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Limit order books

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Limit order books (LOBs) match buyers and sellers in more than half of the world's financial markets. This survey highlights the insights that have emerged from the wealth of empirical and theoretical studies of LOBs. We examine the findings reported by statistical analyses of historical LOB data and discuss how several LOB models provide insight into certain aspects of the mechanism. We also illustrate that many such models poorly resemble real LOBs and that several well-established empirical facts have yet to be reproduced satisfactorily. Finally, we identify several key unresolved questions about LOBs.

Keywords: Limit order books; Data analysis; Modelling; Stylized facts; Complex systems

1. Introduction

More than half of the markets in today's highly competitive and relentlessly fast-paced financial world now use a limit order book (LOB) mechanism to facilitate trade (Roşu 2009). The Helsinki, Hong Kong, Shenzhen, Swiss, Tokyo, Toronto, and Vancouver Stock Exchanges, together with Euronext and the Australian Securities Exchange, all now operate as pure LOBs (Luckock 2001, Gu *et al.* 2008b); the New York Stock Exchange (NYSE), NASDAQ, and the London Stock Exchange (LSE) (Cont *et al.* 2010) all operate a bespoke hybrid LOB system. Thanks to technological advances, traders worldwide have real-time access to the current LOB, providing buyers and sellers alike 'the ultimate microscopic level of description' (Bouchaud *et al.* 2002).

In an LOB, complicated global phenomena emerge as a result of the local interactions between many heterogeneous agents when the system throughput becomes sufficiently large. This makes an LOB an example of a *complex system* (Mitchell 2009). The unusually rich, detailed, and high-quality historic data from LOBs provides a suitable testing ground for theories about well-established statistical regularities common to a wide range of markets (Cont 2001, Farmer and Lillo 2004, Bouchaud *et al.* 2009), as well as for popular ideas in the complex systems literature such as universality, scaling, and emergence.

The many practical advantages to understanding LOB dynamics include: gaining clearer insight into how best to act in given market situations (Harris and Hasbrouck 1996); optimal order execution strategies (Obizhaeva and Wang 2013); market impact minimization (Eisler *et al.* 2012); designing better electronic trading algorithms (Engle *et al.* 2006); and assessing market stability (Kirilenko *et al.* 2011). In this survey, we discuss some of the key ideas that have emerged from the analysis and modelling of LOBs in recent years, and we highlight the strengths and limitations of existing LOB models.

Investigations of LOBs have taken a variety of starting points, drawing on ideas from economics, physics, mathematics, statistics, and psychology. Unsurprisingly, there is no clear consensus on the best approach. This point is exemplified by the contrast between the approach normally taken in the economics literature, in which models focus on the behaviour of individual traders and present LOBs as sequential games (Parlour 1998, Foucault 1999, Roşu 2009), with the approach normally taken in the physics literature, in which order flows are treated as random and techniques from statistical mechanics are used to explore the resulting dynamics (Challet and Stinchcombe 2001, Smith *et al.* 2003, Cont *et al.* 2010). In the present paper, we discuss developments in both the economics and physics literatures, and we emphasize aspects of LOBs that are most relevant to practitioners.

Several other survey articles focus on particular aspects of LOBs. Friedman (2005) reviewed early studies of *double auction* style trading, of which LOBs are an example. Parlour and Seppi (2008) addressed the economic and theoretical aspects of LOB trading. Bouchaud *et al.* (2009) assessed the current understanding of price formation in LOBs. Chakraborti *et al.* (2011a, 2011b) examined the role of econophysics in understanding LOB behaviour. In the present survey,

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we note the similarities and differences between several empirical studies of historical LOB data, discuss LOB models from both the physics and economics literatures, highlight several modelling assumptions that are not well-supported by the empirical findings, and identify several key unresolved questions.

The remainder of the survey is organized as follows. In Section 2, we give formal definitions related to LOBs and formulate a mathematically precise description of LOB trading. In Section 3, we discuss some practical aspects of trading via LOBs and examine the difficulties that arise in quantifying them. In Section 4, we examine the important role of empirical studies of LOBs, highlighting both consensus and disagreement within the literature. We examine a selection of models in Section 5. In Section 6, we discuss key unresolved problems related to LOBs. We conclude in Section 7.

2. A mathematical description of an LOB

In this section, we formulate a precise description of trading that is common to most LOB markets. Of course, some individual exchanges and trading platforms operate slight variations of these core principles. Harris (2003) provided a comprehensive review of specific details governing particular exchanges.

2.1. Preliminaries

Before LOBs grew in popularity, most financial trades took place in *quote-driven* marketplaces, in which a handful of large *market makers* centralize buy and sell orders by publishing the prices at which they are willing to buy and sell the traded asset. The market makers set their sell price higher than their buy price in order to earn a profit in exchange for providing *liquidity*† to the market, for taking on the risk of acquiring an undesirable inventory position, and for being exposed to possible *adverse selection* (i.e. encountering other traders who have better information about the value of the asset and who can therefore make a profit by buying or selling, often repeatedly, with the market maker (Parlour and Seppi 2008)). The only prices available to other traders who want to buy or sell the asset are those made public by the market makers, and the only action available to such traders is to buy or sell at one of the market makers' prices. Ticket touts exemplify a quote-driven market in action.

An LOB is much more flexible because every trader has the option of posting buy (respectively, sell) *orders.*

Definition *An* order $x = (p_x, \omega_x, t_x)$ *submitted at time* t_x *with price* p_x *and size* $\omega_x > 0$ *(respectively,* $\omega_x < 0$*) is a commitment to sell (respectively, buy) up to* $|\omega_x|$ *units of the traded asset at a price no less than (respectively, no greater than)* p_x.

We introduce the vector notation $x = (p_x, \omega_x, t_x)$ because it allows explicit calculation of the *priority* (see Section 3.4) of any order at any time.

†Liquidity is difficult to define formally. Kyle (1985) identified the three key properties of a liquid market to be tightness ('the cost of turning around a position over a short period of time'), depth ('the size of an order-flow innovation required to change prices a given amount'), and resiliency ('the speed with which prices recover from a random, uninformative shock').

For a given LOB, the units of order size and price are set as follows.

Definition *The* lot size σ *of an LOB is the smallest amount of the asset that can be traded within it. All orders‡ must arrive with a size* $\omega_x \in \{\pm k\sigma | k = 1, 2, \ldots\}$.

Definition *The* tick size π *of an LOB is the smallest permissible price interval between different orders within it. All orders must arrive with a price that is specified to the accuracy of* π.

For example, if $\pi = \$0.00001$, then the largest permissible order price that is strictly less than \$1.00 is \$0.99999, and all orders must be submitted at a price with exactly five decimal places.

Definition *The lot size* σ *and tick size* π *of an LOB are collectively called its* resolution parameters.

When a buy (respectively, sell) order x is submitted, an LOB's *trade-matching algorithm* checks whether it is possible to *match* x *to* some other previously submitted sell (respectively, buy) order. If so, the matching occurs immediately. If not, x becomes *active*, and it remains active until either it becomes matched to an incoming sell (respectively, buy) order or it is cancelled. Cancellation usually occurs because the owner of an order no longer wishes to offer a trade at the stated price, but rules governing a market can also lead to the cancellation of active orders. For example, on the electronic trading platform Hotspot FX, all active orders are cancelled at 5pm EST each day to prevent an overly large accumulation of active orders over time (Knight-Hotspot 2013).

It is precisely the active orders in a market that make up an *LOB:*

Definition *An* LOB $\mathcal{L}(t)$ *is the set of all active orders in a market at time* t.

The evolution of an LOB $\mathcal{L}(t)$ is a càdlàg process, i.e. for a limit order $x = (p_x, \omega_x, t_x)$ that becomes active upon arrival, it holds that $x \in \mathcal{L}(t_x)$, $x \notin \lim_{t' \uparrow t_x} \mathcal{L}(t')$. The active orders in an LOB $\mathcal{L}(t)$ can be partitioned into the set of active buy orders $\mathcal{B}(t)$, for which $\omega_x < 0$, and the set of active sell orders $\mathcal{A}(t)$, for which $\omega_x > 0$. An LOB can then be considered as a set of queues, each of which consists of active buy or sell orders at a specified price.

Definition *The* bid-side depth available at price p and at time t *is*

$$n^b(p, t) := \sum_{\{x \in \mathcal{B}(t) | p_x = p\}} \omega_x. \tag{1}$$

The ask-side depth available at price p and at time t, *denoted* $n^a(p, t)$*, is defined similarly using* $\mathcal{A}(t)$.

The depth available is often stated in multiples of the lot size. Because $\omega_x < 0$ for buy orders and $\omega_x > 0$ for sell orders, it follows that $n^b(p, t) \leq 0$ and $n^a(p, t) \geq 0$ for all prices p.

‡In some markets, there are two lot-size parameters: a minimum size σ and an increment ε. In such markets, all orders must arrive with a size $\omega_x \in \{\pm(\sigma + k\varepsilon) | k = 0, 1, 2, \ldots\}$. For simplicity, we assume $\sigma = \varepsilon$.

Definition *The* bid-side depth profile at time t *is the set of all ordered pairs* $\left(p, n^b(p,t)\right)$. *The* ask-side depth profile at time t *is the set of all ordered pairs* $(p, n^a(p,t))$.

Definition *The* mean bid-side depth available at price p between times t_1 and t_2 *is*

$$\overline{n}^b(p, t_1, t_2) := \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} n^b(p,t)\mathrm{d}t. \quad (2)$$

The mean ask-side depth available at price p between times t_1 and t_2, *denoted* $\overline{n}^a(p, t_1, t_2)$, *is defined similarly using the ask-side depth available.*

The terms *bid price*, *ask price*, *mid price*, and *bid-ask spread* are common to much of the finance literature and can be made specific in the context of an LOB.

Definition *The* bid price at time t *is the highest stated price among active buy orders at time* t,

$$b(t) := \max_{x \in \mathcal{B}(t)} p_x. \quad (3)$$

The ask price at time t *is the lowest stated price among active sell orders at time* t,

$$a(t) := \min_{x \in \mathcal{A}(t)} p_x. \quad (4)$$

Definition *The* bid-ask spread at time t *is* $s(t) := a(t) - b(t)$.

Definition *The* mid price at time t *is* $m(t) := [a(t) + b(t)]/2$.

