CHAOS **28**, 013115 (2018)

# Synergistic effects in threshold models on networks

Jonas S. Juul[1,a)] and Mason A. Porter[2,3,4,b)]
[1]Niels Bohr Institute, University of Copenhagen, Blegdamsvej 17, Copenhagen 2100-DK, Denmark
[2]Department of Mathematics, University of California, Los Angeles, Los Angeles, California 90095, USA
[3]Oxford Centre for Industrial and Applied Mathematics, Mathematical Institute, University of Oxford, Oxford OX2 6GG, United Kingdom
[4]CABDyN Complexity Centre, University of Oxford, Oxford OX1 1HP, United Kingdom

(Received 23 January 2017; accepted 1 December 2017; published online 12 January 2018)

Network structure can have a significant impact on the propagation of diseases, memes, and information on social networks. Different types of spreading processes (and other dynamical processes) are affected by network architecture in different ways, and it is important to develop tractable models of spreading processes on networks to explore such issues. In this paper, we incorporate the idea of *synergy* into a two-state ("active" or "passive") threshold model of social influence on networks. Our model's update rule is deterministic, and the influence of each meme-carrying (i.e., active) neighbor can—depending on a parameter—either be enhanced or inhibited by an amount that depends on the number of active neighbors of a node. Such a synergistic system models social behavior in which the willingness to adopt either accelerates or saturates in a way that depends on the number of neighbors who have adopted that behavior. We illustrate that our model's synergy parameter has a crucial effect on system dynamics, as it determines whether degree-$k$ nodes are possible or impossible to activate. We simulate synergistic meme spreading on both random-graph models and networks constructed from empirical data. Using a heterogeneous mean-field approximation, which we derive under the assumption that a network is locally tree-like, we are able to determine which synergy-parameter values allow degree-$k$ nodes to be activated for many networks and for a broad family of synergistic models. *Published by AIP Publishing.*
https://doi.org/10.1063/1.5017962

**Models of cascading processes on networks yield insights into a large variety of processes, ranging from the spread of information and memes in social networks to propagating failures in infrastructure and bank networks.[1–9] In the context of social networks, it is very popular to study models of social influence based on overcoming individuals' stubbornness thresholds with peer pressure or influence.[1–3,10–14] Most such models consider peer pressure only from nearest neighbors, but it is also important to explore the influence of nodes beyond nearest neighbors (e.g., in the context of the "three degrees of influence" that has been reported in some studies).[15] If the combined influence from several nodes is different than the sum of the influences from individual nodes, *synergy* is taking place, and such synergistic effects can exert a major influence on spreading processes on networks. For example, in some systems, the amount of influence per person applying peer pressure may depend on the number of people who are applying peer pressure, and our goal in this paper is to incorporate such ideas into a threshold model of social influence in an analytically tractable way. In our synergistic model, we examine social behavior in which the willingness to adopt either accelerates or saturates in a way that depends on the number of neighbors who have adopted some behavior. We illustrate that a synergy parameter can have a crucial effect on system dynamics (e.g., by determining whether degree-*k* nodes are possible or impossible to activate). We also develop an analytical approximation (in the form of a heterogeneous mean-field theory) that is effective at forecasting both the temporal development of cascades and the sizes of cascades in many networks.**

## I. INTRODUCTION

Examining the spread of opinions, actions, memes, information, and misinformation in a population has received intense scrutiny in sociology, economics, computer science, physics, and many other fields.[1,2,4,6–8,10–24] Such phenomena—including the spread of defaults of banks, norms in populations, and products or new practices in populations—are often modeled as contagion processes that spread from node to node in a network,[25–27] in analogy with the spread of infectious diseases in a population.

In addition to modeling spreading processes themselves, it is important to consider the effect of network structure on contagions.[1,4,5,28] For example, network architecture can have a significant impact on phenomena such as the peak size and temporal development of outbreaks.[5,14,26,29–35]

In the study of contagions, many studies suppose that some small fraction of the nodes is infected initially, and they examine when a meme or disease can spread widely in a network.[4,31] When many nodes have adopted the meme (or become infected, in the context of a disease), it is said that a *cascade* has occurred.[11,23] A cascade can either be good or bad: a game developer may dream about his/her app becoming viral, but bank defaults due to systemic risk is a source

[a)]Electronic mail: jonas.juul@nbi.ku.dk
[b)]Electronic mail: mason@math.ucla.edu

of fear and dread in the financial sector. Seemingly viral spread of misinformation was also a prominent aspect of the 2016 U.S. presidential campaigns and election.

In applications ranging from finance[25] to meme spreading on Twitter,[36] researchers are very interested in trying to identify what causes cascading behavior on networks.[23] In one prominent family of models, known as *threshold models*, nodes survey their neighborhoods and adopt a meme (i.e., change their state) if sufficiently many of their neighboring nodes have already adopted this meme.[2,4,10,11,31] In most such models (and in most compartmental models), nodes are influenced only by their immediate neighbors, but in many situations (e.g., including social media such as Facebook and LinkedIn), individuals are able to observe actions by individuals beyond those to whom they are connected directly by an edge. [In fact, the sizes of the observable neighborhoods are different in different media (e.g., Facebook versus LinkedIn), and this can have profound effects on user experience, company algorithms, and more.[37]] In such situations, *synergistic* effects can occur, as a node can be influenced by multiple nodes at the same time, and the combined influence differs from the sum of the individual influences. Synergistic effects can either increase or decrease the chance that a node will adopt a meme. The aim of our paper is to construct an analytically tractable threshold model that incorporates synergistic effects into spreading processes on networks. We show that synergy has important effects on system dynamics, and we illustrate our model's spreading dynamics on several different networks.

Synergistic effects can contribute to the dynamics of spreading processes in a diverse variety of contexts. Examples include the spread of behavior,[38] the transmission of pathogens,[39] and the spread of new opportunities for farm activities among vineyards that form a wine route together.[40] Other phenomena with synergistic effects include the classical psychological "sidewalk experiment" with people staring up at the sky,[41] increased value from the merging of companies (see, e.g., Ref. 42), and "learning" of delinquent and criminal behavior.[43]

A few years ago, Pérez-Reche *et al.*[44] introduced a simple model of synergistic spreading by augmenting a compartmental model for a biological contagion, and they examined its dynamics on a square lattice in two dimensions. Their model was based on the standard susceptible–infectious–removed (SIR) model,[4,5] in which an *infectious* (I) node infects a *susceptible* (S) neighbor at a constant rate $r_{\mathrm{SI}} = \alpha$. In this SIR model, an infectious node is infectious for a time $\tau$ before it switches states to *removed* (R) (or "recovered", if one is less fatalistic), and then it can never become susceptible or infectious again. Pérez-Reche *et al.* generalized this SIR model so that $r_{\mathrm{SI}}$ includes not only the parameter $\alpha$ but also a synergy term $r_{\mathrm{syn}} = \beta m_i$, where $m_i$ is the number of nodes that contribute to the synergy when updating node $i$. They used a linear form of synergy: $r_{\mathrm{SI}} = \max\{\alpha + r_{\mathrm{syn}}, 0\} = \max\{\alpha + \beta m_i, 0\}$. For $\beta < 0$, the synergy is *interfering*, as synergy decreases the chance that node $i$ becomes infectious; for $\beta > 0$, the synergy is *constructive*, as synergy increases the chance that node $i$ becomes infectious. For $\beta = 0$, the model in Ref. 44 reduces to the standard SIR model; there is no synergy.

Pérez-Reche *et al.* defined two types of synergistic dynamics: (1) *r-synergy*, in which $m_i + 1$ is the total number of infectious nearest neighbors that simultaneously attempt to infect a focal susceptible node $i$; and (2) *d-synergy*, in which $m_i$ is the number of infectious nodes that are adjacent to the infectious nearest neighbor that is attempting to infect the susceptible node $i$. In their simulations, only the node at the center of the square grid is infectious at time $t = 0$; all other nodes start in the susceptible state. An important feature that Pérez-Reche *et al.* illustrated is that the value of the synergy parameter can affect whether an infectious host can infect more than one node.

Several papers have built on Ref. 44 and produced additional insights on synergistic spreading dynamics on networks.[45–48] To our knowledge, all previous studies considered update rules for node states that include stochasticity, and most of them examined spreading on lattices rather than on more general network structures. To facilitate analytical treatment of problems and to help isolate the effects of novel features in a model, it is often convenient to use deterministic update rules,[4] so we will do this in our exploration of synergistic effects. Specifically, we examine a two-state deterministic model in the form of a linear threshold model[2,10,11] in which a node can be either *active* or *inactive*. In the context of social contagions, "inactive" nodes are susceptible, and "active" nodes are infected. Upon becoming active, a node remains active forever. This facilitates analytical treatment, which we will use to shed light on synergistic spreading processes on networks. We focus on what Pérez-Reche *et al.*[44] called "r-synergy" (which includes only nearest-neighbor interactions), although our approach can be generalized for models with next-nearest-neighbor interactions (what Pérez-Reche *et al.* called "d-synergy"). It can also be generalized to incorporate interactions in even larger neighborhoods.

