Variability in Fermi-Pasta-Ulam-Tsingou arrays can prevent recurrences

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In 1955, Fermi, Pasta, Ulam, and Tsingou reported recurrence over time of energy between modes in a one-dimensional array of nonlinear oscillators. Subsequently, there have been myriad numerical experiments using homogenous FPUT arrays in the form of chains of ideal, nonlinearly coupled oscillators. However, inherent variations (e.g., due to manufacturing tolerance) introduce heterogeneity into the parameters of any physical system. We demonstrate that such tolerances degrade the observance of recurrences, often leading to complete loss in moderately-sized arrays. We numerically simulate heterogeneous FPUT systems to investigate the effects of tolerances on dynamics. Our results illustrate that tolerances in real nonlinear oscillator arrays may limit the applicability of results from numerical experiments on them to physical systems, unless appropriate heterogeneities are taken into account.

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I. INTRODUCTION

Fermi, Pasta, Ulam, and Tsingou undertook what is widely believed to be the first set of systematic "mathematical experiments" of a nonlinear system in their now-famous computations that they presented in their May 1955 technical report [1]. They excited a one-dimensional (1D) system of particles, with contiguous elements coupled to each other through a nonlinear interaction (see Fig. 1), in the first mode. They expected that the system's nonlinearity would lead to equipartition of energy between all modes. However, to their surprise, their numerical experiments instead showed that although energy does start to transfer from the first mode to higher modes, over time it actually appears to return to the first mode (see Fig. 2). This phenomenon constitutes a *recurrence* of energy between modes. Such "FPUT recurrences" have been the subject of much research in the past half century, and the FPUT numerical experiments have led to a wealth of work on recurrences, nonlinear lattice systems, and other prominent topics [2–10].

The equations of motion for a quadratically coupled 1D FPUT lattice (i.e., the FPUT-α model) are

$$\ddot{x}_i = (x_{i+1} - x_i) + (x_{i-1} - x_i) + \alpha[(x_{i+1} - x_i)^2 - (x_i - x_{i-1})^2], \tag{1}$$

where the index i (with $i \in \{1, \dots, N\}$) denotes the ith particle. The two ends of the lattice are fixed, so we take $x_0 = x_{N+1} = 0$. Applying this to the example of a mass–spring system in Fig. 1, the masses, springs, and spacing are all identical, with homogeneous masses m, spring constants k, and equilibrium spacing s all equal to 1. In Fig. 2, we show the first four modes of a 1D array of $N = 64$ oscillators with a nonlinear coupling coefficient of $\alpha = 0.25$.

The recurrence phenomenon in these simulations has significant implications in nonlinear dynamics, and extensive computational and theoretical work has examined FPUT recurrences [2,5,9,11,12]. However, there has been very little experimental evidence of such recurrences [13,14]. We propose that this paucity of evidence may be because of variability in system components due to manufacturing tolerances. Most analytical and computational studies of FPUT systems have assumed that the component particles and the couplings between them are homogeneous [5,7,11,15–17].

Because we are approaching this problem from a practical perspective, we subsequently refer to the particles as "oscillators". We do this to convey the idea that one can implement an FPUT system in more than one medium (e.g., as an electronic circuit or a mechanical structure).

It is typically simpler to study homogeneous lattice systems than heterogeneous ones, but it is difficult (if not impossible) to manufacture a large array of identical oscillators. Inherent manufacturing variations always introduce a tolerance, as is observed in any mechanical or electronic system [18–25]. Consequently, the individual oscillators of any real array will not be identical. Additionally, most physical parameters associated with real devices also vary with environmental conditions such as temperature, pressure, and humidity. Therefore, even if one could build an array of identical oscillators, it is

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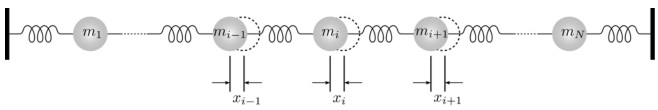

FIG. 1. A 1D system of masses coupled via nonlinear springs in a similar manner to that used by Fermi, Pasta, Ulam, and Tsingou in their numerical experiments.

very likely that the individual oscillators would change while the system is operating. Moreover, individual oscillators in a system would likely change in different ways, leading to further variability between the oscillators.

Although such tolerances are well studied in many physical systems [26,27], we are not aware of any investigations on their effects on FPUT lattices. Impurities and defects in lattices have attracted attention, including research on topics like Anderson phenomena and defect modes, but this has an extremely different focus from our engineering-oriented work [10,28]. Additionally, it is not clear how to extrapolate the results of these studies to phenomena such as tolerance-based inhomogeneities in arrays.

To experimentally observe recurrences in practical lattice systems, it is important as a model scenario to understand the effects of tolerance on an FPUT system. This will enable quantification of the constraints under which one can build such a system, if indeed it is possible. To do this, we study the FPUT system under the influence of typical manufacturing tolerances of physical systems.

II. METHODOLOGY FOR OUR NUMERICAL EXPERIMENTS

We consider the most basic scenario that was described in the original FPUT report [1]. We focus on a 1D FPUT-α array with an initial condition in which all of the energy is in the first mode. For each oscillator in a chain of N oscillators, the initial condition is $x_i = \sin[\pi i/(N+1)]$ and $\dot{x}_i = 0$. In an FPUT lattice described by Eq. (1), each oscillator has its own linear stiffness, and the oscillators interact with each other through nonlinear coupling. Unless we state otherwise, our simulations have $N = 64$ oscillators and a coefficient of $\alpha = 0.25$ for the nonlinear coupling terms. A mass–spring system, such as the one in Fig. 1, has tolerances t_j as in the following equation of motion:

$$\ddot{x}_i = t_{i+1}x_{i+1} + t_{i-1}x_{i-1} - 2t_i x_i + \alpha[(t_{i+1}x_{i+1} - t_i x_i)^2 - (t_i x_i - t_{i-1}x_{i-1})^2]. \quad (2)$$

We have normalized for mass, so any variation in masses is incorporated in the other tolerance terms. We study the system in Eq. (2) to investigate the impact of system heterogeneity due to tolerances on FPUT recurrences. Typical passive electronic components have tolerances of $\pm 0.1\%$, $\pm 1\%$, $\pm 5\%$, and $\pm$ 10% [29], so we use these values in our simulations.