In an LOB, $b(t)$ is the highest price at which it is immediately possible to sell at least the lot size of the traded asset at time t, and $a(t)$ is the lowest price at which it is immediately possible to buy at least the lot size of the traded asset at time t. It is sometimes helpful to consider prices *relative* to $b(t)$ and $a(t)$.†

Definition *For a given price* p, *the* bid-relative price *is* $\delta^b(p) := b(t) - p$ *and the* ask-relative price *is* $\delta^a(p) := p - a(t)$.

Observe the difference in signs between the two definitions: $\delta^b(p)$ measures how much smaller p is than $b(t)$ and $\delta^a(p)$ measures how much larger p is than $a(t)$.

It is often desirable to compare orders on the bid side and the ask side of an LOB. In these cases, the concept of a single *relative price of an order* is useful.

Definition *For a given order* $x = (p_x, \omega_x, t_x)$, *the* relative price of the order *is*

$$\delta^x := \begin{cases} \delta^b(p_x), & \text{if the order is a buy order,} \\ \delta^a(p_x), & \text{if the order is a sell order.} \end{cases} \quad (5)$$

Because $b(t)$ and $a(t)$ vary, it is rarely illuminating to consider the depth available at a specific price over time. However, relative pricing provides a useful alternative.

Definition *The* bid-side depth available at relative price p and at time t *is*

$$N^b(p,t) := \sum_{\{x \in \mathcal{B}(t) | \delta^x = p\}} \omega_x. \quad (6)$$

†Many different naming and sign conventions are used by different authors to describe slightly different definitions of relative price. We introduce an explicit distinction between bid-relative price and ask-relative price to avoid potential confusion.

The ask-side depth available at relative price p and at time t, *denoted* $N^a(p,t)$, *is defined similarly using* $\mathcal{A}(t)$.

Definition *The* bid-side relative depth profile at time t *is the set of all ordered pairs* $\left(p, N^b(p,t)\right)$. *The* ask-side relative depth profile at time t *is the set of all ordered pairs* $(p, N^a(p,t))$.

Definition *The* mean bid-side depth available at relative price p between times t_1 and t_2 *is*

$$\overline{N}^b(p, t_1, t_2) := \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} N^b(p,t)\mathrm{d}t. \quad (7)$$

The mean ask-side depth available at relative price p between times t_1 and t_2, *denoted* $\overline{N}^a(p, t_1, t_2)$, *is defined similarly using the ask-side relative depth available.*

Definition *The* mean bid-side relative depth profile between times t_1 and t_2 *is the set of all ordered pairs* $(p, \overline{N}^b(p, t_1, t_2))$. *The* mean ask-side relative depth profile between times t_1 and t_2 *is the set of all ordered pairs* $(p, \overline{N}^a(p, t_1, t_2))$.

Most traders assess the state of $\mathcal{L}(t)$ via the relative depth profile, and several studies have concluded that order arrival rates depend on relative prices rather than actual prices (see, e.g. Biais *et al.* (1995), Bouchaud *et al.* (2002), Potters and Bouchaud (2003) and Zovko and Farmer (2002)). However, relative depth profiles provide no information about the absolute prices at which trades occur. Additionally, they do not contain information about the bid-ask spread or mid price, so it is common to consider the relative depth profiles and $b(t)$ and $a(t)$ simultaneously to obtain a complete picture of the temporal evolution of an LOB.

Figure 1 shows a schematic of an LOB at some instant in time, illustrating the definitions in this section. The horizontal lines within the blocks at each price level denote how the depth available at that price is composed of different active orders.

Time series of prices arise often during the study of LOBs. As discussed in Section 4.7, it is a recurring theme that the behaviour of such a time series depends significantly on how it

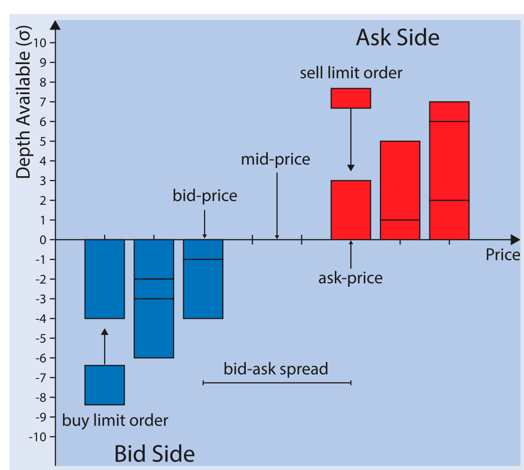

Figure 1. Schematic of an LOB.

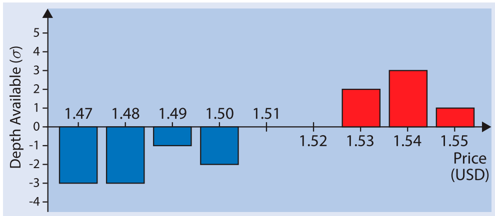

Figure 2. An example LOB.

is sampled. For example, consider the time series $m(t_1), \ldots, m(t_k)$, for some times $t_1, \ldots, t_k$.

- When the t_i are spaced regularly in time, with τ seconds between successive samplings, such a time series is said to be sampled on a τ*-second timescale*.
- When the t_i are chosen to correspond to arrivals of orders, the t_i may be spaced irregularly in time. Such a time series is said to be sampled on an *event-by-event timescale.*
- When the t_i are chosen to correspond to trades (i.e. matchings in an LOB), the t_i may also be spaced irregularly in time. Such a time series is said to be sampled on a *trade-by-trade timescale.*

2.2. Orders: the building blocks of an LOB

The actions of traders in an LOB can be expressed solely in terms of submissions or cancellations of orders of the lot size. For example, a trader who immediately sells 4σ units of the traded asset in the LOB displayed in figure 2 can be considered as submitting 2 sell orders of size σ at the price \$1.50, 1 sell order of size σ at the price \$1.49, and 1 sell order of size σ at the price \$1.48. Similarly, a trader who posts a sell order of size 4σ at the price \$1.55 can be considered as submitting 4 sell orders of size σ at a price of \$1.55 each.

Almost all of the published literature on LOBs adopts the following terminology. Orders that result in an immediate matching upon submission are known as *market orders*. Orders that do not, instead becoming active orders, are known as *limit orders*.† However, it is important to recognize that this terminology is used only to emphasize whether an incoming order triggers an immediate matching or not.

Some trading platforms allow traders to specify that they wish to submit a buy (respectively, sell) market order without explicitly specifying a price. Instead, such a trader specifies only a size, and the trade-matching algorithm sets the price of the order appropriately to initiate the required matching.

2.3. Price changes in LOBs

In LOBs, the rules that govern matchings dictate how prices evolve through time. Consider a buy (respectively, sell) order $x = (p_x, \omega_x, t_x)$ that arrives immediately after time t.

- If $p_x \le b(t)$ (respectively, $p_x \ge a(t)$), then x is a limit order that becomes active upon arrival. It does not cause $b(t)$ or $a(t)$ to change.
- If $b(t) < p_x < a(t)$, then x is a limit order that becomes active upon arrival. Upon arrival, $b(t_x) = p_x$ (respectively, $a(t_x) = p_x$).
- If $p_x \ge a(t)$ (respectively, $p_x \le b(t)$), then x is a market order that immediately matches to one or more active sell (respectively, buy) orders upon arrival. Whenever such a matching occurs, it does so at the price of the active order, which is not necessarily equal to the price of the incoming order. Whether or not such a matching causes $a(t)$ (respectively, $b(t)$) to change at time t_x depends on $n^a(a(t), t)$ (respectively, $n^b(b(t), t)$) and ω_x. In particular, the new bid price $b(t_x)$ upon arrival of a sell market order x is

$$\max(p_x, q), \text{ where } q = \arg\max_{k'} \sum_{k=k'}^{b(t)} \left| n^b(k, t) \right| > \omega_x.$$

Similarly, the new ask price $a(t_x)$ upon arrival of a buy market order x is

$$\min(p_x, q), \text{ where } q = \arg\min_{k'} \sum_{k=a(t)}^{k'} n^a(k, t) > |\omega_x| .$$

Put another way, the incoming order x matches to the highest priority active order y of opposite type. If $|\omega_x| > |\omega_y|$, then any residue size of x is considered for matching to the next highest priority active order of opposite type, and so on until either there are no further active orders with prices that make them eligible for matching, in which case the residue of x becomes active at the price p_x, or x is fully matched. The new bid (respectively, ask) price is then equal to the price of the highest priority active buy (respectively, sell) order after the matching occurs.

Table 1 lists several possible market events that could occur to the LOB displayed in figure 2 and the resulting values of $b(t_x)$, $a(t_x)$, $m(t_x)$, and $s(t_x)$ that they would cause.

In the financial literature, price changes are commonly studied via *returns.*

Definition *The* bid-price return between times t_1 and t_2 *is* $R^b(t_1, t_2) := (b(t_2) - b(t_1))/b(t_1)$. *The* ask-price return between times t_1 and t_2, *denoted* $R^a(t_1, t_2)$*, and the* mid-price return between times t_1 and t_2, *denoted* $R^m(t_1, t_2)$*, are defined similarly.*

Definition *The* bid-price logarithmic return between times t_1 and t_2 *is* $r^b(t_1, t_2) := \log(b(t_2)/b(t_1))$. *The* ask-price logarithmic return between times t_1 and t_2, *denoted* $r^a(t_1, t_2)$*, and the* mid-price logarithmic return between times t_1 and t_2, *denoted* $r^m(t_1, t_2)$*, are defined similarly.*

2.4. The economic benefits of LOBs

In an LOB, traders are able to choose between submitting limit orders and submitting market orders. Limit orders stand a chance of matching at better prices than do market orders, but they also run the risk of never being matched. Conversely, market orders never match at prices better than $b(t)$ and $a(t)$,

†Some practitioners use the terms *aggressive orders* and *resting orders*, respectively, but this terminology is far less common in the published literature.

Table 1. How each order arrival would affect prices in the LOB displayed in figure 2.