The rest of our paper is organized as follows. In Secs. II–IV, respectively, we introduce our models for synergistic spreading on networks, examine this model on two empirical networks, and develop an analytical approximation to describe the fraction of active nodes with degree $k$ and threshold $\phi$ in a network as a function of time. We also demonstrate that we expect certain values of a synergy parameter in the models to lead to abrupt changes in the dynamics. In Sec. V, we study synergistic spreading processes on several families of random networks. In Secs. V A and V B, we simulate synergistic spreading on 3-regular and Erdős–Rényi (ER) random networks and compare our analytical approximation to the simulated spreading processes. In Sec. V C, we simulate synergistic spreading on networks that we construct using a configuration model with degree distributions from two empirical networks. We conclude in Sec. VI.

## II. SYNERGISTIC THRESHOLD MODELS

Probably the most popular type of deterministic model of meme spreading is *threshold models* of social influence.[1,2,4,8,10–12,14] In the simplest type of threshold model, which is a generalization of bootstrap percolation,[49,50] one chooses a threshold $\phi_i$ for each node independently from a probability distribution $f(\phi)$ at time $t=0$ (in traditional bootstrap percolation, all nodes have the same threshold), and a node becomes "active" (i.e., it adopts the meme) if the fraction of its neighbors (or, in some variants, the number of its neighbors) that are active is at least this threshold. In the so-called *Watts threshold model* (WTM),[11] one considers the fraction of active neighbors. An inactive node $i$ with degree $k$, threshold $\phi_i$, and number $n_i$ of active neighbors becomes active when it is updated if and only if $n_i/k_i \geq \phi_i$. Because of the simplicity of basic threshold models, one can derive analytical approximations for cascade conditions in a variety of settings and in various extensions of the model.[12,31,34,51–53]

We seek to develop a synergistic threshold model. We focus on r-synergy and hence on nearest-neighbor interactions. (It is also worth thinking about models with d-synergy, but we leave this for future work.) We examine networks that consist of unweighted, undirected $N$-node graphs. At each point in time, a node can be in one of two states: *inactive* ($S_0$) or *active* ($S_1$). Inactive nodes exert no influence on their neighbors, and active nodes exert some amount of influence on their neighbors. The total amount of influence exerted by all neighbors of a node $i$ gives the *peer pressure* experienced by node $i$. Each node $i$ has a stubbornness threshold $\phi_i$ drawn from a distribution $f(\phi)$ at time $t=0$. We also activate a seed set of nodes at $t=0$. In all of our simulations, the seed consists of a single node chosen uniformly at random. Whenever we consider updating node $i$ (which we do in discrete time with synchronous updating), it becomes active if and only if the peer pressure on it is at least $\phi_i$.

We now construct a response function $F(n_i, k_i, \phi_i, \beta)$ that depends on the number $n_i$ of node $i$'s active neighbors, its degree $k_i$, its threshold $\phi_i$, and a global synergy parameter $\beta$ that we will explain below. The response function, a non-decreasing function of $n_i$, encodes when a node switches from the inactive state to the active one.[32] One can use such a response function to describe numerous models of binary-state dynamics, such as bond and site percolation and the WTM.[31] We express the response function using a peer-pressure function $\Xi(n_i, \beta)$ by writing

$$F(n_i, k_i, \phi_i, \beta) = \begin{cases} 0\,, & \text{if} \quad \Xi(n_i, \beta) < \phi_i k_i\,, \\ 1\,, & \text{otherwise}\,. \end{cases} \tag{1}$$

We want to incorporate synergistic effects into $\Xi(n_i, \beta)$. With inspiration from Pérez-Reche *et al.*,[44] we require that

$$\Xi(n_i, \beta) \begin{cases} = 0\,, & \text{if} \quad n_i = 0\,, \\ > n_i\,, & \text{if} \quad \beta > 0 \quad \text{and} \quad n_i \geq 2\,, \\ = n_i\,, & \text{if} \quad \beta = 0 \quad \text{or} \quad n_i = 1\,, \\ < n_i\,, & \text{if} \quad \beta < 0 \quad \text{and} \quad n_i \geq 2\,. \end{cases} \tag{2}$$

The first line of (2) encodes the requirement that the peer pressure experienced by a node is 0 if it does not have any active neighbors. From the second line, we see that for a positive synergy parameter $\beta > 0$ with $n_i \geq 2$ active neighbors, the peer pressure experience by node $i$ is larger than that in the WTM. This, therefore, amounts to a "constructive synergy." The third line encodes the fact that our synergistic model reduces to the WTM either when the synergy parameter $\beta = 0$ or when the number of active neighbors is $n_i = 1$. (Similarly, the synergistic SIR model of Pérez-Reche *et al.*[44] reduces to the standard SIR model for $\beta = 0$.) From the last line, we see that for a negative synergy parameter $\beta < 0$ and $n_i \geq 2$ active neighbors, the peer pressure experienced by node $i$ is smaller than that in the WTM. This, therefore, amounts to "interfering synergy."

We consider the following two peer-pressure functions that satisfy these requirements:

$$\Xi_{\text{multiplicative}} = (1+\beta)^{n_i - 1} n_i\,, \tag{3}$$

$$\Xi_{\text{power}} = n_i^{1+\beta}\,. \tag{4}$$

Naturally, these are not the only two functions that satisfy the requirements in Eq. (2). Additionally, in Sec. V A, we will argue that any synergistic peer-pressure function that is non-decreasing and continuous in the synergy parameter $\beta$ exhibits the same qualitative behavior as these two functions, in the sense of experiencing the same types of bifurcations.

If a node is *vulnerable* (i.e., it can be activated by a single active neighbor), it remains vulnerable if one introduces synergy using Eq. (3) or Eq. (4). Moreover, no non-vulnerable node can become vulnerable as a result of the synergy introduced using Eq. (3) or Eq. (4). We seek to examine when synergy effects, as encapsulated by the parameter $\beta$, change the number of active neighbors that can activate a degree-$k$ node. That is, we seek to examine when synergy can assist or hinder the spread of a meme through a network. Let us calculate when a specific change like this occurs. Suppose that a node $i$ with degree $k_i$ can be activated when there are at least $m_i$ active neighbors for $\beta = 0$. We wish to determine the values of $\beta$ for which $l_i$ active neighbors are sufficient to activate node $i$. For the power synergy model (4), we calculate

$$(l_i)^{1+\beta} \geq \phi_i k_i, \tag{5}$$

$$\Rightarrow \beta \geq \frac{\ln(\phi_i k_i)}{\ln(l_i)} - 1\,. \tag{6}$$

For the multiplicative synergy model (3), we obtain

$$\beta \geq \left(\frac{\phi_i k_i}{l_i}\right)^{1/(l_i - 1)}. \tag{7}$$

More generally, except for $m_i = 1$ or $l_i = 1$ (by construction, nodes cannot become vulnerable or stop being vulnerable due to synergistic effects), we can solve for the value at which any $l_i \in \mathbb{N}$ active neighbors can activate a node with degree $k_i$ and threshold $\phi_i$, given the synergy parameter $\beta$. We thereby examine how synergy alters the difficulty of activating nodes.

When we initiate our simulations with only a single node as a seed, there is a risk that this seed is surrounded—or is part of a small number of vulnerable nodes that are surrounded—by non-vulnerable nodes. Because such situations arise from the choice of threshold distribution $f(\phi)$ rather than from synergistic effects, we discard such simulations throughout this paper.

## III. SYNERGY IN TWO EMPIRICAL NETWORKS

We start by examining the synergistic threshold model with power synergy (4) on the network of condensed-matter physics paper coauthorships from Ref. 54. (This network is available at https://snap.stanford.edu/data/.) In this network, a node represents an author, and there is an undirected edge between nodes $i$ and $j$ if the authors coauthored at least 1 paper. We suppose for simplicity that all nodes have a threshold of $\phi = 1/10$.

We show the results of our simulations in Fig. 1. We use power synergy (4), and we show results for interfering synergy (with $\beta = -0.80$) in panel (a) and constructive synergy (with $\beta = 0.15$) in panel (b). Data points correspond to the mean fraction of degree-$k$ nodes that are active at each time step. Among our simulations, we include only realizations in which the meme activates at least 0.5% of the nodes in the network. For each degree, a smaller or equal fraction of nodes is activated for interfering synergy than for constructive synergy. In panel (b), we show the $k = 2$ curve from panel (a) for comparison. We see that it takes longer for the meme to spread in the network for interfering synergy than for constructive synergy.