The code that we use to simulate the FPUT equations is based on that published by Dauxois *et al.* [30], and it uses a standard method for solving such equations [31]. To ensure that our results are not the product of the chosen software, we double-checked our computations by using both MATLAB (with a standard Runge-Kutta algorithm in its ODE45 solver) and Python (with the SCIPY ordinary-differential-equation solver that implements a standard LSODA algorithm). We randomly generate the tolerance values for each simulation from a Gaussian distribution. We use this choice because it is the most common one in manufacturing [32–34]. For a tolerance of τ %, this entails a value of t_i that we draw from a Gaussian distribution with a mean of 1 and a standard deviation of $\sigma = 1/3 \times 0.01\tau$. (For example, for 1% tolerance, the standard deviation is 0.0033.) With a 6σ width (i.e., $\pm 3\sigma$), we note that 99.73% of the values of t_i lie within the interval $[1 - 0.01\tau, 1 + 0.01\tau]$. We place any outlying values of t_i on the corresponding edge of this interval.

In light of manufacturing tolerances, the exact value of each parameter is difficult to determine. Therefore, to help understand the effects of a random tolerance spread, we use Monte Carlo simulations and study 100 different sets of tolerance values applied to the FPUT system, while keeping all other system parameters and the initial condition fixed. We run multiple simulations that each use an independent draw of the tolerance values, and we then examine the dynamics of the oscillator arrays. The figures in this article are representative of the majority of our simulation results [35]. In Fig. 3, we show example results for a tolerance of $\pm 1\%$, and we note that Fig. 3(a) is representative of the response in more than 85% of the cases. However, for some combinations of tolerance values, recurrence is occasionally weak [see Fig. 3(b)] or breaks down [see Fig. 3(c)].

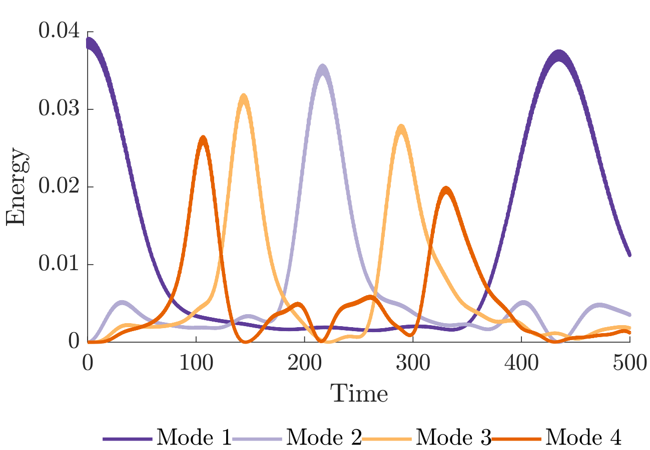

FIG. 2. The FPUT recurrence phenomenon in a 64-particle system (1) with nonlinearity coefficient $\alpha = 0.25$. We show the first four modes of the system. Observe that the energy moves between the modes, almost disappearing from the first mode before returning to the first mode some time later.

III. RESULTS

A. Effect of tolerance on recurrence

With a tolerance of $\pm 0.1\%$, we observe that the FPUT system in Eq. (2) usually exhibits a similar recurrence to that of the ideal system in Eq. (1) (see Fig. 4). However, the addition of tolerance does introduce a slight change in the recurrence timescale. In this and subsequent figures, we show a typical result from a Monte Carlo simulation with one

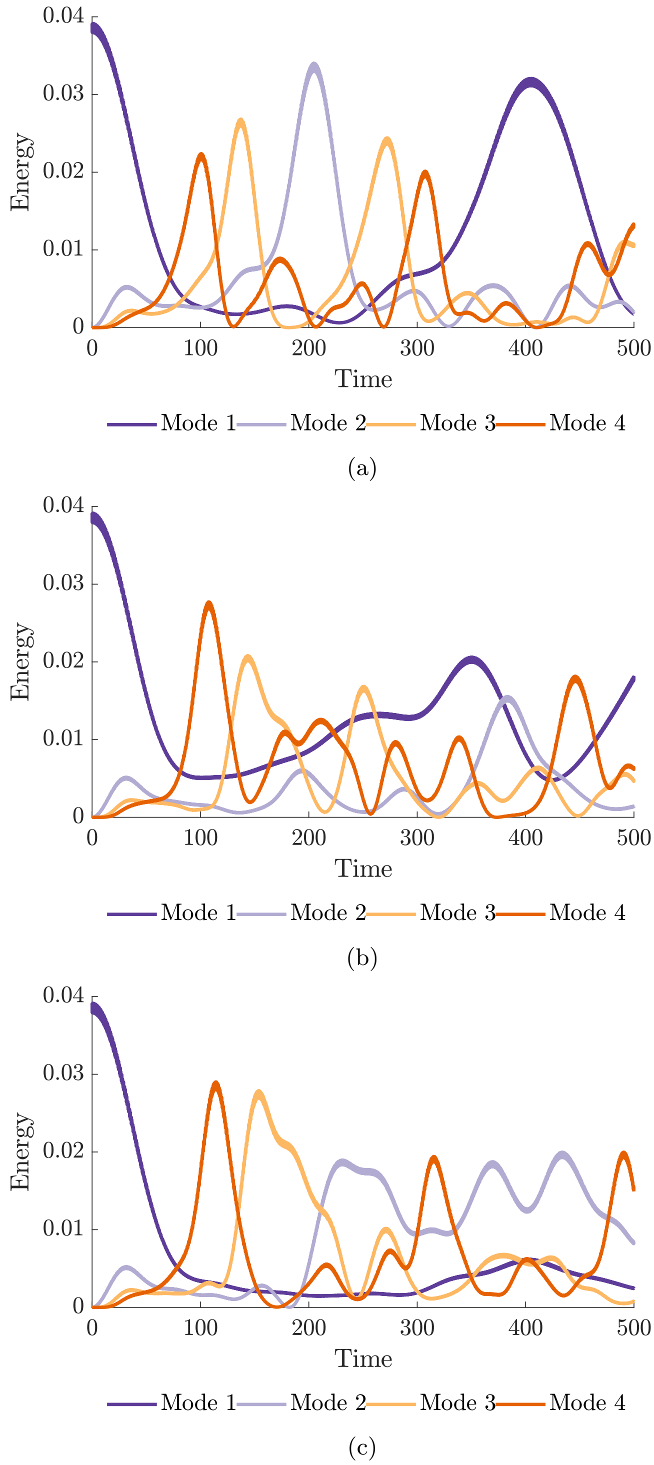

FIG. 3. Simulations of a 1D FPUT-α array with $N = 64$ oscillators and a nonlinearity coupling coefficient of $\alpha = 0.25$ showing examples of recurrence at a tolerance of $\pm 1\%$. These are samples of the range of responses when we use a set of tolerance values that we generate randomly from a Gaussian distribution. Panel (a) is representative of the response in more than 85% of the cases with $N = 64$, $\alpha = 0.25$, and a tolerance of $\pm 1\%$. One can observe that recurrence is present, but it is a bit different from what occurs in the ideal scenario. In some cases, recurrence is weaker [as in panel (b)] or the energy fails to return to the first mode [as in panel (c)].

