WHAT IS...

a Multilayer Network?

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We are surrounded by networks. People communicate with other people using online social networks like Facebook and Twitter (and occasionally even in person in offline social networks). They also travel or perform daily routines using transportation networks. Animals interact with each other in numerous ways in their social networks. Plants and fungi transport nutrients through networks.

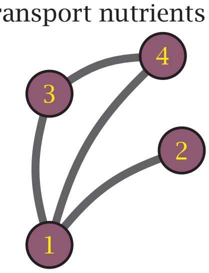

Figure 1. A graph consists of nodes (which I show as disks) that are connected to each other by edges (which I show as arcs).

The simplest type of network, which I show in Figure 1, is a *graph* $G = (V, E)$ [3], where the *nodes* (or "vertices") are elements of the set V of N entities in a network and $E \subseteq V \times V$ is a set of *edges* (or "links" or "ties") that encode pairwise interactions between the entities. A graph can be either undirected or directed. One can encode the information in a graph G as an $N \times N$ adjacency matrix $\mathbf{A}$, whose entry A_{ij} is equal to 1 if there is an edge from node i to node j and is otherwise equal to 0. In an undirected network, $A_{ij} = 1$ if and only if $A_{ji} = 1$. One can learn a lot about a graph G, and about many dynamical processes on it, by studying the properties (e.g., the eigenvalues) of its associated adjacency matrix $\mathbf{A}$. One can also assign weights to edges to represent connections with different strengths (e.g., stronger friendships, larger transportation capacity, and so on) by defining $w : E \longrightarrow X$. The most common choice is $X = \mathbb{R}_+$, so that all edge weights are positive real numbers.

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The study of networks in the form of graphs has a long, rich history in a variety of fields, including mathematics, statistics, computer science, sociology, physics, ecology, economics, and many others [3]. However, most real networks are much more complicated than ordinary graphs. For example, the nodes, edges, and edge weights can change in time; there can be multiple types of nodes or multiple types of relationships; and nodes can represent entities at different levels of granularity (e.g., a mathematics department, an applied-mathematics group, or an individual). One common type is a multirelational network, such as the air-transportation and social networks in Figure 2, in which there are multiple types of edges between nodes. We depict the social network in Figure 2b as an edge-colored multigraph, where different colors (i.e., annotations or "labels") represent different types of relationships: friendship, arguments, horseplay, and so on.

The formalism of *multilayer networks* [2], a generalization of graphs, was developed recently to help study multitudinous types of networks and to unify them into one framework. Many of these, such as multirelational networks in sociology and interconnected networks of different subsystems in engineering, have been studied for decades, but the development of the multilayer-network formalism to analyze such systems is very recent.

Even with its relatively short history, the study of multilayer networks has become very prominent. In briefly introducing this idea, I mostly follow the terminology and conventions from the review article [2].

A multilayer network $M = (V_M, E_M, V, \mathsf{L})$, as illustrated in Figure 3, has an underlying set V of N physical nodes (representing entities), often labeled $1, 2, 3, \ldots, N$, that manifest on layers in L that are constructed from