We now examine our synergistic threshold model on another empirical network, the NORTHWESTERN25 network from the FACEBOOK100 data set.[55] This data set consists of the complete set of people and friendships of 100 different U.S. universities from one day in autumn 2005. NORTHWESTERN25 is the data for Northwestern University. We show results of our numerical simulations on the largest connected component of this network in Fig. 2. We suppose that all nodes have a threshold of $\phi = 1/33$, and we again examine power synergy with interfering synergy (with $\beta = -0.80$) in panel (a) and constructive synergy (with $\beta = 0.15$) in panel (b). For

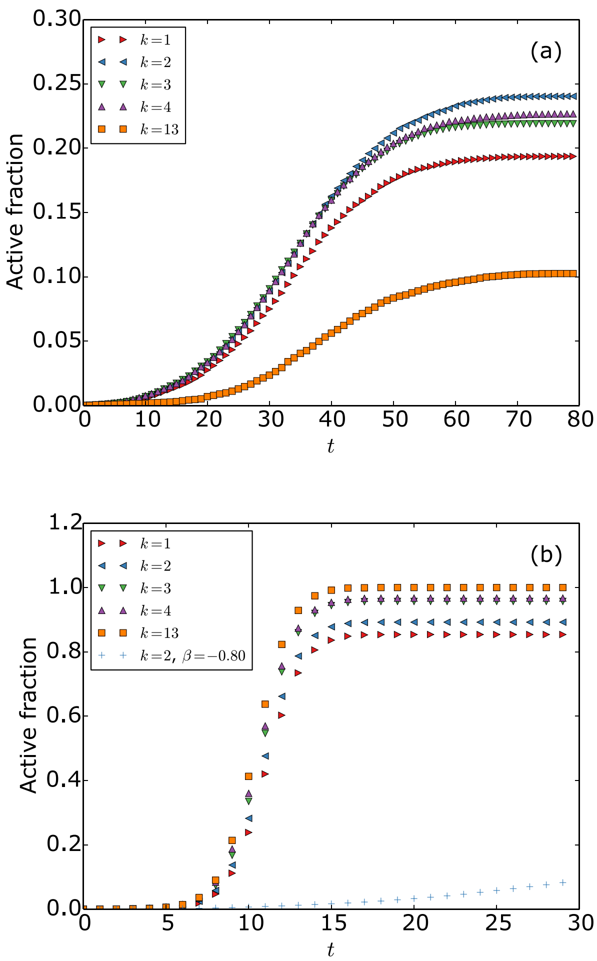


FIG. 1. Example behavior of the synergistic threshold model with power synergy (4) using (a) interfering synergy (with $\beta = -0.80$) and (b) constructive synergy (with $\beta = 0.15$). In panel (b), we show part of the curve for $k = 2$ from the case of interfering synergy for comparison. Because we choose the seed active node uniformly at random, there is a chance that only the seed is activated, and we do not take such runs into consideration. For the interfering synergy plot, only the seed was activated in 94 of our 110 runs; for constructive synergy, this occurred in 31 of 110 runs. For the simulations in this figure, we run the synergistic threshold model on the condensed-matter physics coauthor network from Ref. 54, and the threshold for each node is $\phi = 1/10$. For each degree, a smaller or equal fraction of nodes becomes active for interfering synergy than for constructive synergy. It also takes longer for the meme to spread in the network for interfering synergy than it does for constructive synergy.

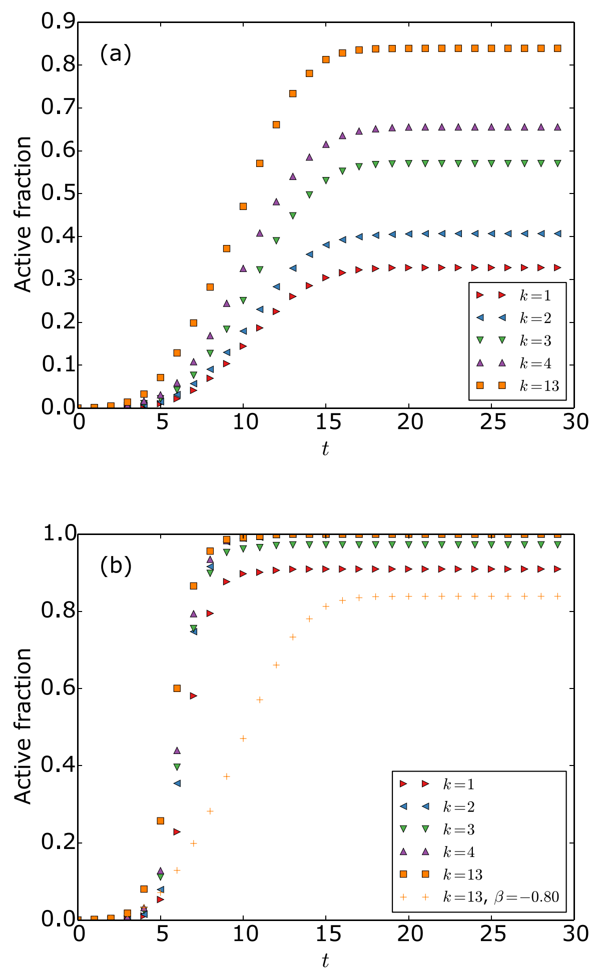


FIG. 2. Example behavior of the synergistic threshold model with power synergy (4) using (a) interfering synergy (with $\beta = -0.80$) and (b) constructive synergy (with $\beta = 0.15$). In panel (b), we show the curve for $k = 13$ for the case of interfering synergy for comparison. Because we choose the seed active node uniformly at random, there is a chance that only the seed is activated, and we do not take such runs into consideration. For the interfering synergy plot, only the seed was activated in 30 of 110 runs; for constructive synergy, this occurred in 24 of 110 runs. For the simulations in this figure, we run the synergistic threshold model on the NORTHWESTERN25 network from the FACEBOOK100 data set,[55] and the threshold for each node is $\phi = 1/33$. For each degree, a smaller or equal fraction of nodes becomes active for interfering synergy than for constructive synergy. It also takes longer for the meme to spread in the network for interfering synergy than it does for constructive synergy.

comparison, we include the curve for degree $k = 13$ for constructive synergy among our plots for interfering synergy. We again see that it takes longer for the meme to spread in the network for interfering synergy than it does for constructive synergy and that, for each degree, a smaller or equal fraction of nodes is activated for interfering synergy than for constructive synergy.

## IV. ANALYTICAL APPROXIMATION OF THE NUMBER OF ACTIVE NODES VERSUS TIME

We now develop an analytical approximation that describes the fraction of active nodes in a network as a function of time, given a peer-pressure function, degree distribution, and threshold distribution. This approximation is a heterogeneous mean-field approximation,[56] and it assumes that neighbors of a node are independent of each other. In our derivation, we assume that networks are locally tree-like,[4,57] which treats such pairs of neighbors as independent (because, in the approximation, they are not adjacent to each other).

Recall that we employ synchronous updating in our simulations. Because our update rule is deterministic, synchronous updating and asynchronous updating yield the same final (i.e., steady state) fraction of active nodes.[58] At time $t = 0$, we activate one seed node of the $N$ total nodes. For our theoretical analysis, this entails that the expected initially active fraction of nodes with degree $k$ and threshold $\phi$ is $\psi_k^\phi = 1/N$ for all choices of $k$ and $\phi$.[59] See Refs. 32 and 60 for a discussion of the effects on cascade size of using a single active node (as opposed to a specified fraction of active nodes) as a seed for the WTM, and see Ref. 61 for a recent discussion of issues regarding synchronous versus asynchronous updating (where asynchronous updating, such as through a Gillespie algorithm, is meant to model continuous-time dynamics) for dynamical processes on networks.

To calculate the fraction $\rho_k^\phi(n+1)$ of active nodes with degree $k$ and threshold $\phi$ at time $n+1$, we write the recursive formula (as in, e.g., Refs. 32, 34, and 60)

$$\rho_k^\phi(n+1) = \psi_k^\phi + (1-\psi_k^\phi)\sum_{j=0}^{k} B_j^k(\bar{q}_k{}^\phi(n))F(j,k,\phi,\beta)\,, \tag{8}$$

where $\bar{q}_k^\phi(n)$ is the probability that a neighbor (chosen uniformly at random) of a uniformly-randomly chosen inactive node with degree $k$ and threshold $\phi$ is active at time $n$, and

$$B_j^k(p) = \binom{k}{j} p^j (1-p)^{k-j}\,. \tag{9}$$

The first term in Eq. (8) is the fraction of nodes that are active at time $t = 0$. The second term represents the nodes that are activated at a later time. The factor $1-\psi_k^\phi$ is present because these nodes are not part of the seed, the sum encompasses the probabilities that a degree-$k$ node can have $0, 1, \ldots, k$ active neighbors at time $n$, and the response function $F(j, k, \phi, \beta)$ encodes when an inactive node becomes active when its state is updated. The sum of the two terms in Eq. (8) gives the fraction of nodes with degree $k$ and threshold $\phi$ that are active at time $n+1$. We write $\bar{q}_k^\phi(n)$ as a function of $q_{k'}^{\phi'}(n)$, the probability that, for a given inactive node, a neighbor with degree $k'$ and threshold $\phi'$ is active at time $n$. This probability is

$$\bar{q}_k^\phi(n) = \frac{\sum_{k',\phi'} P\big((k,\phi),(k',\phi')\big) q_{k'}^{\phi'}(n)}{\sum_{k',\phi'} P\big((k,\phi),(k',\phi')\big)}\,, \tag{10}$$

where $P((k,\phi),(k',\phi'))$ is the probability that a node with degree $k$ and threshold $\phi$ is adjacent to a node with degree $k'$ and threshold $\phi'$. For an inactive node, the probability that a neighboring node with degree $k$ and threshold $\phi$ is active is

$$q_k^\phi(n+1) = \psi_k^\phi + (1-\psi_k^\phi)\sum_{j=0}^{k-1} B_j^{k-1}(\bar{q}_k{}^\phi(n))F(j,k,\phi,\beta)\,. \tag{11}$$