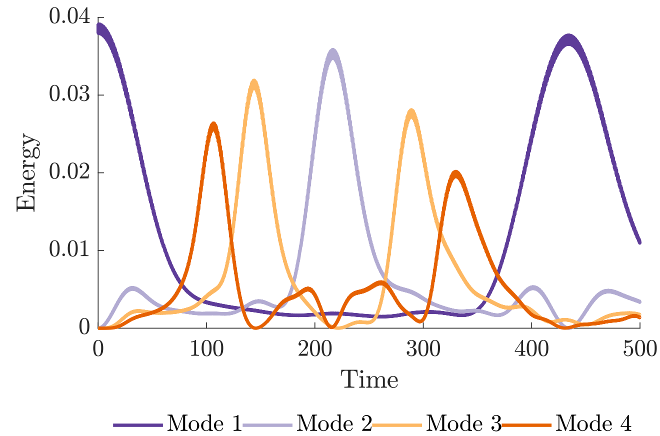

FIG. 4. Simulation of a 1D FPUT-α array with symmetric tolerances, 64 oscillators, and a coupling coefficient of $\alpha = 0.25$ at a tolerance of $\pm 0.1\%$. Comparing this to Fig. 2, we see a slight variation from the response of an ideal system.

randomly selected set of values of t_i from the distribution. On increasing the tolerance to $\pm 1\%$, the recurrence phenomenon starts to differ considerably from the ideal scenario of homogeneous system components [see Fig. 5(a)]. Increasing the tolerance further to $\pm 5\%$ [see Fig. 5(b)], we see even less energy transfer to the higher modes, and very little recurrence of energy appears to be taking place. At a tolerance of $\pm 10\%$ [see Fig. 5(c)], this steady decline in the quality of the recurrence is even more prominent.

These results are noteworthy. Any real oscillatory system has mismatches between the components. We have illustrated that with extremely tight tolerances of $\pm 0.1\%$, the output is close to the ideal, in the sense that the FPUT recurrences resemble those from the case of homogeneous oscillators. However, even at a tolerance of $\pm 5\%$, the recurrence phenomenon departs substantially from that in the ideal scenario, and it is already noticeably imperfect at $\pm 1\%$. To produce a system with a tolerance of $\pm 1\%$ requires a high degree of precision in manufacturing, which is expensive and often impractical.

Additionally, due to accumulation of tolerances, individual oscillators in electronic and mechanical systems often have stacked tolerances of 5%–10% even when they are constructed of individual components with very tight tolerances. At these values, recurrence starts to break down. This suggests that with typical mechanical and electronic systems, one may never see FPUT recurrences at all.

B. Significance of tolerances on linear versus nonlinear terms

The classical FPUT system has equations with two parts: a discretized linear diffusion term and a nonlinear coupling term. To identify whether tolerances in the linear or nonlinear terms contribute more to the breakdown of recurrence, we independently apply tolerances to the linear and nonlinear terms. As the nonlinear coupling leads to recurrence, our initial expectation was that any tolerance in nonlinear coupling parameters should lead to a more severe breakdown of recurrence than incorporating tolerance in the linear parts.