Arriving order x	Values after arrival (USD)			
	$b(t_x)$	$a(t_x)$	$m(t_x)$	$s(t_x)$
Initial values	1.50	1.53	1.515	0.03
($1.48, -3σ, t_x)	1.50	1.53	1.515	0.03
($1.51, -3σ, t_x)	1.51	1.53	1.52	0.02
($1.55, -3σ, t_x)	1.50	1.54	1.52	0.04
($1.55, -5σ, t_x)	1.50	1.55	1.525	0.05
($1.54, 4σ, t_x)	1.50	1.53	1.515	0.03
($1.52, 4σ, t_x)	1.50	1.52	1.51	0.02
($1.47, 4σ, t_x)	1.48	1.53	1.505	0.05
($1.50, 4σ, t_x)	1.49	1.50	1.495	0.01

but they do not face the inherent uncertainty associated with limit orders. An LOB's bid-ask spread $s(t)$ can be considered as a measure of how highly the market values the immediacy and certainty associated with market orders versus the waiting and uncertainty associated with limit orders. Foucault *et al.* (2005) argued that the popularity of LOBs was due in part to their ability to allow some traders to demand immediacy, while simultaneously allowing others to supply it to those who later require it. He conjectured that arbitrageurs, technical traders, and indexers were most likely to place market orders (due to the fast-paced nature of their work) and that portfolio managers were most likely to place limit orders (because their strategies are generally more focused on the long term). In reality, most traders use a combination of both limit orders and market orders; they select their actions for each situation based on their individual needs at that time (Anand *et al.* 2005).

Glosten (1994) argued that LOBs are an effective way for patient traders to provide liquidity to less patient traders, even when liquidity is scarce. Luckock (2003) concluded that the volume traded in an LOB would always exceed that of a Walrasian market,† given the same underlying supply and demand.

Copeland and Galai (1983) noted that a limit order can be considered as a derivative contract written to the whole market, via which the order's owner offers to buy or sell the specified quantity of the asset at the specified price to any trader wishing to accept. For example, a trader who submits a sell limit order $x = (p_x, \omega_x, t_x)$ is offering the entire market a call option to buy ω_x units of the asset at price p_x for as long as the order remains active. Traders offer such derivative contracts— i.e. submit limit orders—in the hope that they will be able to trade at better prices than if they simply submitted market orders. However, whether or not such a contract will be accepted by another trader (i.e. whether or not the limit order will eventually become matched) is uncertain.

3. Challenges of studying LOBs

In this section, we discuss some of the challenges that LOBs present researchers. In particular, we discuss technical issues associated with the study of empirical LOB data and present several challenges inherent in modelling LOBs.

†A *Walrasian market* is a market in which all traders send their desired buy or sell orders to a specialist, who then determines the market value of the asset by selecting the price that maximizes the volume of trade.

3.1. Perfect rationality versus zero intelligence

Constructing a useful model of an LOB entails making several assumptions. One such assumption concerns the reason that order flows exist at all. Much of the economics literature assumes that orders are submitted because *perfectly rational* traders attempt to maximize their 'utility' by making trades in markets driven by 'information' (Parlour and Seppi 2008). However, this assumption has come under scrutiny because utility maximization is often inconsistent with direct observations of individual behaviour (Gode and Sunder 1993, Kahneman and Tversky 2000, Lux and Westerhoff 2009).

At the other extreme lies a *zero-intelligence* approach, in which aggregated order flows are assumed to be governed by specified stochastic processes whose rate parameters are conditional on other variables such as $\mathcal{L}(t)$ (Daniels *et al.* 2003, Smith *et al.* 2003, Cont *et al.* 2010). In this way, order flow can be regarded as a consequence of traders blindly following a set of rules without strategic considerations. Much like perfect rationality, zero-intelligence assumptions are extreme simplifications that are inconsistent with empirical observations. However, a zero-intelligence framework has the appeal of leading to models that can yield quantifiable and falsifiable predictions without the need for auxiliary assumptions. It is, therefore, a useful starting point for building models.‡

Between the two extremes of perfect rationality and zero intelligence lies a broad range of other approaches that make weaker assumptions about traders' behaviour and order flows, at the cost of resulting in models that are more difficult to study. Many such models rely exclusively on Monte Carlo simulation to produce output. Although such simulations still motivate interesting observations, it is often difficult to trace exactly how specific model outputs are affected by input parameters. Such Monte Carlo approaches are also computationally expensive, so they are of little use to traders who assess $\mathcal{L}(t)$ in real time.

3.2. State-space complexity

It is a well-established empirical fact that current order flows depend on both $\mathcal{L}(t)$ and on recent order flows (Biais *et al.* 1995, Sandås 2001, Ellul *et al.* 2003, Hollifield *et al.* 2004,

‡In Section 5, we explore how some authors have attempted to quantify perfect rationality for modelling purposes and discuss the often highly unrealistic assumptions that such formulations require to be tested empirically. A detailed treatment can be found in Foucault *et al.* (2005).

Hall and Hautsch 2006, Lo and Sapp 2010). From a perfect-rationality perspective, this can be regarded as traders reacting to the changing state of a market; from a zero-intelligence perspective, it can be considered as order-flow rates depending on $\mathcal{L}(t)$ and on their recent history. Either way, a key task is to uncover the structure of such conditional behaviour, either to understand what information traders evaluate when making decisions or to quantify the conditional structure of order flows.

A problem with studying conditional behaviour is that the state space of an LOB is huge: if there are P different choices for price in a given LOB, then the state space of the current depth profile alone, expressed in units of the lot size σ, is $\mathbb{Z}^P$. Therefore, a key modelling task is to find a way to simplify the evolving, high-dimensional state space, while retaining an LOB's important features. Some authors have proposed ways to reduce dimensionality (see, e.g. Cont and de Larrard (2011), Eliezer and Kogan (1998) and Smith *et al.* (2003)), but there is no consensus about a simplified state space upon which very general LOB models can be constructed.

3.3. Feedback and coupling

In addition to traders' actions depending on $\mathcal{L}(t)$, the state of $\mathcal{L}(t)$ also clearly depends on traders' actions. These mutual dependences induce feedback between $\mathcal{L}(t)$ and trader behaviour. Also, as described in Section 2.2, $b(t)$ determines the boundary condition for sell limit order placement because any sell order placed at or below $b(t)$ at least partially matches immediately. A similar role is played by $a(t)$ for buy orders. Therefore, order flow creates a strong coupling between $b(t)$ and $a(t)$. Smith *et al.* (2003) observed how such coupling makes LOB modelling a difficult problem.

3.4. Priority

As shown in figure 1, several active orders can have the same price at a given time. Much like priority is given to active orders with the best (i.e. highest buy or lowest sell) price, LOBs also employ a priority system for active orders within each individual price level.

By far, the most common priority mechanism currently used is *price-time*. That is, for active buy (respectively, sell) orders, priority is given to the active orders with the highest (respectively, lowest) price, and ties are broken by selecting the active order with the earliest submission time t_x. Price-time priority is an effective way to encourage traders to place limit orders (Parlour 1998). Without a priority mechanism based on time, there is no incentive for traders to show their hand by submitting limit orders earlier than is absolutely necessary.

Another priority mechanism, commonly used in futures markets, is *pro-rata* (Field and Large 2008). Under this mechanism, when a tie occurs at a given price, each relevant active order receives a share of the matching proportional to the fraction of the depth available that it represents at that price. For example, if a buy market order with size 3σ arrived at the LOB displayed in figure 3, then σ of it would match to active order x_1 and 2σ of it would match to active order x_2, because they correspond, respectively, to $1/3$ and $2/3$ of the depth available at $a(t)$. Traders in pro-rata priority LOBs are faced with the substantial difficulty of selecting optimal limit order sizes, because posting limit orders with larger sizes than the quantity that is really desired for trade becomes a viable strategy to gain priority.

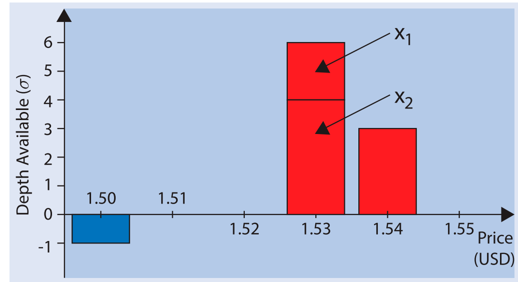

Figure 3. An LOB with pro-rata priority.

Another alternative priority mechanism is *price-size*, in which ties are broken by selecting the active order of largest size among those at the best price. Until recently, no major exchanges used this priority mechanism. However, in October 2010, the first price-size trading platform, NASDAQ OMX PSX, was launched (NADSAQ 2010). Some exchanges also allow traders to specify a minimum match size when submitting orders. Orders with a size smaller than this are not considered for matching to such orders. This is similar to a price-size priority mechanism: small active orders are often bypassed, effectively giving higher priority to larger orders.

Different priority mechanisms encourage traders to behave in different ways. Price-time priority encourages traders to submit limit orders early; price-size and pro-rata priority reward traders for placing large limit orders and thus for providing greater liquidity to the market. Traders' behaviour is closely related to the priority mechanism used, so LOB models need to take priority mechanisms into account when considering order flow. Furthermore, priority plays a pivotal role in models that attempt to track specific orders.

3.5. Incomplete sampling and hidden liquidity

An LOB $\mathcal{L}(t)$ reflects only the subset of trading intentions that traders have announced up to time t. However, the fact that no traders have submitted a limit order at a given price does not imply that none of them want to trade at this price, because they could be keeping their intentions private by submitting orders only when absolutely necessary (Tóth *et al.* 2011). Bouchaud *et al.* (2009) noted that a typical snapshot of $\mathcal{L}(t)$ at a given time is often very sparse, containing few active orders.†

3.5.1. Hidden orders. Many exchanges allow traders to conceal the extent of their intentions to trade, often at the

†Arbitrageurs provide an example of traders who behave in this way, because their strategies depend on simultaneously buying and selling in an attempt to make instant profit. Limit orders are of little use to them because it is uncertain when (if ever) they will be matched.

cost of paying some penalty in terms of priority or price. For example, many exchanges allow traders to submit *iceberg orders* (also known as *hidden-size orders*), a type of limit order that specifies not only a total size and price but also a *visible size*. Other traders only see the visible size. Rules regarding the treatment of the hidden quantity vary greatly from one exchange to another. In some cases, once a quantity of at least the visible size matches to an incoming market order, another quantity equal to the visible size becomes visible. This quantity has priority equal to that of a standard limit order placed at this price at this time. This sort of iceberg order is similar to a trader first submitting a limit order, then watching the market carefully and submitting a new limit order at the same price and size at the exact moment that the previous limit order matches to an incoming market order. A trader acting in this way is sometimes deemed to be constructing a *synthetic iceberg order*.