The only difference between Eq. (11) and Eq. (8) stems from the fact that the degree-$k$ neighbor that we consider in (11) has a maximum of $k-1$ active neighbors if it is adjacent to at least one inactive node. In these equations, we have assumed that each neighbor of node $i$ is independent of the others, because (as indicated above) we are assuming that the network is locally tree-like.[4,57] We also assume that all nodes with degree $k$ and threshold $\phi$ have the same dynamics, so our approach constitutes a heterogeneous mean-field approximation.[56]

## V. SYNERGY IN SYNTHETIC NETWORKS

To illustrate our theoretical results, we examine synergistic spreading in several families of random graphs. For each family of networks, we draw a new network from the ensemble (which is a probability distribution on graphs) for each simulation of a synergistic threshold model. For all networks except Erdős–Rényi (ER) networks, we specify a degree distribution $p(k)$. We use this to determine a degree for each of 10 000 nodes, and we then connect these nodes to each other using a configuration model (connecting ends of edges to each other uniformly at random).[62]

### A. Synergy in 3-regular configuration-model networks

We first examine 3-regular random networks, in which every node has degree 3, which we construct by matching stubs (i.e., ends of edges) uniformly at random. We study how synergy affects meme spreading on these networks by examining several values of the parameter $\beta$ for both multiplicative and power synergy. In our numerical simulations, we suppose that a fraction $p_0 = 0.8$ of the nodes have threshold $\phi = 0.32 < 1/3$ and that a fraction $1 - p_0 = 0.2$ of the nodes have threshold $\phi = 1$. Although our numerical simulations illustrate a rather specific scenario, having only two types of nodes facilitates a detailed investigation of the fraction of active nodes of each type as a function of $\beta$ and time. Our particular choice of $p_0$ ensures that there is a large fraction of vulnerable nodes, but it is otherwise arbitrary. It is also worthwhile to do numerical explorations for a wide variety of threshold distributions, but we leave those for future work.

We choose a single node uniformly at random as a seed and update nodes synchronously at each discrete time step. We stop the simulations when we reach steady state (i.e., when no more nodes can eventually activate). In Fig. 3, we consider multiplicative synergy and plot the steady-state active fractions of high-threshold and low-threshold nodes as a function of the synergy parameter $\beta$. Each data point is a mean over 10 realizations of the spreading process. For each realization, we create a new 3-regular configuration-model network.

When $\beta$ surpasses the values 0 and 0.5, the final fraction of active nodes with threshold $\phi = 1$ increases dramatically. We can see this from Eqs. (7) and (1). For $\beta < 0$, it is not possible to satisfy $\phi_i k_i \geq (1+\beta)^{n_i-1} n_i$, because $n_i \leq k_i$. For $\beta \in [0, 0.5)$, the relation $\phi_i k_i \geq (1+\beta)^{n_i-1} n_i$ holds only for $n_i = k_i$. In this case, nodes with $\phi = 1$ can be activated, but they are never able to help activate a neighbor (unless they are part of the seed set of active nodes), as all of their neighbors are necessarily already active once they have been activated. For $\beta \geq 0.5$, the relation $\phi_i k_i \geq (1+\beta)^{n_i-1} n_i$ holds for $n_i = k_i$ and $n_i = k_i - 1$. In this case, nodes with $\phi = 1$ can be activated even when they still have an inactive neighbor. Hence, nodes with $\phi = 1$ can help spread the meme, resulting in more active nodes with both $\phi = 1$ and $\phi = 0.32$ than what occurs for $\beta < 0.5$. Rephrasing these observations, bifurcations occur at special values of $\beta$ (which are $\beta = 0$ and $\beta = 0.5$ in this example) for the multiplicative peer-pressure function (3), and we calculate the bifurcation points by solving $\Xi(n_i, \beta) = k_i \phi_i$ for $n_i \in \{2, \ldots, k_i\}$ (where we exclude $n_i = 1$ because it corresponds to a vulnerable node, which by design, is vulnerable for any value of $\beta$). Such values of $\beta$ exist for any non-decreasing peer-pressure function $\Xi(n_i, \beta)$ that is continuous in $\beta$. For different peer-pressure functions, the value of $\beta$ that makes it possible for a specific number of active neighbors to activate a specific node can differ, but there is some value of $\beta$ for each function. Hence, in this sense, all continuous, non-decreasing synergistic peer-pressure functions behave in qualitatively the same way. By contrast, the peer-pressure function $\Xi = n_i^{1+|\beta|}$ is not non-decreasing. This function is similar to $\Xi_{\text{power}}$, but with $\beta \to |\beta|$. For this peer-pressure function, we do not obtain the leftmost step that we observe in Fig. 3, so this choice entails different qualitative behavior than what we observe with $\Xi_{\text{power}}$.

In Figs. 4(a) and 4(b), we show how the meme spreads for $\beta = 0.4999$ and $\beta = 0.5001$, respectively. Each data point is a mean over 100 realizations of the spreading process. For each realization, we create a new 3-regular configuration-model network.

For any response function, such as ones that use the peer-pressure functions (3) or (4), one can compute when $n_i \leq k_i$ nodes can activate a node with threshold $\phi_i$ by solving the equation $\Xi(n_i, \beta) = \phi_i k_i$. Therefore, different response

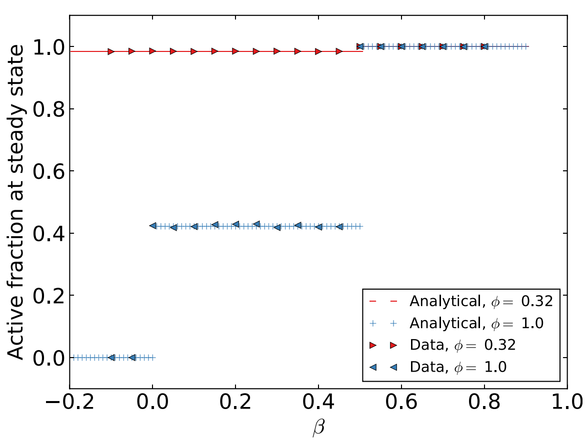


FIG. 3. Steady-state fraction of active nodes in 3-regular random networks of 10 000 nodes for our synergistic threshold model with the multiplicative synergistic peer-pressure function (3). A fraction $p_0 = 0.8$ of the nodes have threshold $\phi = 0.32 < 1/3$, and a fraction $1 - p_0 = 0.2$ of the nodes have threshold $\phi = 1$. Each data point is a mean of 10 realizations of the synergistic threshold model on 10 different 3-regular random networks, which we create using a configuration model. For each value of $\beta$, we construct 10 networks. (In doing these simulations, we discarded two total realizations due to the choice of seed node; the contagion did not spread enough in those cases.).

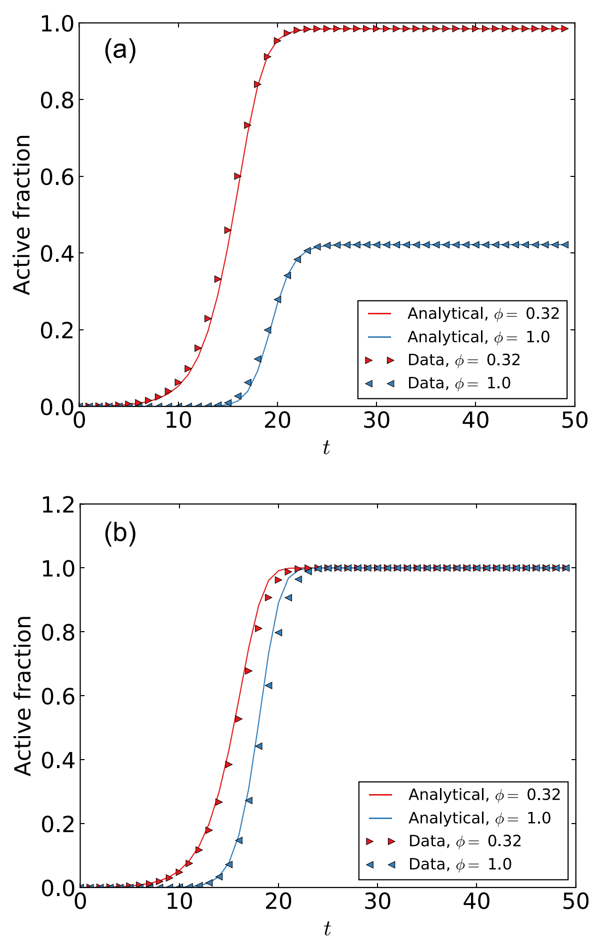


FIG. 4. Active fraction of nodes as a function of time for our synergistic threshold model with peer-pressure function (3) with constructive synergy in 3-regular random networks of 10 000 nodes. A fraction $p_0 = 0.8$ of the nodes have threshold $\phi = 0.32 < 1/3$, and a fraction $1 - p_0 = 0.2$ have threshold $\phi = 1$. In panel (a), the synergy parameter is $\beta = 0.4999$; in panel (b), it is $\beta = 0.5001$. In each panel, each data point is a mean over 100 realizations of the threshold model. We observe good agreement between the analytical approximation (8) and our simulations. (In these simulations, we did not need to discard any realizations due to the choice of seed node.) For each realization, we create a 3-regular random network using a configuration model. The sets of 100 networks are different in the two panels.

functions can have sudden increases in the steady-state fraction of active nodes at critical values of $\beta$ for the same reason: at these values of $\beta$, it becomes possible for some nodes to be activated with fewer active neighbors than is the case for smaller values of $\beta$. Although these critical values of $\beta$ can differ for different response functions, our two synergistic response functions exhibit qualitatively similar behavior, so we henceforth use only the response function that is specified by the power peer-pressure function (4).