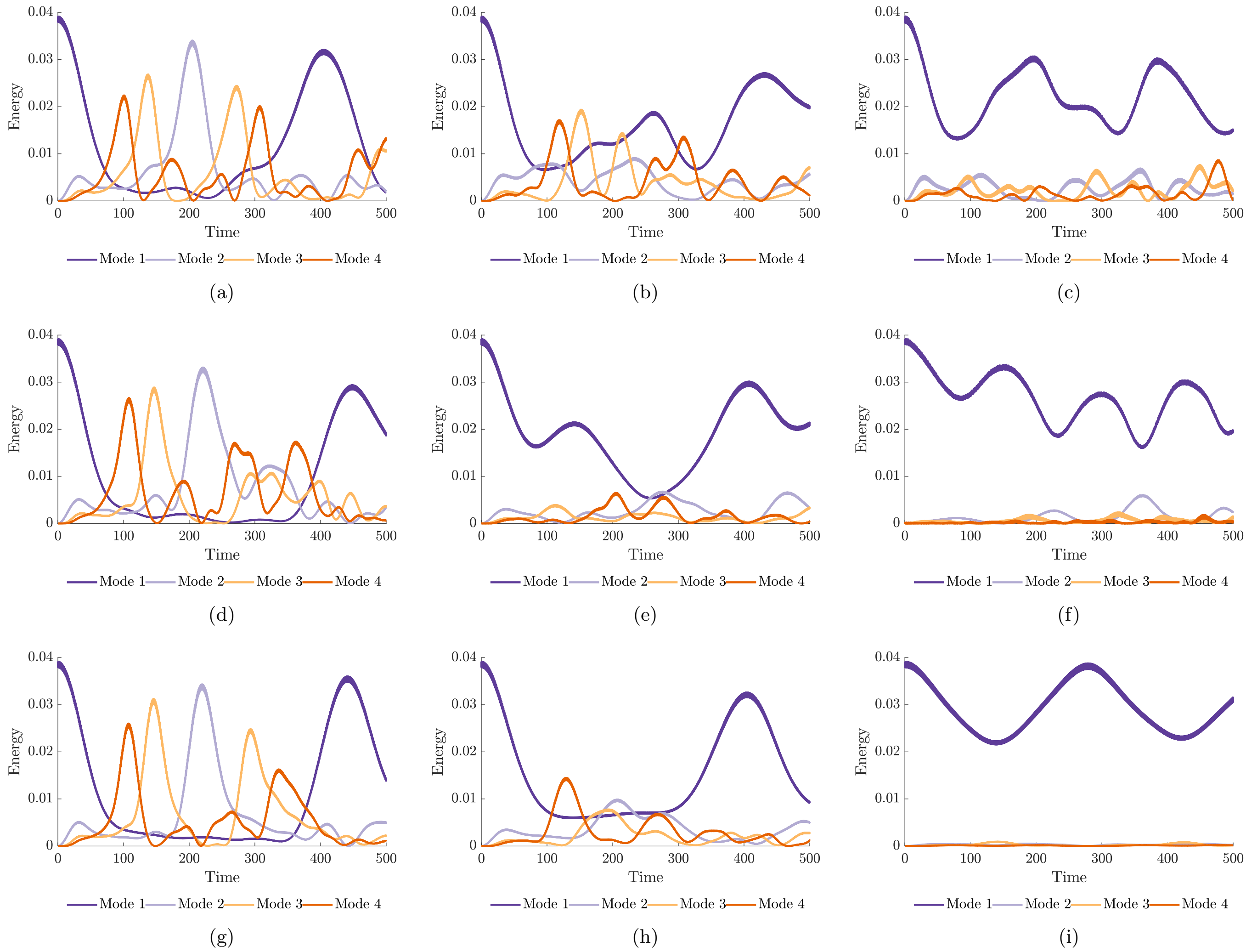

FIG. 5. The effect of tolerance on different parts of a 1D FPUT-α array with a coupling coefficient of $\alpha = 0.25$ and $N = 64$ oscillators. Panels (a)–(c) show examples with tolerance in both the linear and nonlinear terms, panels (d)–(f) show examples with tolerance in only the linear terms, and panels (g)–(i) show examples with tolerance in only the nonlinear terms. Observe that adding tolerance to only the linear terms has a comparable effect on recurrence to adding tolerance to only the nonlinear terms. Tolerance in both linear and nonlinear terms: (a) $\pm 1\%$, (b) $\pm 5\%$, and (c) $\pm 10\%$. Tolerance in the linear terms: (d) $\pm 1\%$, (e) $\pm 5\%$, and (e) $\pm 10\%$. Tolerance in the nonlinear terms: (g) $\pm 1\%$, (h) $\pm 5\%$, and (i) $\pm 10\%$.

However, our observations (see Fig. 5) suggest that this is not the case.

In Figs. 5(d)–5(f), we show our results of incorporating tolerance only in the linear parts of an FPUT-α array. We observe that the system is not exhibiting full recurrence, though there does appear to be some partial recurrence. As we show in Figs. 5(g)–5(i), incorporating tolerance in only the nonlinear terms has, for a fixed amount of tolerance, a comparable effect to incorporating it in only the linear terms. When we incorporate tolerance in both the linear and nonlinear terms in an FPUT-α array, we observe more energy transfer into the lower modes than in the above two scenarios [see Figs. 5(a)–5(c)]. This is particularly evident at larger tolerance values.

C. Impact of tolerance on arrays with different numbers of oscillators

As we have seen, recurrence breaks down as we consider progressively larger tolerances in a 1D FPUT array of 64 oscillators. It is worth examining what happens in arrays with different numbers of oscillators. In our exploration, we compare the results for 1D arrays with 8, 16, 32, 64, and 128 oscillators using a coupling coefficient of $\alpha = 0.25$ and a tolerance of $\pm 5\%$.

In Fig. 6, we illustrate that 1D arrays with a larger number of oscillators experience a larger impact from incorporating tolerances. One can see clearly that recurrence is strong with only eight oscillators [see Fig. 6(c)], whereas recurrence has broken down completely when there are 128 oscillators [see

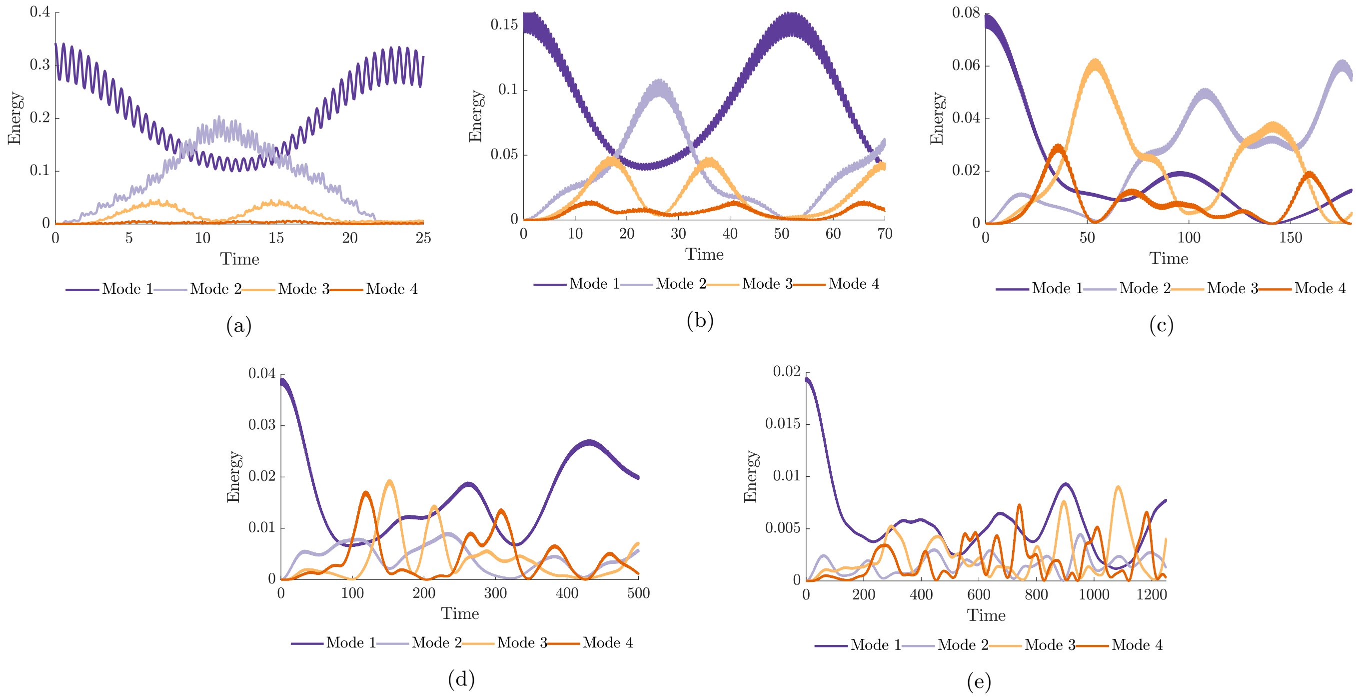

FIG. 6. Simulations of 1D FPUT-α arrays with different numbers of oscillators (N) at a tolerance of $\pm5\%$ with a coupling coefficient of $\alpha = 0.25$. (a) $N = 8$, (b) $N = 16$, (c) $N = 32$, (d) $N = 64$, and (e) $N = 128$. [Panel (d) appeared previously in Fig. 5(b).] This figure illustrates that, for a given tolerance, arrays with fewer oscillators are more likely to exhibit recurrence (which does not occur if there are too many oscillators).

