302 (2024).

[38] V. D. Blondel, J. M. Hendrickx, and J. N. Tsitsiklis, On Krause's multi-agent consensus model with state-dependent connectivity, IEEE Trans. Autom. Control **54**, 2586 (2009).

[39] J. C. Dittmer, Consensus formation under bounded confidence, Nonlin. Anal. Theory Methods Appl. **47**, 4615 (2001).

[40] J. Lorenz, A stabilization theorem for dynamics of continuous opinions, Physica A **355**, 217 (2005).

[41] G. Chen, W. Su, W. Mei, and F. Bullo, Convergence of the heterogeneous Deffuant–Weisbuch model: A complete proof and some extensions, IEEE Trans. Autom. Control **70**, 877 (2024).

[42] X. F. Meng, R. A. Van Gorder, and M. A. Porter, Opinion formation and distribution in a bounded-confidence model on various networks, Phys. Rev. E **97**, 022312 (2018).

[43] W. Chu, BCMs with random-time interactions, Bitbucket (2026), https://bitbucket.org/chuwq/bcms-with-random-time-interactions/src/main/.

[44] T. Hoffmann, M. A. Porter, and R. Lambiotte, Generalized master equations for non-Poisson dynamics on networks, Phys. Rev. E **86**, 046102 (2012).

[45] L. Speidel, R. Lambiotte, K. Aihara, and N. Masuda, Steady state and mean recurrence time for random walks on stochastic temporal networks, Phys. Rev. E **91**, 012806 (2015).

[46] U. Krause, in *Communications in Difference Equations: Proceedings of the Fourth International Conference on Difference Equations*, edited by S. Elyadi, G. Ladas, J. Popenda, and J. Rakowski (Gordon and Breach Science Publishers, Amsterdam, the Netherlands, 2000), pp. 227–236.

[47] G. J. Li and M. A. Porter, Bounded-confidence model of opinion dynamics with heterogeneous node-activity levels, Phys. Rev. Res. **5**, 023179 (2023).

[48] V. Guttal and I. D. Couzin, Social interactions, information use, and the evolution of collective migration, Proc. Natl. Acad. Sci. USA **107**, 16172 (2010).

[49] C. K. Hemelrijk, Models of, and tests for, reciprocity, unidirectionality and other social interaction patterns at a group level, Anim. Behav. **39**, 1013 (1990).

[50] P. G. Fennell, S. Melnik, and J. P. Gleeson, Limitations of discrete-time approaches to continuous-time contagion dynamics, Phys. Rev. E **94**, 052125 (2016).

[51] N. Masuda and L. E. C. Rocha, A Gillespie algorithm for non-Markovian stochastic processes, SIAM Rev. **60**, 95 (2018).

[52] D. T. Gillespie, A general method for numerically simulating the stochastic time evolution of coupled chemical reactions, J. Comput. Phys. **22**, 403 (1976).

[53] M. Boguñá, L. F. Lafuerza, R. Toral, and M. Á. Serrano, Simulating non-Markovian stochastic processes, Phys. Rev. E **90**, 042108 (2014).

[54] F. Chung and L. Lu, The average distances in random graphs with given expected degrees, Proc. Natl. Acad. Sci. USA **99**, 15879 (2002).

[55] B. K. Fosdick, D. B. Larremore, J. Nishimura, and J. Ugander, Configuring random graph models with fixed degree sequences, SIAM Rev. **60**, 315 (2018).

[56] W. Chu and M. A. Porter, A density description of a bounded-confidence model of opinion dynamics on hypergraphs, SIAM J. Appl. Math. **83**, 2310 (2023).