### B. Synergy in Erdős–Rényi networks

We now simulate the spread of memes with power synergy (i.e., using the peer-pressure function (4)) on ER networks. Specifically, we use $\mathcal{G}(N,p)$ networks, where $N$ is the number of nodes and $p$ is the probability that there is an edge between a pair of nodes. The expected mean degree of such an ER network is $z = p(N-1) \approx pN$. First, we consider ER networks with expected mean degree $z = 3$, and we then consider ER networks with expected mean degree $z = 8$. In both cases, all nodes are assigned the same threshold $\phi = 1/7$. In our simulations, we use $N = 10\,000$. These networks do not in general consist of a single component, and this is especially relevant for $z = 3$. However, components other than the largest connected component (LCC) are so small that if the seed node is part of one of these small components, the total number of activated nodes is so small that such a simulation is one that we discard. We also confirm with computations that the LCC is very large even for $z = 3$. For example, in one set of 100 realizations of ER networks with $N = 10\,000$ nodes and expected mean degree $z = 3$, the mean size of the LCC is $9409.61 \pm 4.99$ nodes, and the mean size of the second-largest component is $4.07 \pm 0.95$ nodes.

#### 1. Expected Mean Degree z = 3

We use our analytical approximation (8) to find the expected steady-state active fraction of nodes as a function of their degree and the synergy parameter $\beta$ for the response function with power peer-pressure function (4). We plot these quantities in Fig. 5. In Fig. 6, we plot the time series of the fraction of active nodes when the synergy parameter is $\beta = -0.93$, for which our model predicts different steady-state active fractions for nodes with degrees 1, 2, 3, and 8. We observe very good agreement between our simulations and the analytical approximation (8) for these four node degrees.

#### 2. Expected Mean Degree z = 8

We now examine ER networks with expected mean degree $z = 8$. We simulate synergistic spreading with the power peer-pressure function (4) with a parameter value of $\beta = -0.835$. We choose this value of $\beta$ so that the steady-state fraction of active nodes is different for nodes with different degrees. In Fig. 7, we show the fraction of active nodes as a function of time, and we observe good agreement between our computations and our analytical approximation (8).

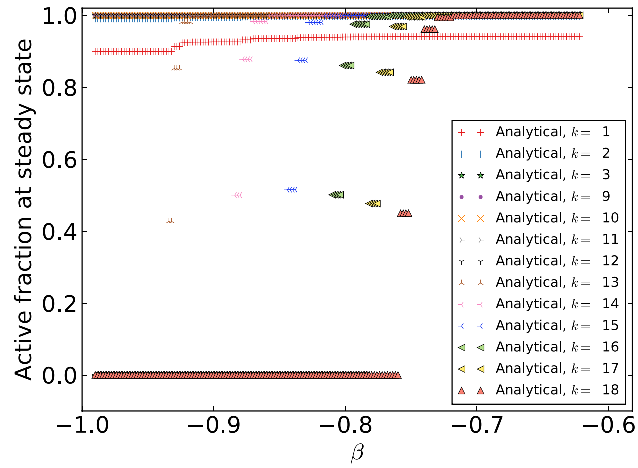


FIG. 5. Steady-state active fraction of degree-$k$ nodes as a function of the synergy parameter $\beta$ for a meme that spreads on ER networks with expected mean degree $z = 3$, homogeneous threshold $\phi = 1/7$, and a response function with the power peer-pressure function (4). Using Eq. (6), our analytical approximation (8) gives abrupt jumps that agree well with our numerical calculations (see Fig. 6).

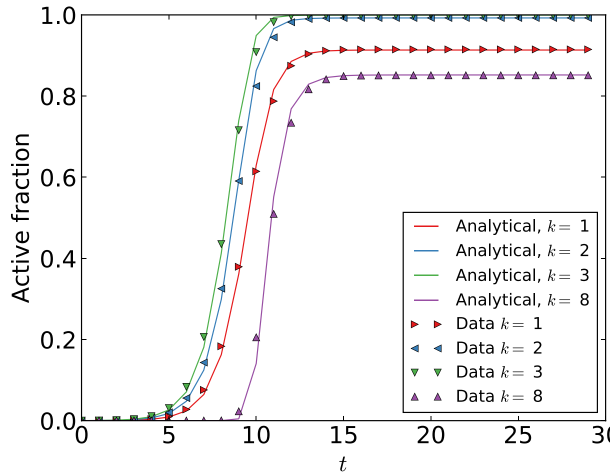


FIG. 6. Active fraction of nodes of degrees 1, 2, 3, and 8 as a function of time for synergistic spreading with the power peer-pressure function (4) in ER networks with interfing synergy $\beta = -0.93$, expected mean degree $z = 3$, and homogeneous threshold $\phi = 1/7$. Each data point is a mean over 31 realizations of the spreading process. Our analytical approximation (8) of the temporal activation of nodes of degrees 1, 2, 3, and 8 agrees very well with the results of our simulations. We obtain good matches for all node degrees that we examined in this way. We construct a new random ER network for each realization. (In doing these simulations, we discarded nine realizations due to the choice of seed node; the contagion did not spread enough in those cases.).

### C. Synergy on networks with degree distributions from empirical data

We now simulate the spread of synergistic memes on two networks with degree distributions from empirical data. In Sec. V C 1, we consider random networks created using a configuration model with a degree distribution determined by the degree sequence of the network of coauthorships in condensed-matter physics papers[54] that we examined in Sec. III. This network has a mean degree of $z \approx 8$. In Sec. V C 2, we simulate the spread of synergistic memes on configuration-model networks with a degree distribution from the degree sequence of the Northwestern25 network

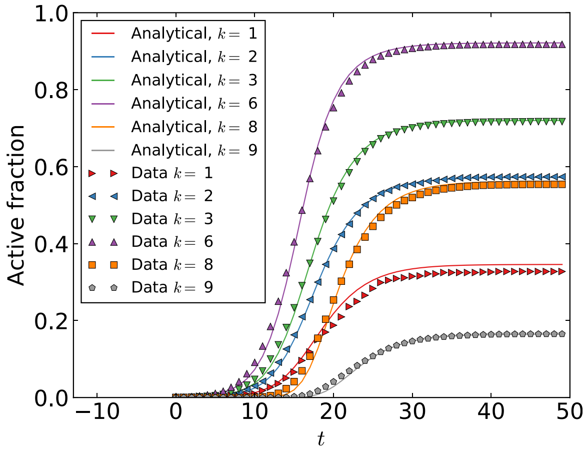


FIG. 7. Fraction of active degree-$k$ nodes as a function of time for our synergistic threshold model with power peer-pressure function (4) and interfering synergy $\beta = -0.835$ for ER networks with expected mean degree $z = 8$. Each node has a threshold of $\phi = 1/7$. We average our numerical computations over 31 realizations of the dynamics. We observe a good match between our numerical computations and our analytical approximation, although there is a slight discrepancy for nodes with $k = 1$. (In doing these simulations, we discarded nine realizations due to the choice of seed node; the contagion did not spread enough in those cases.).

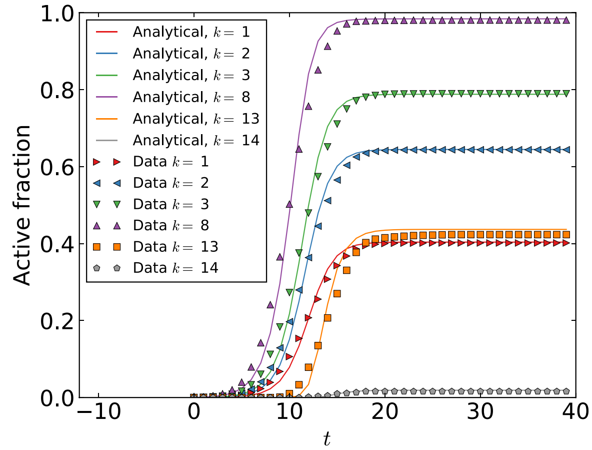


FIG. 8. Fraction of active nodes with degrees 1, 2, 3, 8, 13, and 14 as a function of time for our synergistic threshold model with power peer-pressure function (4) and interfering synergy $\beta = -0.85$ in configuration-model networks with a degree distribution determined from the degree sequence of the condensed-matter theory collaboration network from Ref. 54. Each node has a threshold of $\phi = 1/10$. We average the results over nine realizations of the dynamics, and we create a new configuration-model network for each realization. We observe good agreement between our analytical approximation and our numerical simulations. (In doing these simulations, we discarded one realization due to the choice of seed node; the contagion did not spread enough in that case.).

from the Facebook100 data set.[55,63] This Facebook network has a mean degree of $z \approx 92$. For each realization, we create a new 10 000-node network using a configuration model and degree sequences drawn from the associated degree distribution.