Fig. 6(e)]. In light of typical manufacturing tolerances, this implies that one can safely observe recurrences only for arrays with very few oscillators. Additionally, for progressively larger arrays, the recurrence stops occurring at ever smaller tolerances; in practice, one may never be able to see tolerances in any large array.

D. Asymmetric coupling

In our previous simulations, we showed results from symmetrically coupled FPUT systems. However, in a variety of physical realizations of such a system, including an electronic one, forward coupling (i.e., between x_i and x_{i+1}) and backward coupling (i.e., between x_i and x_{i-1}) is not always the same. This asymmetry leads to another source of error, which we incorporate into the following asymmetric FPUT-α array:

$$\ddot{x}_i = (t_{i+1}x_{i+1} + t_{i-1}x_{i-1} - 2t_i x_i) + \alpha\big[f_i^{\text{nlin}}(x_{i+1} - x_i)^2 - b_i^{\text{nlin}}(x_i - x_{i-1})^2\big], \quad (3)$$

where we represent the tolerances on the "forward" and "backward" nonlinear terms by f_i^{nlin} and b_i^{nlin}, respectively. We conduct simulations of the FPUT system in Eq. (3) and show our results in Figs. 7–9.

Comparing the results of this asymmetric system to those of the symmetric system in Eq. (2), we observe that the impact of tolerance is comparable in the two systems when we incorporate tolerance into both the linear and nonlinear terms (see Fig. 9). However, recurrence in the asymmetric FPUT system breaks down for a smaller tolerance than it does in the symmetric FPUT system. Additionally, as we increase the number of oscillators in a system, we observe that recurrence breaks down for a smaller number of oscillators in the asymmetric FPUT system than in the symmetric system. One interesting observation is that incorporating tolerance in only the nonlinear terms of the symmetric FPUT system has a comparable effect to that of adding it to only the linear terms (see Fig. 5), whereas incorporating tolerance in only the nonlinear terms in the asymmetric FPUT system seems

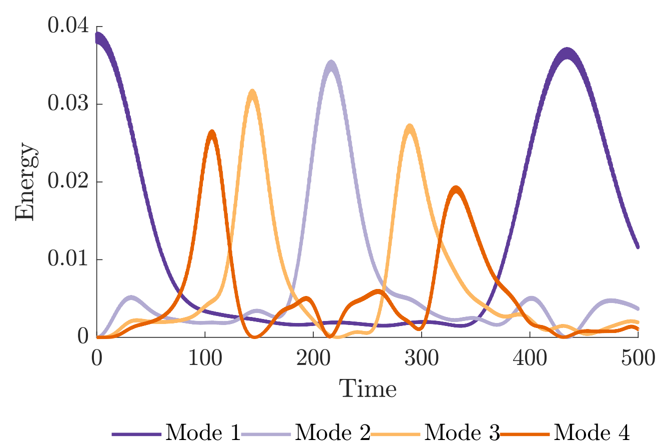

FIG. 7. Simulation of a 1D FPUT-α array with asymmetric tolerances, 64 oscillators, and a coupling coefficient of $\alpha = 0.25$ at a tolerance of $\pm0.1\%$. Comparing this simulation to that in Fig. 4, we observe that this asymmetric system has a comparable deviation from ideal recurrence (i.e., when there is no tolerance) to the symmetric system.