Some exchanges have an alternative structure for iceberg orders. Whenever a quantity equal to at least the visible size of an iceberg order is matched by an incoming market order, the rest of the order (i.e. the portion of the hidden component that is not matched by the same incoming market order) is cancelled. Iceberg orders can thereby match larger incoming market orders than is apparent, but otherwise they behave like any other order. This is the system currently used by the Reuters trading platform (Thomson-Reuters 2011).

Some other trading platforms allow entirely hidden limit orders. These orders are given priority behind both entirely visible active orders at their price and the visible portion of iceberg orders at their price, but they give traders the ability to submit limit orders without revealing any information whatsoever to the market.

3.5.2. Displayed liquidity. Even in LOBs with no hidden liquidity, traders are not always able to view the set of all active orders in real time. Many exchanges display only active orders that lie within a certain range of relative prices. Furthermore, some electronic trading platforms only transmit updates to $\mathcal{L}(t)$ at a specific frequency, so all activity that has taken place since the most recent refresh signal is invisible to traders.

3.5.3. Dark pools. Recently, there has been an increase in the popularity of so-called *dark pools* (see, e.g. Carrie (2006) and Hendershott and Jones (2005)). The matching rules governing trade in dark pools vary greatly from one exchange to another (Mittal 2008). Some dark pools are essentially LOBs in which all active orders are entirely hidden. Other dark pools do not allow traders to specify prices for their orders. Instead, traders submit orders describing their desired quantity and whether they wish to buy or sell, and the dark pool holds all such requests in an entirely hidden, time-priority queue until they are matched to orders of the opposite type. Upon matching, trades occur either at the mid-price $m(t)$ of another specified standard (i.e. non-dark) LOB for the same asset or at a price that is later negotiated by the two traders involved.

3.6. Volatility

Loosely speaking, volatility is a measure of the variability of returns of a traded asset (Barndorff-Nielsen and Shephard 2010). The volatility of an asset provides some indication of how risky it is. All else held equal, an asset with higher volatility is expected to undergo larger price changes over a given time interval than an asset with lower volatility. For traders who wish to manage their risk exposure, volatility is an important consideration when choosing the assets in which to invest, and, therefore, often forms the basis of optimal portfolio construction (Rebonato 2004).

Many different measures of volatility exist, and the exact form of volatility studied in a given situation depends on both the data available and the purpose of the calculation (Shephard 2005). Even when estimated on the same data, different measures of volatility sometimes exhibit different properties. For example, different measures of volatility follow different intraday patterns in a wide range of different markets (see Cont *et al.* (2011) and references therein). Therefore, many empirical studies report results using several different measures of volatility.

In an LOB, traders have access to far more information than just $b(t)$ and $a(t)$. In particular, information such as $n^b(b(t), t)$ and $n^a(a(t), t)$ is useful to predict how prices are likely to change (Biais *et al.* 1995, Ellul *et al.* 2003, Bortoli *et al.* 2006, Hall and Hautsch 2006, Lo and Sapp 2010). As discussed in Section 4.5, several empirical studies from a wide range of LOBs have reported links between volatility and other LOB properties. However, to our knowledge, there does not yet exist an estimate of volatility that takes into account the full state of $\mathcal{L}(t)$. Instead, most estimates of volatility consider only changes in price series such as $b(t)$, $a(t)$, and $m(t)$. For further discussion of practical issues regarding volatility estimation, see Liu *et al.* (1999).

3.6.1. Model-free estimates of volatility. There is an extensive literature on the use of price-series data to perform direct, model-free estimates of volatility (see, e.g. Aït-Sahalia *et al.* (2011), Andersen and Todorov (2010), Bandi and Russell (2006) and Zhou (1996)). In this section, we discuss three methods for performing such estimates.

Definition *Given the bid-price logarithmic return series* $r^b(t_1, t_2), r^b(t_2, t_3), \ldots, r^b(t_{k-1}, t_k)$ *sampled at regularly spaced times, the* bid-price realized volatility *is*

$$v^b(t_1, \ldots, t_k) := \textit{st. dev.}\left(\left\{r^b(t_i, t_{i+1}) \mid i = 1, \ldots, k-1\right\}\right).$$

The ask-price realized volatility*, denoted* $v^a(t_1, t_2, \ldots, t_k)$*, and the* mid-price realized volatility*, denoted* $v^m(t_1, t_2, \ldots, t_k)$*, are defined similarly.*

Realized volatility depends on the frequency at which price series are sampled. It is a useful measure for comparing the variability of return series sampled with the same frequency, but it is not appropriate to compare the realized volatility of a once-daily price series for one stock to a once-hourly price series for another.

Definition *Given the bid-price logarithmic return series* $r^b(t_1, t_2), r^b(t_2, t_3), \ldots, r^b(t_{k-1}, t_k)$ *sampled at the times at which k consecutive sell market orders arrive, the* bid-price realized volatility per trade *is*

$$V^b(t_1, \ldots, t_k) := \textit{st. dev.}\left(\left\{r^b(t_i, t_{i+1}) \mid i = 1, \ldots, k-1\right\}\right).$$

The ask-price realized volatility per trade*, denoted* $V^a(t_1, t_2, \ldots, t_k)$*, is defined similarly using k consecutive buy market* *order arrival times. The* mid-price realized volatility per trade*, denoted* $V^m(t_1, t_2, \ldots, t_k)$*, is defined similarly using k consecutive market order arrival times (irrespective of whether they are buy or sell market orders).*

Realized volatility per trade is useful for comparing how prices vary on a trade-by-trade basis.

Definition *Given the bid-price series* $b(t)$ *over an entire trading day D, the* bid-price intra-day volatility *is* $\rho^b(D) := \log\left(\max_{t\in D} b(t)/\min_{t\in D} b(t)\right)$*. The* ask-price intra-day volatility*, denoted* $\rho^a(D)$*, and the* mid-price intra-day volatility*, denoted* $\rho^m(D)$*, are defined similarly.*

Intra-day volatility is useful for estimating the probability of very large price changes in a given day. It is particularly important for *day traders*, who unwind their trading positions before the end of each trading day.

3.6.2. Model-based estimates of volatility. A difficulty that arises when estimating any measure of volatility in an LOB is that many traders submit then immediately cancel limit orders. This can occur for a variety of reasons, but it is often the result of electronic trading algorithms searching for hidden liquidity. Such behaviour can cause $b(t)$ and $a(t)$ to fluctuate rapidly without any trades occurring, and it can be considered as *microstructure noise* rather than a meaningful change in price. One way to address this problem is to assume that the observed data is governed by a model from which an estimate of volatility that is absent of microstructure noise can be derived. The parameters of the model are then estimated from the data, and the volatility estimate is derived explicitly from the model. However, a drawback of this method is that it depends heavily on the model, and models that poorly mimic important aspects of the trading process may, therefore, give misleading estimates of volatility.

3.7. Resolution parameters

Values of σ and π vary greatly between different trading platforms. Expensive stocks are often traded with $\sigma = 1$ share; cheaper shares are often traded with $\sigma \gg 1$ share. In foreign exchange (FX) markets, some trading platforms use values as large as $\sigma = 1$ million units of the base currency, whereas others use values as small as $\sigma = 0.01$ units of the base currency.† A given currency pair is often traded with different values of π on different trading platforms. For example, $\pi = \$0.00001$ for the GBP/USD LOB and $\pi = 0.001$ for the USD/JPY LOB on the electronic trading platform Hotspot FX, whereas $\pi = \$0.00005$ for the GBP/USD LOB and $\pi = 0.005$ for the USD/JPY LOB on the electronic trading platform EBS (EBS, 2012; Hotspot, 2013). In equity markets, π is often 0.01% of the stock's mid price $m(t)$, rounded to the nearest power of 10. For example, $m(t)$ for Apple Inc. fluctuated between approximately \$400 and approximately \$700 in 2012, during which time it traded with $\pi = \$0.01$.

It is a recurring theme in the literature (see, e.g. Biais *et al.* (1995), Foucault *et al.* (2005), Seppi (1997) and Smith *et al.* (2003)) that an LOB's resolution parameters σ and π greatly affect trade within it. An LOB's lot size σ dictates the smallest permissible order size, so any trader who wishes to trade in quantities smaller than σ is unable to do so. Furthermore, as we discuss in Section 4.6, traders who wish to submit large market orders often break them into smaller chunks to minimize their market impact. The size of σ controls the smallest permissible size of such chunks and therefore directly affects traders who implement such a strategy. An LOB's tick size π dictates how much more expensive it is for a trader to gain the priority (see Section 3.4) associated with choosing a higher (respectively, lower) price for a buy (respectively, sell) order (Parlour and Seppi 2008). In markets where π is extremely small, there is little reason for a trader to submit a buy (respectively, sell) limit order at a price p where there are already other active orders. Instead, he/she can gain priority over such active orders very cheaply, by choosing the price $p + \pi$ (respectively, $p - \pi$) for the limit order. Such a set-up leads to LOBs that undergo extremely frequent changes in $b(t)$ and $a(t)$ due to the small depths available. This makes it difficult for other traders to monitor the state of the market in real time. In September 2012, the electronic FX trading platform EBS increased the size of π for most of its currency pairs' LOBs. Their reason for doing so was 'to help thicken top of book price points, increase the cost of top of book price discovery, and improve matching execution in terms of percent fill amounts' (EBS 2012).

3.8. Bilateral trade agreements

On some exchanges, each trader maintains a *block-list* of other traders with whom he/she is unwilling to trade. A trade can only occur between traders θ_i and θ_j if θ_i does not appears on θ_j's block-list and vice-versa. The exchange shows each trader θ_i a personalized LOB that contains only the active orders owned by traders with whom it is possible for θ_i to trade. When a trader submits a market order, it can only match to active orders in their personalized LOB, bypassing any higher priority active orders from traders on their block-list.

On exchanges that use such *bilateral trade agreements*, it is possible for a buy (respectively, sell) market order to bypass all active orders at the globally lowest (respectively, highest) price available in $\mathcal{L}(t)$ and to match to an active order with a strictly higher (respectively, lower) price. Furthermore, given two traders θ_i and θ_j who do not have a bilateral trade agreement, it is possible for $\mathcal{L}(t)$ to simultaneously contain both an active buy order $x = (p_x, \omega_x, t_x)$ owned by θ_i and an active sell order $y = (p_y, \omega_y, t_y)$ owned by θ_j, with $p_y \leq p_x$, without a matching occurring. Therefore, it is possible for such markets to have a negative bid-ask spread.