### *1. Condensed-matter physics collaboration network*

We draw the degree of each of the 10 000 nodes from the degree distribution of the condensed-matter physics collaboration network,[54] and we place edges using a configuration model. In Fig. 8, we plot the fraction of active nodes of degree $k$ as a function of time. We average over nine simulations (we discarded one simulation because there was insufficient spreading from the seed node) of the spreading of a meme with the power synergy peer-pressure function (4) on these networks. For each of these realizations, we create a new random network using a configuration model.

As in Sec. III, we use the peer-pressure function (4) and a homogeneous threshold $\phi = 1/10$ for our simulations. We first consider interfering synergy with $\beta = -0.85$, which makes it impossible to activate any node whose degree is 16 or larger. Our analytical approximation gives good agreement with our numerical simulations. In Fig. 9, we examine the effect of constructive synergy with the peer-pressure function (4). In this case, we use $\beta = 0.20$ and $\phi = 1/7$. For all node degrees that we checked, the steady-state active fractions from our analytical predictions and numerical simulations are indistinguishable. However, in our analytical approximation, the active fraction increases earlier than what we observe in our simulations.

### *2. A Facebook network*

We simulate the spread of synergistic memes on configuration-model networks that we construct using the degree sequence of the Northwestern25 network from the

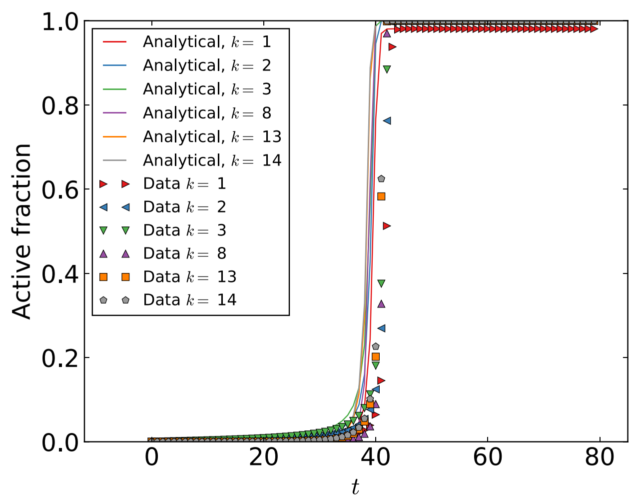


FIG. 9. Fraction of active nodes with degrees 1, 2, 3, 8, 13, and 14 as a function of time for our synergistic threshold model with power peer-pressure function (4) and constructive synergy $\beta = 0.20$ in configuration-model networks with a degree distribution determined from the degree sequence of the condensed-matter theory collaboration network from Ref. 54. Each node has a threshold of $\phi = 1/7$. We average the results over 10 realizations of the dynamics, and we create a new configuration-model network for each realization. In our analytical approximation, the fraction of active nodes increases slightly earlier than what we observe in our numerical simulations, but the resulting steady-state fractions of active nodes are visually indistinguishable. (In these simulations, we did not need to discard any realizations due to the choice of seed node.).

Facebook100 data set.[55] The network has a mean degree of $z \approx 92$, a minimum degree of $d = 1$, and a maximum degree of $d = 2105$. We assign all nodes a degree from a degree distribution based on the degree sequence of the Northwestern25 network, and we again create edges using a configuration model. We suppose that each node has a threshold of $\phi = 1/33$. In Fig. 10, we plot the fraction of active degree-$k$ nodes as a function of time. As in our other simulations, each realization is a different draw of one of these configuration-model networks. We show results for both interfering synergy (with power peer-pressure function (4) and $\beta = -0.05$) and constructive synergy (with $\beta = 0.15$ and peer-pressure function (4)). For this family of networks, our analytical approximation departs from our numerical simulations for both the steady-state fractions of active nodes and the times at which the active fractions of degree-$k$ nodes saturate. Additionally, our analytical approximation suggests that interfering synergy slows down the spreading process much more than is actually the case in our simulations.

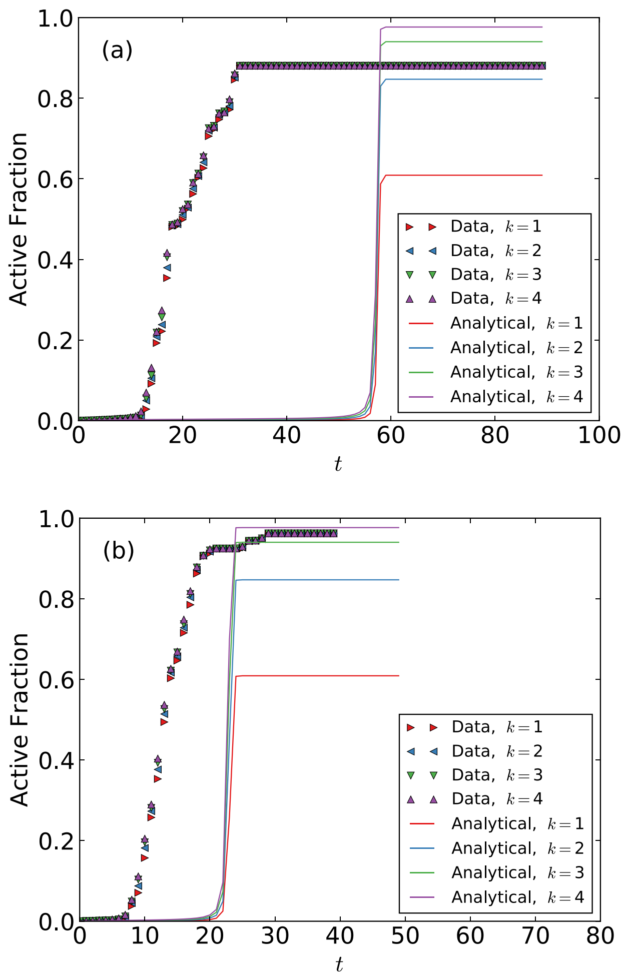


FIG. 10. Simulations of synergistic spreading on 10 000-node networks with a degree distribution determined from the degree sequence of the NORTHWESTERN25 network from the FACEBOOK100 data set.[55] The nodes have a homogeneous threshold of $\phi = 1/33$. (a) We examine interfering synergy (with power synergy (4) and $\beta = -0.05$) and plot the fraction of active nodes with degrees 1, 2, 3, and 4 as a function of time. All nodes with degree $k \geq 5$ exhibit similar behavior to those with the plotted degrees, and the final fractions of activated nodes are between 0.79 and 0.88. The time until the cascade occurs is very different in our analytical approximation (8) and numerical simulations, and there are also discrepancies in the steady-state fraction of active nodes between our analytics and numerics. We average our results over 51 realizations. (In doing these simulations, we discarded 149 realizations due to the choice of seed node; the contagion did not spread enough in those cases.) (b) We examine constructive synergy (with power synergy (4) and $\beta = 0.15$) and plot the fraction of active nodes with degrees 1, 2, 3, and 4 as a function of time. All nodes with degree $k \geq 5$ eventually have fractions of active nodes that are larger than 0.92. For this case as well, the time until the cascade occurs is very different in our analytical approximation (8) and our numerical simulations, and there are also discrepancies in the steady-state fraction of active nodes between our analytics and numerics. We average our results over 53 realizations. (In doing these simulations, we discarded 147 realizations due to the choice of seed node; the contagion did not spread enough in those cases.).

Our analytical approximation assumes that we are considering dynamics on a locally tree-like network, although such methodology can yield results that produce "unreasonably" effective matches between theory and computations (e.g., of the locations of phase transitions) even in many situations in which the hypotheses used to derive the theoretical approximations do not hold.[57] Melnik *et al.*[57] discussed various reasons why a tree-based theory may not provide a good description of the actual dynamics on a network (for a given dynamical system, such as a particular type of spreading process). For the FACEBOOK100 networks, they found for several spreading processes (including the WTM) that simulations with a homogeneous threshold distribution yield different results than what one obtains from a tree-based theory. (In our work, we usually use a homogenous threshold.) In contrast, they found for a Gaussian distribution of thresholds that WTM simulations with a seed consisting of all nodes with $\phi < 0$ yields results that are well-described by their tree-based approximation. In Ref. 57, all nodes with $\phi < 0$ were active at the beginning of simulations, because nodes with $\phi < 0$ are activated by any nonnegative fraction of active neighbors.[60] In our case, however, when using this threshold distribution, we obtain different results in simulations versus analytical approximations of cascades.