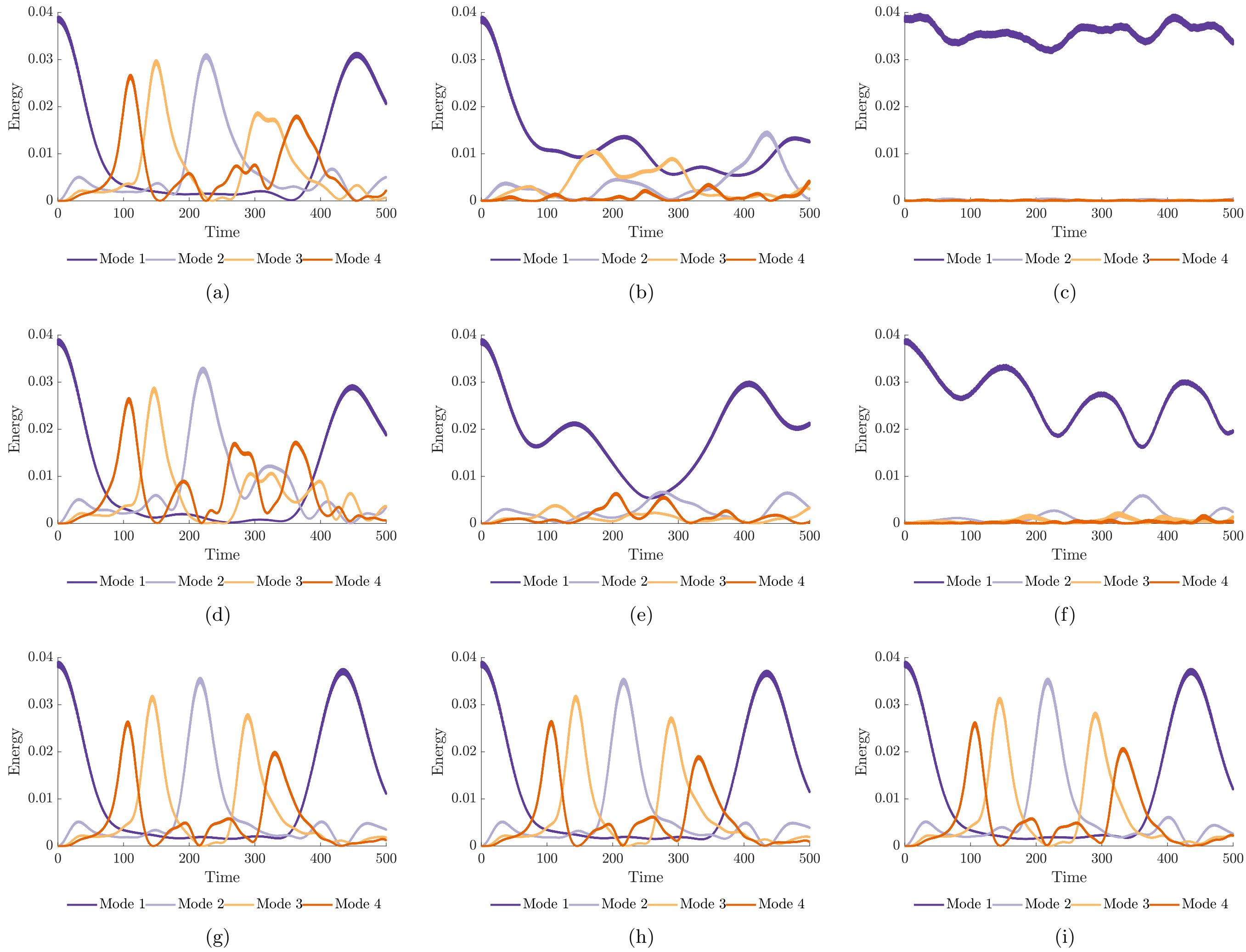

FIG. 8. The effect of tolerance on different parts of an asymmetric 1D FPUT-α array with $N = 64$ oscillators and a coupling coefficient of $\alpha = 0.25$. Comparing this simulation to that in Fig. 5, we see that incorporating tolerance in only the nonlinear terms has a much smaller effect on the qualitative dynamics than is the case in the symmetric system. Tolerance in both linear and nonlinear terms: (a) $\pm1\%$, (b) $\pm5\%$, and (c) $\pm10\%$. Tolerance in the linear terms: (d) $\pm1\%$, (e) $\pm5\%$, and (f) $\pm10\%$. Tolerance in the nonlinear terms: (g) $\pm1\%$, (h) $\pm5\%$, and (i) $\pm10\%$.

to have significantly less impact than we expected, as we observe recurrence that is close to that of an ideal asymmetric system (see Fig. 8). One possible explanation for this may be the difference in structure of the equations of motion when we incorporate tolerances. We explore this briefly in Appendix A.

IV. CONCLUSIONS AND DISCUSSION

The numerical experiments reported by Fermi, Pasta, Ulam, and Tsingou in 1955 have led to an extensive body of computational, theoretical, and experimental work in nonlinear systems. However, it is commonly assumed in studies of FPUT arrays and other lattice systems that the units are homogeneous. Physical systems, by contrast, are heterogeneous by nature; and this can affect recurrence phenomena.

In the present paper, we examined the effect of incorporating heterogeneity on a 1D FPUT array. Such heterogeneity arises from the inherent tolerances of various manufacturing processes, so one must take it into consideration in laboratory experiments. The results of our simulations illustrate that tolerance has a significant impact on recurrences in an FPUT system, destroying it in a 64-element system for tolerance values that lie within the typical range for manufacturing tolerances. However, by reducing the number of oscillators in an FPUT system, one retains recurrence for some reasonable amount of tolerance before it breaks down. Thus, by controlling manufacturing tolerance between nominally identical components, it may be possible to observe recurrence in an FPUT system with a small number of oscillators. However, tight tolerances are hard to achieve and are often very expensive. Therefore, producing such a system may not be practical. In large arrays of oscillators, even very small tolerances will surely eliminate any chance of seeing recurrence in practice.

In this article, we used a typical sinusoidal initial condition in a 1D FPUT array; we have not yet explored other initial conditions. There are numerous studies that investigate FPUT systems with other initial conditions, alongside other

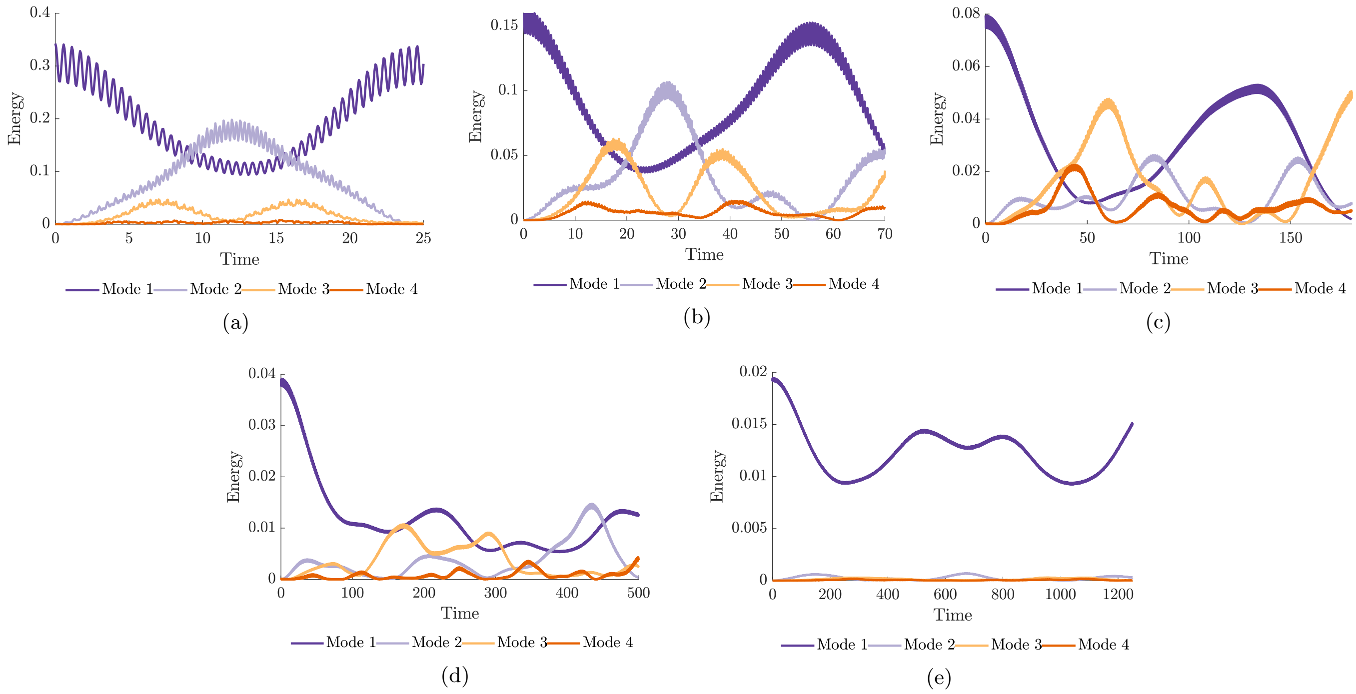

FIG. 9. Simulations of 1D FPUT-α arrays with different numbers of oscillators (N) at a tolerance of $\pm 5\%$ with a coupling coefficient of $\alpha = 0.25$. (a) $N = 8$, (b) $N = 16$, (c) $N = 32$, (d) $N = 64$, and (e) $N = 128$. [Panel (d) appeared previously in Fig. 8(b).]

considerations (such as driven or damped systems [5,16]). One can also examine similar considerations in other lattice systems, such as granular crystals, which have Hertzian interactions between components [36,37]. Incorporating tolerances into parameters for other initial conditions (and other variants of FPUT systems and other types of nonlinear lattices) can lead to behaviors other than those that we have discussed in this article. Because tolerance has such a major impact on a standard FPUT system with sinusoidal input, it would be unreasonable to assume that other scenarios will only be affected minimally. Consequently, it is necessary to revisit investigations of FPUT systems and other nonlinear lattices with tolerance in mind.