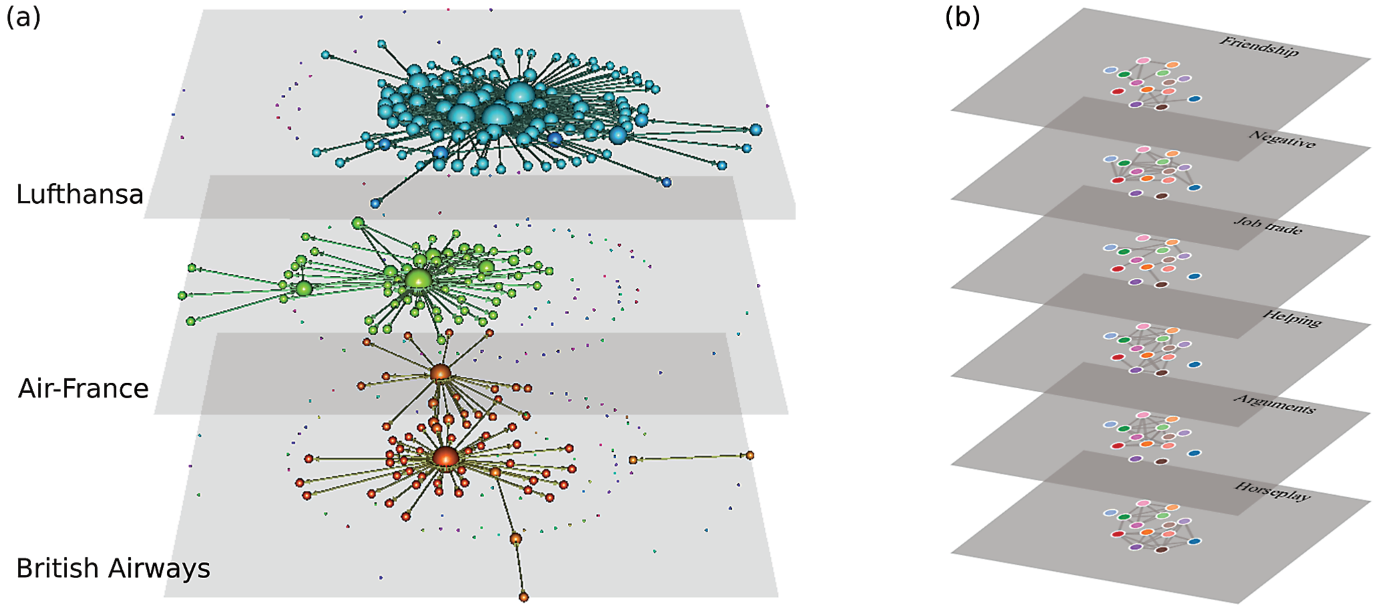

Figure 2. Two examples of multilayer networks: (a) part of an air-transportation network, in which each layer has flights from a different airline; and (b) a social network of individuals, with layers representing different types of relationships between them, in a bank-wiring room.

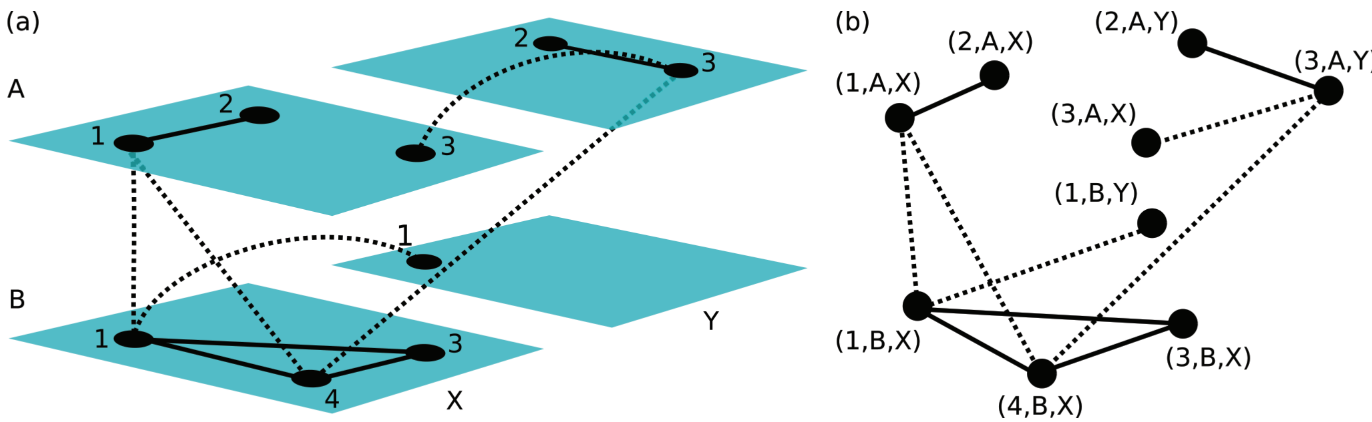

Figure 3. (a) An example of a multilayer network $M = (V_M, E_M, V, \mathsf{L})$ with four physical nodes and two aspects, which have corresponding elementary-layer sets $L_1 = \{A, B\}$ and $L_2 = \{X, Y\}$. The four layers of M are (A, X), (A, Y), (B, X), and (B, Y). Each layer includes a subset of the physical nodes in V. The set of state nodes is $V_M = \{(1, A, X), (2, A, X), (3, A, X), (2, A, Y), (3, A, Y), (1, B, X), (3, B, X), (4, B, X), (1, B, Y)\}$. One can connect nodes to each other in a pairwise manner both within layers and across layers. I show intralayer edges as solid lines and interlayer edges as dotted lines and arcs. (b) The graph $G_M = (V_M, E_M)$ associated the multilayer network M. I again show intralayer edges as solid lines and interlayer edges as dotted lines. The adjacency matrix of this graph, which has accompanying labels (in both nodes and edges) from the layer information, is the multilayer network's supra-adjacency matrix. (See Figure 4 for an example.) Intralayer edges, which correspond to an ordinary type of edge in a graph, are associated with nonzero entries in the diagonal blocks of a supra-adjacency matrix, whereas interlayer edges are associated with nonzero entries in off-diagonal blocks.