Two properties that may provide some indication of the effectiveness of tree-based theories for studying dynamical processes on a network are the mean geodesic (i.e., shortest) path length between nodes and the mean local clustering coefficient of the network. Although this is not something that is required mathematically (as there are counterexamples, such as a star graph), we expect that a "typical" tree-like network—in the extreme case, consider an ensemble of networks drawn uniformly at random from the set of all trees with a given number of nodes—to have larger mean geodesic path lengths than networks of the same size that are not tree-like. One also expects a locally-tree-like network to have a smaller mean local clustering coefficient than a network with the same number of nodes that is not locally tree-like. Averaging the mean geodesic path length between nodes in a set of 10 randomizations (based on a configuration model, as described above) of the NORTHWESTERN25 network yields $2.510 \pm 0.007$, which is somewhat smaller than in the original network and is much smaller than any other random network in our study (see Table I). Averaging the local clustering coefficient for the same 10 networks yields $0.02828 \pm 0.00109$, which is reasonably small but is much larger than for any other random network that we examine in this paper. This suggests that the randomized NORTHWESTERN25 networks are less tree-like than our other random networks. Additionally, the mean local clustering coefficient and the mean geodesic path length in the original NORTHWESTERN25 and condensed-matter collaboration networks are larger than those of the randomized networks that we construct from those networks. Unsurprisingly,

TABLE I. Mean geodesic path length between nodes and mean local clustering coefficient in the four random-network families that we examine. We construct the 3-regular random graphs using a configuration model (with stubs connected uniformly at random), and we also construct configuration-model networks using degree sequences (with associated degree distributions) from the condensed-matter collaboration network and NORTHWESTERN25 Facebook network. In each case, we average our results over 10 networks, and we indicate the mean values and the standard deviations of those means. We also indicate the values for the original NORTHWESTERN25 and condensed-matter collaboration networks. Observe that the mean geodesic path length between nodes is noticeably smaller in the NORTHWESTERN25 networks than in the other networks. Among the random networks, the mean local clustering coefficient is by far the largest in the NORTHWESTERN25 network, although the random network constructed from the condensed-matter collaboration network also has a much larger mean local clustering coefficient than the ER networks and 3-regular networks. The values of the mean geodesic path length and mean local clustering coefficients in the corresponding original empirical networks are larger (considerably so, for the clustering coeffecients) than those in the configuration-model networks with degrees drawn from the associated degree distributions.

| Network | Mean geodesic path length | Mean local clustering coefficient |
|---|---|---|
| 3-regular configuration model | $6.359 \pm 0.001$ | $0.00033 \pm 0.00011$ |
| Erdős–Rényi, $z = 3$ | $8.366 \pm 0.043$ | $0.00020 \pm 0.00014$ |
| Erdős–Rényi, $z = 8$ | $4.664 \pm 0.009$ | $0.00079 \pm 0.00008$ |
| Condensed-matter collaborations (original network) | 5.352 | 0.64173 |
| Condensed-matter collaborations (random networks) | $4.091 \pm 0.024$ | $0.00471 \pm 0.00049$ |
| NORTHWESTERN25 (original network) | 2.723 | 0.23828 |
| NORTHWESTERN25 (random networks) | $2.509 \pm 0.007$ | $0.02828 \pm 0.00109$ |

randomization considerably decreases the value of the mean local clustering coefficients, especially for the condensed-matter collaboration network.

## VI. CONCLUSIONS

It is important to study when diseases, information, memes, or other things (e.g., misinformation or "alternative facts") spread to a large number of nodes in a network.[4,15] For example, prior studies have suggested that some organisms and tumors spread via synergistic effects[64,65] and that synergistic effects can also be important for the spread of information on networks,[35] the spread of behavior in online social networks,[38] the transmission of pathogens,[39] and the spread of opportunities among vineyards on wine routes.[40]

In the present paper, we developed a deterministic threshold model with synergistic spreading, and we illustrated that constructive synergy speeds up the spreading process and that interfering synergy slows down the spreading process. Using both computations and a heterogeneous mean-field approximation (which assumes that a network is locally tree-like), we investigated the fraction of nodes, resolved by degree and as a function of a synergy parameter, that are activated for two empirical networks and several families of random graphs. We illustrated that the synergy functions (3) and (4) lead to critical values of a synergy parameter $\beta$, and we showed that such values also arise for any peer-pressure function that is continuous and non-decreasing in $\beta$. We found for non-vulnerable nodes with a specified degree $k$ that there exist $k-1$ critical synergy parameter values that indicate when a node is activated by at least $m \in \{2, 3, \ldots, k\}$ active neighbors. In all cases, we observed that constructive synergy speeds up the spreading process and that interfering synergy slows down the spreading process.

Investigating the influence of synergistic effects on spreading processes on networks is a promising area of study. It is an important feature to consider when studying the spread of information (and misinformation) on social networks,[35] the dynamics of certain biological organisms, and social processes in which the propensity for state changes either saturates or increases with the number of individuals who are trying to influence others in a network. It has interesting effects on spreading behavior in various types of networks, such as lattices[35] and modular networks,[47] and it can affect whether or not it is possible for certain nodes to adopt a certain meme or behavior.

In the future, it will be interesting to consider synergistic spreading processes—with both deterministic and stochastic update rules—on other types of networks, such as multilayer networks,[66–68] temporal networks,[69] and adaptive networks.[70]

## ACKNOWLEDGMENTS

Part of this research was carried out at the Mathematical Institute at University of Oxford. We thank James Fowler, James Gleeson, Matthew Jackson, Mikko Kivelä, James Moody, and four anonymous referees for helpful comments. J.S.J. also thanks the Mathematical Institute for their hospitality. J.S.J. received funding through the University of Copenhagen, UCPH 2016 Excellence Programme for Interdisciplinary Research.

[1] *Spreading Dynamics in Social Systems*, edited by S. Lehmann and Y.-Y. Ahn (Springer-Verlag, forthcoming), https://socialcontagionbook.github.io.
[2] T. W. Valente, *Network Models of the Diffusion of Innovations* (Hampton Press, 1995).
[3] E. M. Rogers, *Diffusion of Innovations*, 3rd ed. (Free Press, 1983).
[4] M. A. Porter and J. P. Gleeson, *Dynamical Systems on Networks: A Tutorial (Frontiers in Applied Dynamical Systems: Reviews and Tutorials)* (Springer, 2016), Vol. 4.
[5] R. Pastor-Satorras, C. Castellano, P. Van Mieghem, and A. Vespignani, "Epidemic processes in complex networks," Rev. Mod. Phys. **87**, 925–979 (2015).
[6] J. Borge-Holthoefer, R. A. Baños, S. González-Bailón, and Y. Moreno, "Cascading behavior in complex socio-technical networks," J. Complex Networks **1**, 3–24 (2013).
[7] K. Dey, S. Kaushik, and L. V. Subramaniam, "Literature survey on interplay of topics, information diffusion and connections on social networks," preprint arXiv:1706.00921 (2017).
[8] D. Guilbeault, J. Becker, and D. Centola, "Complex contagions: A decade in review," in *Spreading Dynamics in Social Systems*, edited by S.