Our work is limited in its scope, and there are other areas to explore when applying tolerance to an FPUT system. In particular, we have not yet examined the impact of increasing the energy in the nonlinear terms by increasing the coupling strength. Other salient areas include FPUT-β lattices and the impact of tolerances on the very long "superrecurrences" that were discovered by Tuck and Menzel (née Tsingou) [12].

There has also been work in other areas of nonlinear science from which it is desirable to draw inspiration for additional work. For example, in studies of synchronization on networks, there is also a long history of examining the collective properties of coupled phase oscillators (a rather different type of system from the one that we studied) with natural frequencies drawn from some distribution [38]. With practical laboratory experiments in mind, it is crucial to conduct systematic investigations of incorporating tolerance into those and other systems.

ACKNOWLEDGMENTS

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APPENDIX A: ALTERNATIVE ASYMMETRIC STRUCTURES

In the main body of the paper, we observed that incorporating tolerances into the nonlinear terms had a much more significant effect in symmetric FPUT lattices than in asymmetric ones. We also proposed that this may be due to the way in which we applied the tolerances to the systems and the resultant effect that this has on the structure of the original FPUT chain. Our choices in the main text reflected practical ways in which we envisage incorporating tolerances in experimental systems.

To explore this issue further, we now consider different ways to incorporate tolerances into the arrays. For example, if we apply different forward and backward tolerances to the linear components of an FPUT lattice, while keeping the tolerances on the nonlinear components the same as in Eq. (3), we obtain the following equation of motion:

$$\ddot{x}_i = (f_{i+1}x_{i+1} - f_i x_i + b_{i-1}x_{i-1} - b_i x_i) + \alpha\big[f_i^{\mathrm{nlin}}(x_{i+1} - x_i)^2 - b_i^{\mathrm{nlin}}(x_i - x_{i-1})^2\big], \tag{A1}$$

where the forward and backward tolerances on element i are f_i and b_i, respectively. In Fig. 10, we show our results from simulating Eq. (A1).

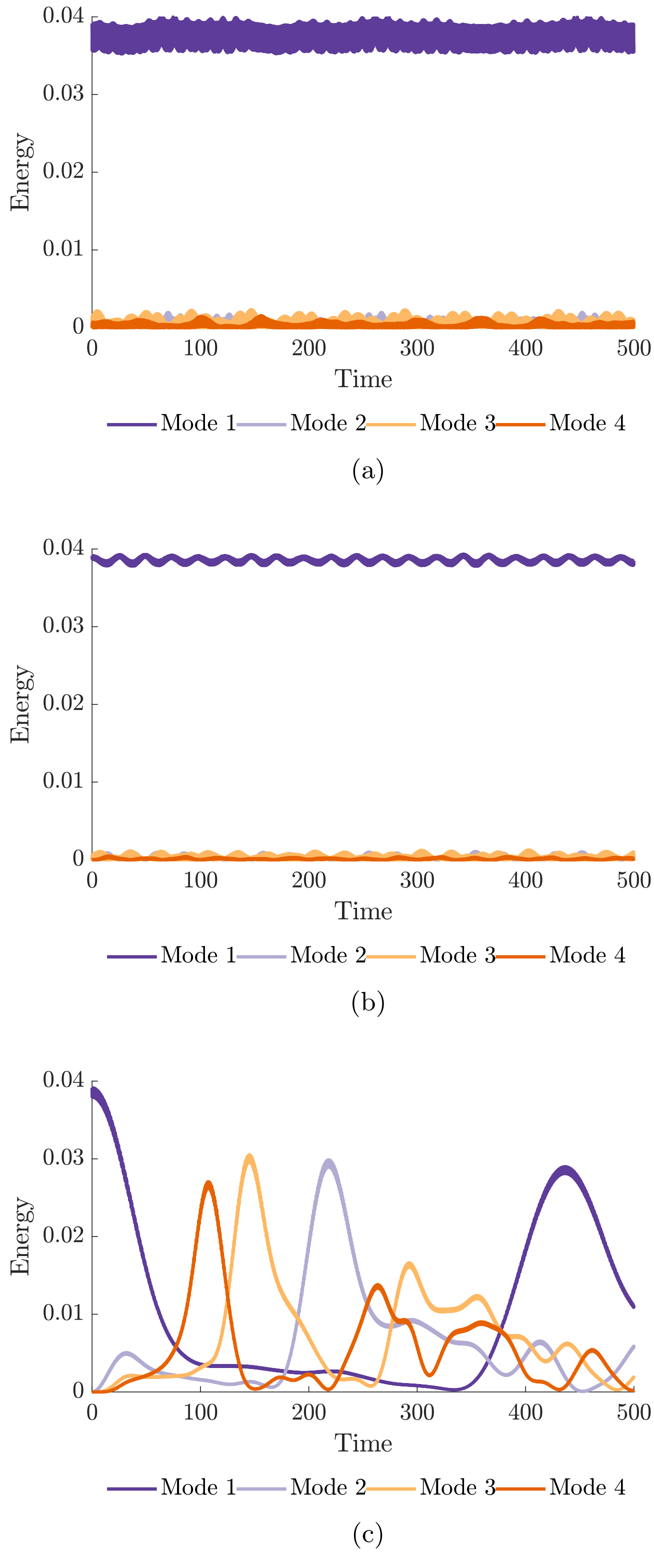

FIG. 10. The effect of tolerance on different parts of an asymmetric 1D FPUT-α array with equations of motion (A1), $N = 64$ oscillators, a coupling coefficient of $\alpha = 0.25$, and a tolerance of $\pm 5\%$. Panel (a) shows a simulation with tolerance in both the linear and nonlinear terms, panel (b) shows a simulation with tolerance in only the linear terms, and panel (c) shows a simulation with tolerance in only the nonlinear terms.