These factors make modelling of specific matchings and of the evolution of $\mathcal{L}(t)$ a very difficult task in LOBs that operate with bilateral trade agreements. Gould *et al.* (2013a) presents a full discussion of these issues, so we do not consider such LOBs further.

†In FX markets, an XXX/YYY LOB matches exchanges of the *base currency* XXX to the *counter currency* YYY. A price in an XXX/YYY LOB denotes how many units of currency YYY are exchanged for a single unit of currency XXX. For example, a trade at the price \$1.52342 in a GBP/USD market corresponds to 1 pound sterling being exchanged for 1.52342 US dollars.

3.9. Opening and closing auctions

Many exchanges suspend standard limit order trading at the beginning and end of the trading day and instead use an auction system to match orders. For example, the LSE's flagship order book SETS (SETS 2011) has three distinct trading phases in each trading day. Between 08:00 and 16:30, the standard LOB mechanism is used in a period known as *continuous trading*. Between 07:50 and 08:00, a 10-min *opening auction* takes place. Between 16:30 and 16:35, a 5-min *closing auction* takes place. During both auctions, all traders can view and place orders as usual, but no orders are matched. Due to the absence of matchings, the highest price among buy orders can exceed the lowest price among sell orders. All orders are stored until the opening auction ends. At this time, for each price p at which there is non-zero depth available, the trade-matching algorithm calculates the total volume C_p of trades that could occur by matching buy orders with a price greater than or equal to p to sell orders with a price less than or equal to p. It then calculates the *uncrossing price*

$$\hat{p} = \arg\max_{p} C_p. \tag{8}$$

In contrast to standard LOB trading, all trades take place at the same uncrossing price $\hat{p}$. Given $\hat{p}$, if there is a smaller depth available for sale than there is for purchase (or vice versa), ties are broken using time priority.

Throughout the opening auction, all traders can see what the value of $\hat{p}$ would be if the auction were to end at that moment. This allows all traders to observe the discovery of the price without any matchings taking place until the process is complete.†

3.10. Statistical issues

As we discuss in Section 4, many authors have reported statistical regularities in LOB data from a wide variety of different markets. However, such statistical analysis is fraught with difficulties because assumptions such as independence and stationarity, which are often required to ensure consistency of estimation, are rarely satisfied by LOB data (Cont 2005, Mantegna and Stanley 1999). Furthermore, suboptimal estimators have been employed commonly in the literature, and have often produced estimates with large variance or bias. For example, there are questions about the validity of many reported power laws throughout the scientific literature (Clauset *et al.* 2009, Stumpf and Porter 2012). Many authors use ordinary least-squares regression on a log–log plot to estimate power-law exponents from LOB data, yet Clauset *et al.* (2009) showed that this method produces significant systematic estimation errors. They also showed that it is inappropriate to use power-law estimators designed for continuous data on discrete data (or vice versa), yet many LOB studies do precisely this.

In this section, we list some of the pitfalls of statistical estimation using LOB data and suggest some useful estimators for data analysis. However, these techniques are not 'one-size-fits-all' approaches, and it is important to verify the necessary assumptions before implementing them on a given data set.

3.10.1. Power laws. Several LOB properties are reported to have *power-law tails:*

Definition *A random variable Z with distribution function* F_Z *is said to have a* power-law tail with exponent α *if there exists some* $\alpha > 0$ *such that* $F_Z(z) \sim O\left(z^{-\alpha}\right)$ *as* $z \to \infty$.

If there exists a $z_{\min} > 0$ such that $F_Z(z)$ is proportional to $z^{-\alpha}$ for all $z \geq z_{\min}$, then clearly Z has a power-law tail.‡ When attempting to fit power-law tails to empirical observations, it is often assumed that such a $z_{\min}$ exists (and resides within the range of the data), because the existence of such a $z_{\min}$ allows simple, closed-form expressions to be derived. Under this assumption, Clauset *et al.* (2009) provided a comprehensive algorithm for consistent estimation of α and $z_{\min}$ via maximum likelihood techniques, as well as for testing the hypothesis that the data really does follow a power law for $z \geq z_{\min}$. Several other consistent estimation procedures also exist (see, e.g. Hill (1975) and Mu *et al.* (2009)), but no single estimator has emerged as the best to adopt in all situations. Therefore, some empirical studies report results using several different estimators and then draw inference based on the whole set of results. However, as Mu *et al.* (2009) highlighted, different estimators often produce vastly different estimates of α, making such inference difficult.

3.10.2. Long-memory processes. As we discuss in Section 4.7, several time series related to LOBs have been reported to exhibit *long memory*. Intuitively, a time series has long memory if values from the present are correlated with values in the distant future. The study of long-memory processes involves considerable challenges, and caution is needed when applying standard statistical techniques to data with long memory (Beran 1994). For example, the effective sample size of a long-memory process is significantly smaller than the number of data points, so statistical estimators often converge at an extremely slow rate (Farmer and Lillo 2004). Furthermore, the correlation structure can cause such estimators to converge to arbitrary values (Beran 1994).

In this section, we discuss several practical challenges of estimating long memory. We denote by $\boldsymbol{X}$ a real-valued, second-order stationary§ time series of length k, $\boldsymbol{X} = X(t_1), X(t_2), \ldots, X(t_k)$.

One way to define long memory is via the asymptotic behaviour of the *autocorrelation function.*

†Biais *et al.* (1999) performed a formal hypothesis test on price-discovery data from the Paris Bourse. Working at the 2.5% level, they did not reject the null hypothesis that traders' conditional expectations of asset price approached the market value of the asset during the final 9 min of the price-discovery process. However, they reported that traders' actions were not significantly different from noise during the early part of the price-discovery process.

‡This is not the only probability density function that has a power-law tail, but it is the most common in the literature.

§A time series is *second-order stationary* if its first and second moments are finite and do not vary with time. For a discussion of issues regarding stationarity in financial time series, see Taylor (2008).

Definition *The* autocorrelation function *A of a time series* $\boldsymbol{X}$ *is given by*

$$A_{\boldsymbol{X}}(l) := \frac{1}{k-l} \sum_{i=1}^{k-l} \left(X(t_i) - \langle \boldsymbol{X} \rangle\right)\left(X(t_{i+l}) - \langle \boldsymbol{X} \rangle\right), \quad (9)$$

where $\langle \boldsymbol{X} \rangle = \frac{1}{k}\sum_{i=1}^{k} X(t_i)$ *is the mean of the series.*

Definition *A time series* $\boldsymbol{X}$ *is said to exhibit* long memory *if*

$$\lim_{n\to\infty} \sum_{l=0}^{n} |A_{\boldsymbol{X}}(l)| = \infty. \quad (10)$$

One way that this can occur is if there exists some $\alpha \in (0, 1)$ such that $A_{\boldsymbol{X}}$ decays like a power law:

$$A_{\boldsymbol{X}}(l) \sim O\left(l^{-\alpha}\right), \text{ as } l \to \infty. \quad (11)$$

The exponent α describes the strength of the long memory: the smaller the value of α, the stronger the long-range autocorrelations (Lillo and Farmer 2004). Because of the slow decay of the autocorrelation function in a long-memory process, present values of the series can have a significant effect on its values in the distant future. It is a recurring mistake in the literature that if $\boldsymbol{X}$ has long memory, its unconditional distribution must exhibit heavy tails. However, Preis *et al.* (2006, 2007) showed that such an implication does not hold in general.

A key difficulty when using equation (11) to assess whether a given series has long memory is that it deals only with asymptotic behaviour. To study the large-l behaviour, it is necessary to observe more than l values of $\boldsymbol{X}$, but clearly any empirically observed time series is finite. Also, the values of the function $A_{\boldsymbol{X}}(l)$ can be small, making estimation of the functional form of A very difficult. Therefore, direct estimation of α from the autocorrelation function often produces very poor results (Lillo and Farmer 2004).

An alternative way to characterize long memory is via the diffusion properties (Beran 1994, Lillo and Farmer 2004) of the integrated series $\boldsymbol{Y}$:

$$Y(l) = \sum_{i=1}^{l} X(t_i). \quad (12)$$

Let

$$V(l) = \mathrm{Var}\left(\{Y(t_{i+1}), Y(t_{i+2}), \ldots, Y(t_{i+l})\}\right) \quad (13)$$

for some $i \in \{0, 1, \ldots, k-l\}$. In the limit $l \to \infty$, if $\boldsymbol{X}$ is a short-memory process, then $V(l)$ scales as $O(l)$, whereas if $\boldsymbol{X}$ is a long-memory process, then $V(l)$ scales as $O(l^{2H})$, for some $H \in (1/2, 1)$ (Beran 1994, Lillo and Farmer 2004). The exponent H is called the *Hurst exponent.* For long-memory processes, H is related to α in equation (11) by

$$H = 1 - \frac{\alpha}{2}. \quad (14)$$

Short-memory processes have a Hurst exponent of $H = 1/2$ (Lillo and Farmer 2004).

Under some assumptions, there are several asymptotically unbiased estimators of H that are more robust to noise in $\boldsymbol{X}$ than is direct estimation of α from the autocorrelation function (Taqqu *et al.* 1995). However, the performance of such estimators on empirical data, which might not conform to the estimators' assumptions, varies considerably (Xu *et al.* 2005, Rea *et al.* 2009). Different disciplines tend to favour different estimators, although the choices are often based on historical reasons, not performance. Some of the most commonly used are:

- the *R/S statistic* and *modified R/S statistic* (Lo 1989, Teverovsky *et al.* 1999);
- *log-periodogram regression* (Geweke and Porter-Hudak 1983); and
- *order-m detrended fluctuation analysis (DFAm)* (Peng *et al.* 1994, Kantelhardt *et al.* 2001, La Spada and Lillo 2011).

As with the estimation of power laws discussed in Section 3.10.1, no single estimator has emerged as the best in all situations. Some empirical studies report results using several different estimators and then draw inference based on them all (Taqqu *et al.* 1995).