elementary-layer sets $L_1, \ldots, L_d$, where d is the number of "aspects" (i.e., types of layering). One layer in L is a combination, through the Cartesian product $L_1 \times \cdots \times L_d$, of an elementary layer from each aspect. In Figure 3, the sets of elementary layers are $L_1 = \{A, B\}$ and $L_2 = \{X, Y\}$. The set of node-layer tuples (sometimes called "state nodes") in M is $V_M \subseteq V \times L_1 \times \cdots \times L_d$, and the set of multilayer edges is $E_M \subseteq V_M \times V_M$. The edge $((i, \alpha), (j, \beta)) \in E_M$ indicates that there is an edge from node i on layer α to node j on layer β (and vice versa, if M is undirected). Each aspect of M represents a type of layering: a type of social tie, a point in time, and so on. For example, a multirelational network that does not change in time, such as the bank-wiring network in Figure 2b, has one aspect; a multirelational network that has layers covering multiple time points has two aspects; and so on. To consider weighted edges, one proceeds as in ordinary graphs by using a function $w : E_M \longrightarrow X$.

For example, suppose that Figure 3 represents a multilayer network of collaborations and citations among scientists. In this network, Buffy (physical node 1), Willow (2), Angel (3), and Wesley (4) are writing papers on the mathematical theory of vampire slaying and using this information in vampire-slaying expeditions. The elementary layers A and B encode different types of interactions: paper coauthorships (A) and joint expeditions (B). Suppose that the elementary layers X (2017) and Y (2018) represent years. Thus, the intralayer edge between state nodes $(1, A, X)$ and $(2, A, X)$ signifies that Buffy and Willow went on a joint vampire-slaying expedition in 2017. Let's suppose that interlayer edges, which are often harder to interpret than intralayer ones, represent the use of information from a paper or an exposition. Such an an interaction is directed, although I don't indicate any directions on the edges in Figure 3. To give an example, the edge from $(3, A, Y)$ to $(4, B, X)$ represents the fact that, in a 2018 paper, Angel used information from one of Wesley's 2017 expeditions.

Each unweighted multilayer network with d aspects and the same number of nodes in each layer has an associated adjacency tensor $\mathcal{A}$ of order $2(d + 1)$. Analogous to the case of ordinary graphs, each directed edge in E_M is associated with a 1 entry of $\mathcal{A}$ (undirected edges are each associated with two such entries) and the other entries (the "missing" edges) are 0. If a multilayer network does not have the same number of nodes in each layer, one can add empty nodes so that it does, but the edges attached to such nodes are "forbidden" edges. When studying multilayer networks, missing edges and forbidden edges need to be treated differently (e.g., when normalizing quantities such as clustering coefficients or measures of node-layer or edge importance). One can flatten $\mathcal{A}$ into a "supra-adjacency matrix" $\mathbf{A}_M$, which is the adjacency matrix of the graph G_M associated with M (as in Figure 3b). Intralayer edges are associated with entries on the diagonal blocks of a supra-adjacency matrix, and interlayer edges are associated with matrix entries on the off-diagonal blocks. Figure 4, which illustrates a type of multilayer network known as a cognitive social structure, gives an example supra-adjacency matrix and associated multilayer network. In practice, most numerical computations with multilayer networks employ supra-adjacency matrices.

Multilayer networks allow one to investigate a diverse set of complicated network architectures and to integrate different types of data into one mathematical object. Two key types of multilayer networks arise from (i) labeling edges or (ii) labeling nodes. When one labels edges, one thinks of edges in different layers as representing different types of relationships. This is the case for a "multiplex network," a type of multilayer network in which the only permitted types of interlayer edges are those that connect manifestations of the same physical node in different layers. A special case of a multiplex network is an edge-colored multigraph, like the one in Figure 2b. Interlayer edges in a multiplex network occur on diagonal elements of off-diagonal blocks in a supra-adjacency matrix (as in Figure 4b). By contrast, when one labels nodes, one can think of different layers as representing different subsystems (in "interconnected networks" or "networks of networks," such as in coupled infrastructure networks), and one can have interlayer edges with nonzero supra-adjacency matrix elements in both the diagonal and off-diagonal entries of the off-diagonal blocks.