Lehmann and Y.-Y. Ahn (Springer-Verlag, forthcoming), https://socialcontagionbook.github.io.
[9]A. G. Haldane and R. M. May, "Systemic risk in banking ecosystems," Nature **469**, 351–355 (2011).
[10]M. Granovetter, "Threshold models of collective behavior," Am. J. Sociol. **83**, 1420–1443 (1978).
[11]D. J. Watts, "A simple model of global cascades on random networks," Proc. Natl. Acad. Sci. U. S. A. **99**, 5766–5771 (2002).
[12]D. Kempe, J. Kleinberg, and E. Tardos, "Maximizing the spread of influence through a social network," in *Proceedings of the 9th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, KDD '03 (ACM, New York, NY, USA, 2003), pp. 137–146.
[13]P. S. Dodds and D. J. Watts, "A generalized model of social and biological contagion," J. Theor. Biol. **232**, 587–604 (2005).
[14]D. Centola and M. Macy, "Complex contagions and the weakness of long ties," Am. J. Sociol. **113**, 702–734 (2007).
[15]N. A. Christakis and J. H. Fowler, "Social contagion theory: Examining dynamic social networks and human behavior," Stat. Med. **32**, 556–577 (2013).
[16]M. O. Jackson and D. López-Pintado, "Diffusion and contagion in networks with heterogeneous agents and homophily," Network Sci. **1**, 49–67 (2013).
[17]M. O. Jackson and Y. Zenou, "Games on networks," in *Handbook of Game Theory*, edited by P. Young and S. Zamir (Elsevier, 2014), Vol. 4, pp. 95–163.
[18]E. Laurence, J.-G. Young, S. Melnik, and L. J. Dubé, "Exact analytical solution of irreversible binary dynamics on networks," preprint arXiv:1711.02721 (2017).
[19]C. Castellano, S. Fortunato, and V. Loreto, "Statistical physics of social dynamics," Rev. Mod. Phys. **81**, 591–646 (2009).
[20]N. A. Christakis and J. H. Fowler, "The spread of obesity in a large social network over 32 years," New England J. Med. **357**, 370–379 (2007).
[21]S. Aral, L. Muchnik, and A. Sundararajan, "Distinguishing influence-based contagion from homophily-driven diffusion in dynamic networks," Proc. Natl. Acad. Sci. U. S. A. **106**, 21544–21549 (2009).
[22]J. Ugander, L. Backstrom, C. Marlow, and J. Kleinberg, "Structural diversity in social contagion," Proc. Natl. Acad. Sci. U. S. A. **109**, 5962–5966 (2012).
[23]S. Goel, A. Anderson, J. Hofman, and D. J. Watts, "The structural virality of online diffusion," Manage. Sci. **62**, 180–196 (2016).
[24]D. Mollison, "Spatial contact models for ecological and epidemic spread," J. R. Stat. Soc. Ser. B **39**, 283–326 (1977).
[25]M. S. Helmut Elsinger and A. Lehar, "Risk assessment for banking systems," Manage. Sci. **52**, 1301–1314 (2006).
[26]D. Centola, R. Willer, and M. Macy, "The emperor's dilemma: A computational model of self-enforcing norms," Am. J. Sociol. **110**, 1009–1040 (2005).
[27]H. P. Young, "Innovation diffusion in heterogeneous populations: Contagion, social influence, and social learning," Am. Econ. Rev. **99**, 1899–1924 (2009).
[28]I. Z. Kiss, J. C. Miller, and P. L. Simon, *Mathematics of Epidemics on Networks: From Exact to Approximate Models* (Springer International Publishing, 2017).
[29]D. Taylor, F. Klimm, H. A. Harrington, M. Kramár, K. Mischaikow, M. A. Porter, and P. J. Mucha, "Topological data analysis of contagion maps for examining spreading processes on networks," Nat. Commun. **6**, 7723 (2015).
[30]E. Valdano, L. Ferreri, C. Poletto, and V. Colizza, "Analytical computation of the epidemic threshold on temporal networks," Phys. Rev. X **5**, 021005 (2015).
[31]J. P. Gleeson, "Binary-state dynamics on complex networks: Pair approximation and beyond," Phys. Rev. X **3**, 021004 (2013).
[32]J. P. Gleeson, "Cascades on correlated and modular random networks," Phys. Rev. E **77**, 046117 (2008).
[33]T. R. Hurd and J. P. Gleeson, "On Watts' cascade model with random link weights," J. Complex Networks **1**, 25–43 (2013).
[34]S. Melnik, J. A. Ward, J. P. Gleeson, and M. A. Porter, "Multi-stage complex contagions," Chaos **23**, 013124 (2013).
[35]M. Zheng, L. Lü, and M. Zhao, "Spreading in online social networks: The role of social reinforcement," Phys. Rev. E **88**, 012818 (2013).
[36]J. P. Gleeson, K. P. O'Sullivan, R. A. Baños, and Y. Moreno, "Effects of network structure, competition and memory time on social spreading phenomena," Phys. Rev. X **6**, 021019 (2016).
[37]C. Borgs, M. Brautbar, J. Chayes, S. Khanna, and B. Lucier, "The power of local information in social networks," in *Proceedings of the 8th International Workshop on Internet and Network Economics, WINE 2012, Liverpool, UK, December 10–12, 2012*, edited by P. W. Goldberg (Springer Berlin Heidelberg, Berlin, Heidelberg, 2012), pp. 406–419.
[38]D. Centola, "The spread of behavior in an online social network experiment," Science **329**, 1194–1197 (2010).
[39]J. J. Ludlam, G. J. Gibson, W. Otten, and C. A. Gilligan, "Applications of percolation theory to fungal spread with synergy," J. R. Soc. Interface **9**, 949–956 (2012).
[40]G. Brunori and A. Rossi, "Synergy and coherence through collective action: Some insights from wine routes in Tuscany," Sociologia Ruralis **40**, 409–423 (2000).
[41]S. Milgram, L. Bickman, and L. Berkowitz, "Note on the drawing power of crowds of different size," J. Pers. Soc. Psychol. **13**, 79–82 (1969).
[42]S. Sudarsanam, P. Holl, and A. Salami, "Shareholder wealth gains in mergers: Effect of synergy and ownership structure," J. Bus. Finance Accounting **23**, 673–698 (1996).
[43]C. Ballester, Y. Zenou, and A. Calvó-Armengol, "Delinquent networks," J. Eur. Econ. Assoc. **8**, 34–61 (2010).
[44]F. J. Pérez-Reche, J. J. Ludlam, S. N. Taraskin, and C. A. Gilligan, "Synergy in spreading processes: From exploitative to explorative foraging strategies," Phys. Rev. Lett. **106**, 218701 (2011).
[45]S. N. Taraskin and F. J. Pérez-Reche, "Effects of variable-state neighborhoods for spreading synergystic processes on lattices," Phys. Rev. E **88**, 062815 (2013).
[46]D. Broder-Rodgers, F. J. Pérez-Reche, and S. N. Taraskin, "Effects of local and global network connectivity on synergistic epidemics," Phys. Rev. E **92**, 062814 (2015).
[47]K. Chung, Y. Baek, D. Kim, M. Ha, and H. Jeong, "Generalized epidemic process on modular networks," Phys. Rev. E **89**, 052811 (2014).
[48]Q.-H. Liu, W. Wang, M. Tang, T. Zhou, and Y.-C. Lai, "Explosive spreading on complex networks: The role of synergy," Phys. Rev. E **95**, 042320 (2017).
[49]J. C. Miller, "Equivalence of several generalized percolation models on networks," Phys. Rev. E **94**, 032313 (2016).
[50]J. Chalupa, P. L. Leath, and G. R. Reich, "Bootstrap percolation on a Bethe lattice," J. Phys. C: Solid State Phys. **12**, L31–L35 (1979).
[51]A. Grönlund and P. Holme, "A network-based threshold model for the spreading of fads in society and markets," Adv. Complex Syst. **8**, 261–273 (2005).
[52]J. S. Juul and M. A. Porter, "Hipsters on networks: How a small group of individuals can lead to an anti-establishment majority," preprint arXiv:1707.07187 (2017).
[53]S.-W. Oh and M. A. Porter, "Complex contagions with timers," preprint arXiv:1706.04252 (2017).
[54]J. Leskovec, J. Kleinberg, and C. Faloutsos, "Graph evolution: Densification and shrinking diameters," ACM Trans. Knowl. Discovery Data **1**, 2 (2007).
[55]A. L. Traud, P. J. Mucha, and M. A. Porter, "Social structure of Facebook networks," Phys. A **391**, 4165–4180 (2012).
[56]S. Gómez, J. Gómez-Gardenes, Y. Moreno, and A. Arenas, "Nonperturbative heterogeneous mean-field approach to epidemic spreading in complex networks," Phys. Rev. E **84**, 036105 (2011).
[57]S. Melnik, A. Hackett, M. A. Porter, P. J. Mucha, and J. P. Gleeson, "The unreasonable effectiveness of tree-based theory for networks with clustering," Phys. Rev. E **83**, 036112 (2011).
[58]However, the temporal pattern of activation depends on the choice of asynchronous or synchronous updating. One can also generalize the WTM to other models in which the final fraction of active nodes does depend on this choice. For example, if nodes can be activated into different states as a function of their neighbors' states at the time of activation, then the steady-state fraction of nodes in a specific state can depend on whether one updates the nodes synchronously or asynchronously.
[59]The reason for this is as follows. We choose the single seed node uniformly at random from all nodes in a network, and we do not know the degree or the threshold of this node. Consider the fraction $g_k^\phi$ of nodes with degree $k$ and threshold $\phi$ that are active, given that the seed is one of these nodes; and then multiply $g_k^\phi$ by the probability that the seed node has degree $k$ and threshold $\phi$. That is, $\psi_k^\phi = g_k^\phi p(k,\phi) = \frac{1}{Np(k,\phi)} p(k,\phi) = 1/N$.
[60]J. P. Gleeson and D. J. Cahalane, "Seed size strongly affects cascades on random networks," Phys. Rev. E **75**, 056103 (2007).
[61]P. G. Fennell, S. Melnik, and J. P. Gleeson, "Limitations of discrete-time approaches to continuous-time contagion dynamics," Phys. Rev. E **94**, 052125 (2016).
[62]B. K. Fosdick, D. B. Larremore, J. Nishimura, and J. Ugander, "Configuring random graph models with fixed degree sequences," SIAM Rev. (in press); preprint arXiv:1608.00607.

[63]A. L. Traud, E. D. Kelsic, P. J. Mucha, and M. A. Porter, “Comparing community structure to characteristics in online collegiate social networks,” SIAM Rev. **53**, 526–543 (2011).
[64]E. Ben-Jacob, O. Schochet, A. Tenenbaum, I. Cohen, A. Czirok, T. Vicsek *et al*., “Generic modelling of cooperative growth patterns in bacterial colonies,” Nature **368**, 46–49 (1994).
[65]L. A. Liotta and E. C. Kohn, “The microenvironment of the tumour–host interface,” Nature **411**, 375–379 (2001).
[66]M. De Domenico, C. Granell, M. A. Porter, and A. Arenas, “The physics of spreading processes in multilayer networks,” Nat. Phys. **12**, 901–906 (2016).
[67]M. Salehi, R. Sharma, M. Marzolla, M. Magnani, P. Siyari, and D. Montesi, “Spreading processes in multilayer networks,” IEEE Trans. Network Sci. Eng. **2**, 65–83 (2015).
[68]M. Kivelä, A. Arenas, M. Barthélemy, J. P. Gleeson, Y. Moreno, and M. A. Porter, “Multilayer networks,” J. Complex Networks **2**, 203–271 (2014).
[69]P. Holme and J. Saramäki, “Temporal networks,” Phys. Rep. **519**, 97–125 (2012).
[70]G. Demirel, F. Vázquez, G. A. Bhöme, and T. Gross, “Moment-closure approximations for discrete adaptive networks,” Phys. D **267**, 68–80 (2014).