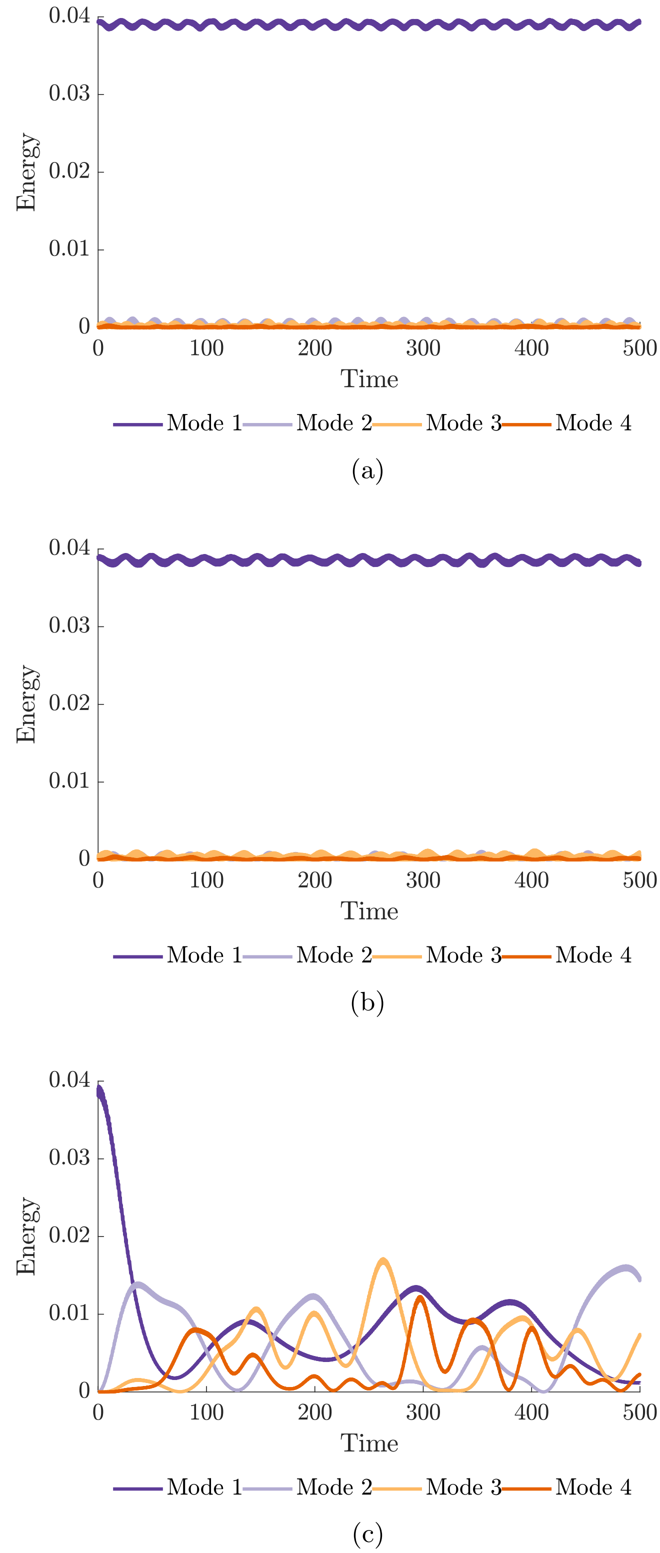

FIG. 11. The effect of tolerance on different parts of an asymmetric 1D FPUT-α array with equations of motion (A2), $N = 64$ oscillators, a coupling coefficient of $\alpha = 0.25$, and a tolerance of $\pm 5\%$. Panel (a) shows a simulation with tolerance in both the linear and nonlinear terms, panel (b) shows a simulation with tolerance in only the linear terms, and panel (c) shows a simulation with tolerance in only the nonlinear terms.

Taking this discussion a step further, if we now split the tolerances in the nonlinear components so that the way we incorporate tolerances in asymmetric FPUT arrays more closely resembles that in symmetric FPUT arrays, we obtain the following equation of motion:

$$\ddot{x}_i = (f_{i+1}x_{i+1} - f_i x_i + b_{i-1}x_{i-1} - b_i x_i) + \alpha\left[\left(f_{i+1}^{\text{nlin}}x_{i+1} - f_i^{\text{nlin}}x_i\right)^2 - \left(b_i^{\text{nlin}}x_i - b_{i+1}^{\text{nlin}}x_{i-1}\right)^2\right]. \tag{A2}$$

We show the results of simulating Eq. (A2) in Fig. 11.

The results in this appendix illustrate that the way in which one constructs an array of nonlinear oscillators—and specifically the way in which one incorporates tolerances into such a system—has a major effect on the qualitative dynamics of such systems. Further investigation of such phenomena, though beyond the scope of the present paper, is an important avenue to pursue.

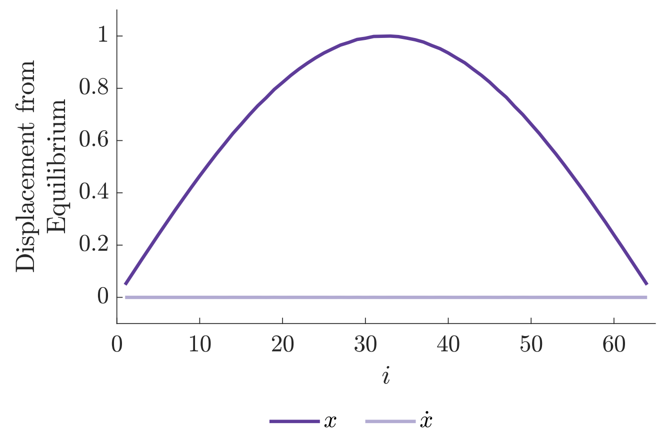

FIG. 12. Output from the time-reversed simulation of Fig. 3(a), showing that we obtain values close to the initial condition (which consists of an excitation of the system's first mode).

APPENDIX B: TEST OF NUMERICAL STABILITY

We tested our simulation methodology to see if our results are time-reversible. Using the same tolerances and values of N and α from the main text, we used our output as an initial condition and simulated the equations of motion backward in time. We then calculate, for each oscillator, the error between the initial condition (an excitation of the system's first mode, as discussed in Sec. II) and the output of the present tests as a percentage. We then report the maximum error among all oscillators. These errors are below 0.5% for $N \leqslant 64$ and below 4% for $N = 128$. In Fig. 12, we show a typical output after the time-reversed simulation; this illustrates that our result is close to the initial condition of an excitation of the first mode.

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