Multilayer networks have rich structural properties, and dynamical processes on them behave in fascinating ways—including experiencing novel phase transitions, where system behavior changes qualitatively. For further discussion of dynamical processes on multilayer networks, see [2] and a recent survey article [1] on spreading processes on multilayer networks. An important idea is that interlayer edges are fundamentally different from intralayer edges, and it is often less straightforward to assign weights from data to interlayer edges than to intralayer ones. A conceptually easy situation is a multimodal transportation network, in which one might calculate an interlayer edge weight based on how long it takes to change modes of transportation (e.g., from the subway to a bus). For communication on a social network, one might construe an interlayer edge as representing a transition probability between different modes of communication. For other applications, interlayer edges can run into significant conceptual difficulties, and researchers struggle with how to make sense of them. For example, in protein interaction networks, a layer can represent a type of interaction; there are dependencies across layers, and interlayer edges can encode them, but how does one determine meaningful values for the weights of those edges?

Examining consequences of the relative weights of intralayer and interlayer edges has also led to interesting theoretical results. A valuable example started with a paper by Filippo Radicchi and Alex Arenas (*Nature Physics*, 2013); it has been built on subsequently by them and others [2]. Considering a multiplex network, Radicchi and Arenas constructed the combinatorial supra-Laplacian matrix $\tilde{\mathbf{L}}_M = \mathbf{D}_M - \mathbf{A}_M$, where $\mathbf{D}_M$ is the diagonal supramatrix that has node-layer strengths along the diagonal. Each diagonal entry of $\tilde{\mathbf{L}}_M$ consists of the sum of the corresponding row in $\mathbf{A}_M$, and each nondiagonal element of $\tilde{\mathbf{L}}_M$ consists of the corresponding element of $\mathbf{A}_M$ multiplied by -1. Using the case in which counterpart nodes in each pair of layers are connected with a homogeneous interlayer edge weight as an illustrative example, Radicchi and Arenas showed that $\tilde{\mathbf{L}}_M$'s smallest nontrivial eigenvalue Λ_2, which is related to many structural and dynamical features of the corresponding multilayer network M, has two distinct regimes when examined as a function of the relative weights of the interlayer and intralayer edges. They also showed that there is a discontinuous phase