4. Empirical observations in LOBs

The empirical literature on LOBs is very large, yet different studies often present conflicting conclusions. Reasons for this include different trade-matching algorithms operating differently, different asset classes being traded on different exchanges, differing levels of liquidity in different markets, and different researchers having access to data of differing quality. Furthermore, as traders' strategies have evolved over time, so too have the statistical properties of the order flow they generate. This has become a particularly important consideration because competition and trading volumes have increased with the widespread uptake of electronic trading algorithms.

To aid comparisons, we present in Appendix A a description of the aims, date range, data source, and data type of each of the empirical studies of LOBs that we discuss in this survey. We now discuss the main findings of these empirical studies in more detail, including a selection of *stylized facts* that have consistently emerged from several different markets. However, we note in Section 6 that there have been few recent data analyses, despite the many recent changes in markets.

4.1. Order size

Given the heterogeneous motivations for trading within a single market, it is unsurprising that incoming order sizes vary substantially. Nevertheless, several regularities occur in empirical data.

For equities traded on the Paris Bourse, Bouchaud *et al.* (2002) reported that the distribution of $\log(|\omega_x|)$ was approximately uniform for incoming limit orders with $|\omega_x| \in (10, 50000)$. For two stocks traded on NASDAQ, Maslov and Mills (2001) reported power-law and log-normal distributions to fit the distribution of incoming limit order sizes $|\omega_x|$. The mean power-law exponent was 1 ± 0.3 (i.e. with standard deviation 0.3). However, the quality of the power-law fits was deemed to be weak, and the log-normal fits were deemed to be applicable over a wider range of limit order sizes than the power-law fits (although the authors stated no precise range of applicability for either). For four stocks on the Island ECN, Challet and Stinchcombe (2001) reported that incoming limit order sizes $|\omega_x|$ clustered strongly at round-number amounts such as 10, 100, and 1000. Mu *et al.* (2009) reported a similar round-number preference for market orders on the Shenzhen Stock Exchange. Mu *et al.* (2009) also studied the distribution of total trade sizes when aggregated over a variety of time windows and found it to exhibit a power-law tail. Different power-law exponent estimators produced different estimates of the tail exponent, but the authors judged the tail exponent to be larger than two. Maslov and Mills (2001) reported similar power-law fits on NASDAQ. Studying five days of data covering three equities, they reported a mean power-law exponent of 1.4 ± 0.1. Although they did not state a range of sizes over which their reported power-law distributions applied, figure 1 in Maslov and Mills (2001) suggests an approximate range of 200–5000. In a study of the 1000 largest equities in the USA, Gopikrishnan *et al.* (2000) also reported power-law fits for the distribution of trade sizes. The mean power-law exponent was 1.53 ± 0.07. However, Bouchaud *et al.* (2009) noted that the data studied by Gopikrishnan *et al.* contains information about trades that were privately arranged to occur off-book. They conjectured that this caused Gopikrishnan *et al.* to overestimate the arrival frequency of very large orders.

In a study of the Stockholm Stock Exchange, Hollifield *et al.* (2004), reported that buy (respectively, sell) market orders that *walked up the book*—i.e. buy market orders with a size $|\omega_x| > n^a(a(t), t)$ (respectively, sell market orders with a size $\omega_x > |n^b(b(t), t)|$)—accounted for only 0.1% of submitted market orders. Therefore, the vast majority of submitted buy (respectively, sell) market orders matched only to active orders at $a(t)$ (respectively, $b(t)$), rather than at other prices.

4.2. Relative price

As discussed in Section 2.1, regularities in price series are best investigated via the use of relative pricing, as $b(t)$ and $a(t)$ themselves evolve through time. Several authors have reported power-law behaviour in the distribution of relative prices (Bouchaud *et al.* 2002, Zovko and Farmer 2002, Potters and Bouchaud 2003, Maskawa 2007, Mike and Farmer 2008, Gu *et al.* 2008b). One possible reason for this behaviour is that some traders place limit orders deep into LOBs, under the optimistic belief that large price swings could occur (Bouchaud *et al.* 2002).

The distributions of relative prices of orders that arrived with a non-negative relative price on the Paris Bourse (Bouchaud *et al.* 2002), NASDAQ (Potters and Bouchaud 2003), the LSE (Zovko and Farmer 2002, Maskawa 2007), and the Shenzhen Stock Exchange (Gu *et al.* 2008b) were all reported to follow such a power law, with different values of the exponent for the different markets. On the Paris Bourse, for buy and sell orders alike, the power-law exponent for relative prices from π to over 100π (even up to 1000π for some stocks) was approximately 0.6. On NASDAQ, the ranges of relative prices over which the distributions followed a power law and the power-law exponents themselves both varied from stock to stock. On the LSE, the value of the power-law exponent was approximately 1.5 for relative prices between 10π and 2000π for both buy and sell orders. In aggregated data describing 23 stocks on the Shenzhen Stock Exchange, the power-law exponent for the distribution of non-negative relative prices† was 1.72 ± 0.03 for buy orders and 1.15 ± 0.02 for sell orders, and the power-law exponent for the distribution of negative relative prices was 1.66 ± 0.07 for buy orders and 1.80 ± 0.07 for sell orders. This asymmetry between buy orders and sell orders contrasts to the other markets that were studied, but the exact matching rules on the Shenzhen Stock Exchange prevent large price changes from occurring within a single day (which could account for this effect).

The maximum arrival rate of incoming orders was reported to occur at a relative price of 0 on the LSE (Mike and Farmer, 2008), the Shenzhen Stock Exchange (Gu *et al.* 2008b), the Paris Bourse (Biais *et al.* 1995, Bouchaud *et al.* 2002) and NASDAQ (Challet and Stinchcombe 2001). However, the maximum arrival rate on the Tokyo Stock Exchange was reported to occur inside the spread (Cont *et al.* 2010).

4.3. Order cancellations

Several empirical studies covering a wide range of different markets have concluded that the vast majority of active orders ended in cancellation rather than matching. The percentage of orders that were cancelled ranged from approximately 70% to more than 80% on the Island ECN (Challet and Stinchcombe 2001, Hasbrouck and Saar 2002), an exchange-traded fund that tracked the NASDAQ 100 (Potters and Bouchaud 2003), S&P 500 futures contracts (Baron *et al.* 2012), and in FX markets (Gereben and Kiss 2010, Lo and Sapp 2010). Therefore, cancellations played a major role in the evolution of $\mathcal{L}(t)$ in all of these markets.

In recent years, electronic trading algorithms have surged in popularity across all markets, and such algorithms often submit and cancel vast numbers of limit orders over short periods as part of their trading strategies (Harris 2002, Hendershott *et al.* 2011). The widespread use of such trading algorithms seems to have further increased the percentage of orders that are cancelled in recent data. In particular, a study of recent FX data found that more than 99.9% of active orders ended in cancellation rather than matching (Gould *et al.* 2013b).

4.4. Mean relative depth profile

Despite their different resolution parameters (see Section 2.1) and the different prices at which trades occur in them, several qualitative regularities are common to the mean relative depth profiles in a wide range of markets.

No significant difference was detected between the mean bid-side and the mean ask-side relative depth profiles on the Paris Bourse (Biais *et al.* 1995, Bouchaud *et al.* 2002), NASDAQ (Potters and Bouchaud 2003) and Standard and Poor's Depositary Receipts (SPY)‡ (Potters and Bouchaud 2003). By contrast, Gu *et al.* (2008c) reported asymmetry between the mean bid-side and the mean ask-side relative depth

†Observe that the notation used by Gu *et al.* (2008b) assigns the opposite signs when measuring relative price than those that we employ.

‡SPY is an exchange-traded fund that allows traders to effectively buy and sell shares in all of the 500 largest stocks traded in the USA.

profiles on the Shenzhen Stock Exchange, but this is unsurprising considering that this market has additional rules restricting price movements each day that essentially impose asymmetric restrictions on the range of relative prices over which traders can submit orders.

Mean relative depth profiles have been reported to exhibit a hump shape† in a wide range of markets, including the Paris Bourse (Bouchaud *et al.* 2002), NASDAQ (Potters and Bouchaud 2003), the Stockholm Stock Exchange (Hollifield *et al.* 2004), and the Shenzhen Stock Exchange (Gu *et al.* 2008c). The location of the hump varied across markets. However, it is difficult to perform direct comparisons between different markets because differences in their tick sizes π affect both the granularity of the price axis and the ways in which traders behave (see Section 3.7). There may also be strategic reasons that the hump occurs in different locations in different markets. For example, traders are more likely to submit limit orders with larger relative prices in markets in which large price changes are relatively common than they are in markets in which such price changes are rare. This increases the relative price at which the hump resides. Roşu (2009) conjectured that a hump would exist in all markets in which large market orders are sufficiently likely; this represents a trade-off between the optimism that a limit order placed away from $b(t)$ or $a(t)$ might eventually match (at a significant profit) and the pessimism that limit orders that are placed too far away from $b(t)$ and $a(t)$ might never match.

4.5. Conditional frequencies of events

The properties that we have discussed thus far in this section have all been calculated unconditionally (i.e. without reference to other events or variables). However, several factors influence how traders interact with LOBs, so it is reasonable to study not only unconditional frequencies, but also the frequencies of those events given that some other condition was satisfied. However, the study of such conditional event frequencies in LOBs is difficult for two main reasons:

(i) The state space is very large. Deciding which of the enormous number of possible events or LOB states on which to condition is very difficult (see Section 3).
(ii) There is a small *latency* between the time that a trader sends an instruction to submit or cancel an order and the time that the exchange server receives the instruction. Furthermore, some exchanges only send refresh signals at fixed time intervals, so traders cannot be certain that LOBs that they observe via their trading platform are perfect representations of the actual LOBs at that instant in time. Therefore, conditioning on the 'most recent' event is problematic, as the most recent event recorded by the exchange (and thus appearing in the market data) may not be the most recent event that a given trader observed via the trading platform.

Nevertheless, several empirical studies of conditional event frequencies in LOBs have identified interesting behaviour. In this section, we review the key findings from several such publications, highlighting both the similarities and differences that have emerged across different markets.

It is important to note that most studies of conditional dependence in LOBs have used data that dates back 10 or more years. Although this alleviates the aforementioned difficulties with latency (as the volume of order flows in LOBs was much smaller in the past than it is today, so the mean inter-arrival times between successive events were substantially longer than the latency times), it also inevitably raises the question of how representative such findings are of today's LOBs. We return to this issue in Section 6.