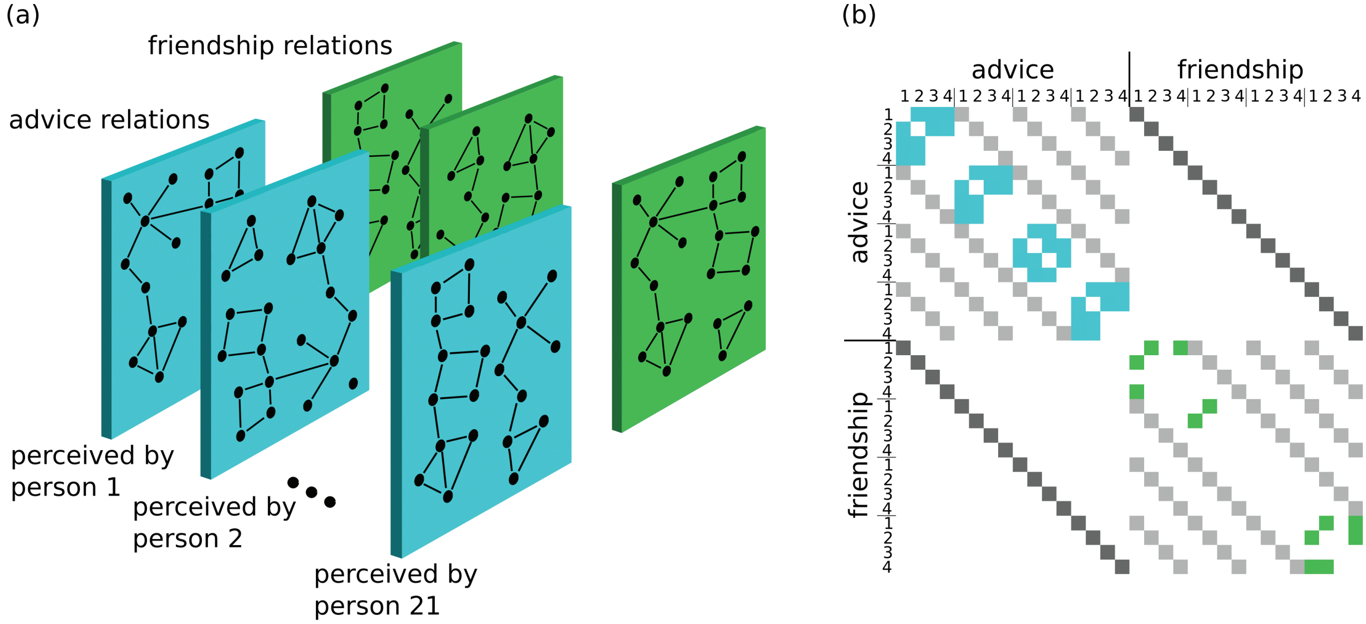

Figure 4. (a) Representation of a cognitive social structure as a multilayer network (without drawing any interlayer edges). In a cognitive social structure, each "perceiver" can have a different view of which edges exist in a network. Such data can arise, for example, from survey questions in which one asks people to fill in edges whenever they think a given type of relationship exists between two people. The depicted network has two aspects: perceiver and type of social connection. (b) Illustration of (a subset of) the supra-adjacency matrix that represents this cognitive social structure. This example has a nested block-diagonal structure: the interior block-diagonal structures (in blue and green) correspond to intralayer connections, and the two large blocks on the block diagonal have edges between different perceivers when the edge-type (i.e., friendship or advice) aspect is fixed to one of the two alternatives. One can represent this multilayer network as a multiplex network, because the only nonzero off-block-diagonal entries occur in the diagonal entries of the depicted blocks. (In the figure, I show nodes and layers only for persons 1–4.)

transition between those regimes. In one regime, Λ_2 is independent of the intralayer network structure and is thus determined by the weight of the interlayer edges. In the other regime, Λ_2 is bounded above by a constant multiplied by the smallest nontrivial eigenvalue of the unweighted superposition of the layers. This thread of work has important implications for interpretation of results in many investigations of multilayer networks, including for the behavior of dynamical processes on multilayer networks, evaluating node importances in such networks, and partitioning such networks into dense "communities" of nodes.

Before closing, it is also worth highlighting that there are two categories of dynamical processes on multilayer networks: (i) a single process that is defined on a multilayer network and (ii) interacting dynamical processes that are defined separately on different layers of such a network [1]. An important example of the first category is a random walk, where the qualitative behavior depends on the relative speeds and probabilities of intralayer versus interlayer steps. To examine the time scales of diffusion of such a dynamical process, one calculates the smallest nontrivial eigenvalues of $\tilde{\mathbf{L}}_M$, related supra-matrices, and matrices that are associated with individual network layers. Another example of this category of process is the spread of memes on social media. An example of the second category of dynamical process is interactions between multiple strains of a disease.

Excellent available software to both visualize and analyze multilayer networks includes MuxViz by Manlio De Domenico (http://muxviz.net; in R) and Pymnet by Mikko Kivelä (http://www.mkivela.com/pymnet/; in Python).

The study of multilayer networks—including their structure, dynamical processes on them, and numerous applications—is among the most vibrant areas of network science. They offer a promising avenue both for further mathematical study and for numerous applications.

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References

[1] Manlio De Domenico, Clara Granell, Mason A. Porter, and Alex Arenas. (2016), The physics of spreading processes in multilayer networks, *Nature Physics* **12**, 901–906.

[2] Mikko Kivelä, Alex Arenas, Marc Barthelemy, James P. Gleeson, Yamir Moreno, and Mason A. Porter. (2014), Multilayer networks, *Journal of Complex Networks* **2**, no. 3, 203–271.

[3] Mark E. J. Newman. (2018), *Networks*, 2nd ed., Oxford University Press.

Photo Credits

Figure 1 was drawn by the author for this article using TikZ-network, by Jürgen Hackl and available at https://github.com/hackl/tikz-network, which allows one to draw networks (including multilayer networks) directly in a LaTeX file.

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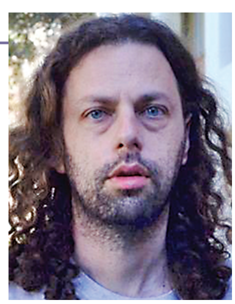

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Mason Porter studies networks, complex systems, nonlinear systems, and their applications. He enjoys playing games of all sorts and is very fond of fantasy, baseball (Go Dodgers!), making witty remarks, and inserting music lyrics and other allusions into his scientific articles.