4.5.1. Order size. A simple example of conditional structure is the relationship reported between the size $|\omega_x|$ and the relative price δ^x of orders on the Paris Bourse (Bouchaud *et al.*, 2002). For the stocks studied, the distribution of $|\omega_x|$ varied substantially according to the relative price of the corresponding orders. In particular, orders with a larger relative price had a smaller absolute size $|\omega_x|$ on average. Maslov and Mills (2001) made a similar observation for limit orders on NASDAQ.

4.5.2. Relative price. Biais *et al.* (1995) noticed on the Paris Bourse that traders placed more orders with a relative price δ^x satisfying $-s(t) < \delta^x < 0$ (i.e. limit orders that arrived inside of the bid-ask spread) at times when $s(t)$ was larger than its median value. Hall and Hautsch (2006) and Cao *et al.* (2008) made similar observations using data from the Australian Stock Exchange. Similarly, on the NYSE (Ellul *et al.* 2003), the percentage of incoming orders that arrived with a relative price $\delta^x > -s(t)$ (i.e. were limit orders) increased as $s(t)$ increased and decreased when $s(t)$ decreased. Biais *et al.* (1995) argued that when $s(t)$ is small, it is less expensive for traders to demand immediate liquidity, so market orders become more attractive. However, it is also possible to explain such an observation via a zero-intelligence approach: if limit order prices are chosen uniformly at random, then it is more likely that an incoming limit order price resides in the interval $(b(t), a(t))$ when the interval is wider.

Biais *et al.* (1995) reported on the Paris Bourse that the percentage of buy (respectively, sell) limit orders that arrived with relative price δ^x satisfying $-s(t) < \delta^x < 0$ was higher at times when $|n^b(b(t), t)|$ (respectively, $n^a(a(t), t)$) was larger. They conjectured that this was caused by traders competing for higher priority than the active orders in the (already long) queue by submitting an order with a better price. Furthermore, Ellul *et al.* (2003) reported on the NYSE that the arrival rate of buy (respectively, sell) limit orders with a relative price δ^x satisfying $-s(t) < \delta^x < 0$ tended to increase as the total size of active buy (respectively, sell) orders increased. They also reported a similar result for the arrival of buy (respectively, sell) market orders. In studies of the Australian Stock exchange, Hall and Hautsch (2006) calculated that the percentage of buy (respectively, sell) orders that were limit orders decreased as the total size of active buy (respectively, sell) orders increased, and Cao *et al.* (2008) reported that the proportion of arriving orders that were market orders increased when $|n^b(b(t), t)|$ and $n^a(a(t), t)$ were larger. In a study of the LSE, Maskawa (2007)

†More precisely, the absolute value of the mean depth available increased over the first few relative prices, and it subsequently decreased.

concluded that traders favoured placing their limit orders at relative prices similar to those where there was already a large number of active orders.

However, such conditional structure has not been found in all markets. Mike and Farmer (2008) reported for the LSE that the distribution of relative prices was independent of $s(t)$. In a study of the Shenzhen Stock Exchange, Gu *et al.* (2008b) reported that the distribution of relative prices was independent of both $s(t)$ and volatility. Biais *et al.* (1995) concluded that $\left|n^b(b(t), t)\right|$ (respectively, $n^a(a(t), t)$) had little impact on the rate of incoming sell (respectively, buy) market orders on the Paris Bourse.

In a study of the Swiss Stock Exchange, Ranaldo (2004) reported that order flow depended on several factors, including volatility, recent order flow, and the state of $\mathcal{L}(t)$. Traders submitted more limit orders and fewer market orders during periods when $s(t)$ or volatility were high. The proportion of orders that arrived with negative relative price decreased as the inter-arrival time between recent orders increased. Traders submitted higher-priced buy orders (respectively, lower-priced sell orders) when the total size of active buy (respectively, sell) orders was greater. Ranaldo (2004) noted that buy order submission seemed to depend on both the sell side and the ask side of $\mathcal{L}(t)$, whereas sell order submission seemed to depend only on the sell side of $\mathcal{L}(t)$. He also noted, however, that market performance during the sample period might have caused such asymmetry, because the percentage change in $m(t)$ was positive for all but one of the stocks studied and exceeded 10% for four of them.

In a study of the LSE, Zovko and Farmer (2002) reported that the relative prices of incoming limit orders were conditional on the bid-price realized volatility per trade. They constructed two time series by calculating the mean relative price of arriving buy limit orders and the bid-price realized volatility per trade over 10 min windows, and then calculated their cross correlation. They rejected (at the 2.5% level) the hypothesis that the two series were uncorrelated and observed that changes in bid-price realized volatility immediately preceded changes in mean relative price for buy limit orders.† They also observed similar behaviour when comparing the time series of ask-price realized volatility and the time series of mean relative price for sell limit orders.

Lo and Sapp (2010) reported that traders in FX markets submitted orders with higher relative prices during periods of high mid-price realized volatility.

4.5.3. Order flows. In a study of the Stockholm Stock Exchange, Sandås (2001) reported that order flows at time t were conditional on both $\mathcal{L}(t)$ and on previous order flows. In their study of FX markets, Lo and Sapp (2010) reported that order flows at time t were conditional on several variables including $s(t)$, $n^b(b(t), t)$, $n^a(a(t), t)$, depth available behind the best prices, time of day, and recent order flows. However, the precise structure of the conditional dependences varied between currency pairs. In a study of the NYSE, Ellul *et al.* (2003) reported that the rate of buy (respectively, sell) limit order arrivals increased after periods of positive (respectively, negative) mid-price returns and that the rate of limit order arrivals also increased late in the trading day.

On the Australian Stock Exchange (Hall and Hautsch 2006), the arrival rates of all market events were reported to increase and decrease together. The authors suggested that other exogenous factors (which they did not measure) might have influenced aggregate LOB activity. In a more recent study of the Australian Stock Exchange, Cao *et al.* (2008) reported that the arrival rates of market events at time t were conditional on $\mathcal{L}(t)$, but not on the state of $\mathcal{L}(t)$ at earlier times. They concluded that traders evaluated only an LOB's most recent state—and not a longer history—when they made order placement and cancellation decisions. Cao *et al.* (2008) found no evidence that mid-price returns had a significant impact on order arrival or cancellation rates.

Using several different financial instruments traded in electronic LOBs, Toke (2011) reported that both buy limit order and sell limit order arrival rates increased following the arrival of a market order, but they found no evidence that market order arrival rates increased following the arrival of a limit order.

4.5.4. Event clustering. Using data from 40 stocks on the Paris Bourse, Biais *et al.* (1995) observed strong clustering through time when studying the 'action classes' (such as 'arrival of buy market order', 'arrival of buy limit order within the spread', and 'cancellation of active sell order') of market events. For all action classes, the conditional frequency with which a market event belonged to the specified action class, given that the previous market event also belonged to the same action class, was higher than the corresponding unconditional frequency. The authors offered numerous possible explanations for this phenomenon: traders might have strategically split large orders into smaller chunks to avoid revealing their full trading intentions or to minimize market impact (see Section 4.6); different traders might have mimicked each other; different traders might have reacted independently to new information; or different traders might have tried to undercut each other (i.e. cancelled active buy (respectively, sell) orders and resubmitted them at a slightly higher (respectively, lower) price solely to gain price priority). Bursts of small, frequent changes in $b(t)$ and $a(t)$ occurred more often when $s(t)$ was large, and they argued that this provided evidence of undercutting. However, Bouchaud *et al.* (2009) concluded that the phenomenon was driven primarily by strategic order splitting and found no evidence that different traders mimicked each other.

In a study of the NYSE, Ellul *et al.* (2003) reported that periods with above-average order arrival rates clustered together in time, as did periods with below-average order arrival rates. They also reported a similar clustering of market events by action classes to that observed by Biais *et al.* (1995) on the Paris Bourse. However, Ellul *et al.* (2003) reported that the number of occurrences of market events from a specific action class in a given 5-min window and the corresponding number

†Zovko and Farmer (2002) noted that it was not clear from the cross-correlation function alone whether traders explicitly considered bid-price realized volatility when choosing the prices for their buy limit orders, or whether some other external factor first affected bid-price realized volatility and then affected traders' actions. If the former could be demonstrated, it would support the widely held belief that many traders consider realized volatility to be an important factor in deciding when to place a limit order (Zovko and Farmer 2002).

of occurrences of market events in the previous 5-min window were negatively correlated. Furthermore, they concluded that the arrival rate of market events from a given action class was more heavily conditional on the action class of the single most recent market event than it was on $\mathcal{L}(t)$, whereas the distribution of the number of occurrences of market events from a given action class in a given 5-min window was more heavily conditional on $\mathcal{L}(t)$ during the previous 5-min window than it was on the number of occurrences of market events from any specific action class in the same window.

4.5.5. Cancellations. Biais *et al.* (1995) reported that cancellations of buy (respectively, sell) active orders on the Paris Bourse occurred more frequently after the arrival of a buy (respectively, sell) market order. They conjectured that this was evidence that traders submitted large orders in the hope of finding hidden liquidity and then cancelled any unmatched portions.

In a study of the Australian Stock Exchange, Cao *et al.* (2008) concluded that priority considerations played a key role for traders when deciding whether or not to cancel their active orders. The cancellation rate for active buy (respectively, sell) orders increased when new, higher-priority buy (respectively, sell) limit orders arrived. In addition, the cancellation rate of active buy (respectively, sell) orders at prices $p < b(t)$ (respectively, $p > a(t)$) increased when $n^b(p-\pi,t)$ (respectively, $n^a(p+\pi,t)$) became zero. The authors proposed that this occurred because traders with active orders at price p could, without substantial loss of priority, cancel and then resubmit them at price $p-\pi$ (respectively, $p+\pi$), to possibly gain a better price if the order eventually matched. No similar increase occurred when $n^b(p+\pi,t)$ (respectively, $n^a(p-\pi,t)$) became zero.

4.5.6. Price movements. In a study of the Paris Bourse, Biais *et al.* (1995) reported that $a(t)$ decreased more frequently (respectively, $b(t)$ increased more frequently) immediately after the arrival of a market order that caused $b(t)$ to decrease (respectively, $a(t)$ to increase). They suggested that such behaviour could have been caused by traders reacting to information, either because external sources of news led to a revaluation of the underlying asset or because traders interpreted the downward movement of $b(t)$ (respectively, upward movement of $a(t)$) itself as news. Indeed, Potters and Bouchaud (2003) found evidence on NASDAQ that each new trade was interpreted by traders as new information that directly affected the flow of incoming orders.

4.5.7. Volatility. For Canadian stocks, Hollifield *et al.* (2006) reported that several different volatility measures were correlated with order flow rates. Furthermore, on Euronext (Chakraborti *et al.* 2011b) and for German Index Futures (Kempf and Korn 1999), mid-price realized volatility increased with the number of arriving market orders. Jones *et al.* (1994) reported a similar finding in a study of the NYSE; however, Ellul *et al.* (2003) later reported a positive correlation between higher mid-price realized volatility and the percentage of arriving orders that were limit orders.

In a study of the Australian Stock Exchange, Hall and Hautsch (2006) reported that the number of arrivals and cancellations of large limit orders (i.e. those whose size was in the upper quartile of the unconditional empirical distribution of order sizes) in any given 5-min window was positively correlated with mid-price realized volatility during both that window and the previous 5-min window. However, in a more recent study, Cao *et al.* (2008) concluded that mid-price realized volatility per trade had only a minimal effect on order flows.

A weak but positive correlation between $s(t)$ and realized mid-price volatility has been observed in a wide range of markets (see Wyart *et al.* (2008) and references therein). However, a much stronger positive correlation between $s(t)$ and mid-price volatility was observed at the trade-by-trade timescale on the Paris Bourse (Bouchaud *et al.* 2004), the FTSE 100 (Zumbach 2004), and the NYSE (Wyart *et al.* 2008). In a recent study of stocks traded on the NYSE, Hendershott *et al.* (2011) reported that the once-daily time series of bid-price realized volatility was positively correlated with the daily mean spread. Stocks with a lower mid price had higher bid-price realized volatilty on average. Lo and Sapp (2010) reported that the variance of the depth available at any given price in FX markets increased during periods of high mid-price realized volatility. Hasbrouck and Saar (2002) investigated links between volatility and various aspects of the depth profile on the Island ECN, but they found only weak relationships.

As discussed in Section 3.1, Bortoli *et al.* (2006) reported that mid-price intra-day volatility on the Sydney Futures Exchange varied according to how much information about the depth profile traders could view in real time.

4.6. Market impact and price impact

A key consideration for a trader who wishes to buy or sell a large quantity of an asset is how his/her actions might affect the asset's LOB (Almgren and Chriss 2001, Bouchaud *et al.* 2009, Cont *et al.* 2011, Eisler *et al.* 2012, Obizhaeva and Wang 2013). For example, if trader θ_i wishes to buy 20σ shares using the LOB displayed in figure 4, then submitting a single market order of size $\omega_x = -20\sigma$ would result in purchasing 2σ shares at \$1.5438, 5σ shares at \$1.5439, 6σ shares at \$1.5440, and 7σ shares at \$1.5441. However, if θ_i were initially to submit only a market order of size $\omega_x = -2\sigma$, then it is possible that other traders might submit new limit orders, because by purchasing the 2σ shares with highest priority in the LOB, θ_i would have made it more attractive for other participants to submit new sell limit orders than it was immediately before such a purchase. If this occurs, then θ_i could submit a market order that matches to these newly submitted limit orders and then repeat this process until all 20σ shares are purchased. Empirical observations suggest that such *order splitting* is very common in a wide range of different markets (Bouchaud *et al.* 2009). Of course, there is no guarantee that the initial market order of size 2σ would stimulate such submissions of limit orders from other traders. Indeed, it could even cause other traders to cancel their existing limit orders or to submit buy market orders, further increasing $a(t)$ and thereby ultimately causing θ_i to pay a higher price for the total purchase of 20σ shares.

The change in $b(t)$ and $a(t)$ caused by a trader's actions is called the *price impact* of the actions. The necessity for traders to monitor and control price impact predates the widespread adoption of LOBs. In a quote-driven market, for example, any

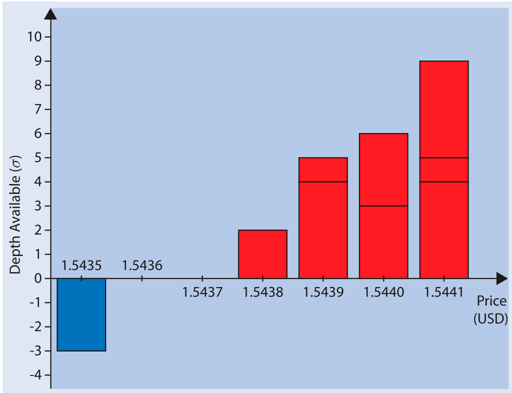

Figure 4. An example LOB to illustrate price impact and market impact.

single market maker only has access to a finite inventory, so there is a limit on the size that is available for trade at the quoted prices. Furthermore, purchasing or selling large quantities of the asset in such a market could cause market makers to adjust their quoted prices. Both of these outcomes are examples of price impact.

In an LOB, it is also possible to consider the impact of an action on the entire state of $\mathcal{L}(t)$. This more general type of impact is called *market impact*. To date, the terms 'price impact' and 'market impact' have often been used interchangeably to refer only to changes in $b(t)$ or $a(t)$, but recent work (Hautsch and Huang 2011) has shed light on how traders' actions can affect the depths available at other prices, suggesting that it is appropriate to separate the two notions.

Bouchaud *et al.* (2009) provided a detailed review of studies of both price impact and market impact. Both are difficult to quantify formally, as they each consist of two components:

- *instantaneous* (or *immediate*) *impact,* which consists of the immediate effects of a specified action and
- *permanent impact,* which consists of the long-term impact due to a specified action causing other traders to behave differently in the future.

For example, the instantaneous price impact of a buy market order of size 2σ in the LOB in figure 4 is a change in $a(t)$ from \$1.5438 to \$1.5439. An example of permanent market impact of this buy market order might be another trader deciding to submit a new sell limit order at the price \$1.5442. The various forms of impact are defined as follows.

Definition *The* instantaneous bid-price impact of a market event at time t' *is*

$$b(t') - \lim_{t \uparrow t'} b(t). \tag{15}$$

Definition *The* instantaneous bid-price logarithmic return impact of a market event at time t' *is*

$$\log b(t') - \lim_{t \uparrow t'} [\log b(t)]\,. \tag{16}$$

Definition *The* instantaneous bid-price impact function $\phi^b(\omega_x)$ *outputs the mean instantaneous bid-price impact for a buy market order of size* ω_x.

Definition *The* instantaneous bid-price logarithmic return impact function $\Phi^b(\omega_x)$ *outputs the mean instantaneous bid-price logarithmic return impact for a buy market order of size* ω_x.

Definitions for the ask price, using sell market orders of size ω_x (respectively, mid price, using both buy and sell market orders of size $|\omega_x|$) are similar.

Definition *The instantaneous market impact of a market event at time* t' *is*

$$\mathcal{L}(t') \setminus \lim_{t \uparrow t'} \mathcal{L}(t), \tag{17}$$

where $\setminus$ *denotes the difference of the two sets.*

Instantaneous impact exists because the arrival or cancellation of any order affects $\mathcal{L}(t)$ directly. Bouchaud *et al.* (2009) described three reasons that permanent impact might exist. First, trades themselves might convey information to other traders.† Second, traders might successfully forecast short-term price movements and choose their actions accordingly.‡ Third, purely random fluctuations in supply and demand might lead to permanent impact.

It is not possible to quantify precisely the permanent price or market impact of an action, because doing so would involve calculating the differences between scenarios in which the action did occur and those in which it did not occur. Clearly, all actions either occur or do not occur, so such comparisons are impossible in practice.

4.6.1. Instantaneous price impact. To date, instantaneous price impact for individual market orders has been studied primarily via instantaneous price impact and instantaneous logarithmic return impact functions. In a study of the NYSE and American Stock Exchange, Hasbrouck (1991) found ϕ^m to be a concave function of $|\omega_x|$. This implies that the instantaneous price impact of a single market order of size $|\omega_x|$ was, on average, larger than the sum of the instantaneous price impacts of two market orders x_1 and x_2 of sizes $|\omega_{x_1}|$ and $|\omega_{x_2}|$, with $|\omega_{x_1}| + |\omega_{x_2}| = |\omega_x|$.

Lillo *et al.* (2003) studied the stocks of 1000 different companies traded on the NYSE and sorted them into 20 groups according to their market capitalization (i.e. according to the total value of all of a given company's shares). Within each group, they then merged their data and fitted a single curve to $\Phi^m(|\omega_x|)$. For all 20 groups, they concluded that Φ^m followed a power law $\Phi^m(|\omega_x|) \approx |\omega_x|^\alpha$, with an exponent α that depended on the group and varied between approximately 0.2 and 0.5. However, the authors did not present goodness-of-fit tests with their results, and it is not clear how well the fits performed for individual stocks. After the change of variables

$$\omega_x' := \frac{\omega_x}{C^\eta}, \qquad p' := pC^\gamma, \tag{18}$$

†Grossman and Stiglitz (1980) introduced this idea for a general market, and it has since been discussed extensively in an LOB context (see, e.g. Almgren and Chriss (2001), Bouchaud *et al.* (2009), Hasbrouck (1991), and Potters and Bouchaud (2003)).

‡This explanation suggests that it is not traders' actions that cause the value of an asset to rise or fall. Instead, such changes in valuation happen exogenously and traders align their actions with them to maximize profits. Bouchaud *et al.* (2009) did not find evidence that this was a good reflection of reality.

where C denotes the mean market capitalization for stocks in the group and η and γ are fitted constants, the $\Phi^{m'}\left(\left|\omega_x'\right|\right)$ curves for each of the 20 groups collapsed onto a single curve.

Farmer *et al.* (2005) reported a similar collapse of Φ^m onto a single power-law curve $\Phi^{m'}\left(\left|\omega_x'\right|\right) \approx \left|\omega_x'\right|^{0.25}$ for 11 stocks traded on the LSE after using the change of variables

$$\omega_x' := \frac{\omega_x \alpha}{\mu}, \qquad p' := \frac{p\lambda}{\mu}, \tag{19}$$

where μ, λ, and ν denote the mean arrival rate of market orders, the mean arrival rate of limit orders, and the mean cancellation rate of active orders per unit size σ, respectively.

Using data from the Shenzhen Stock Exchange, Zhou (2012) partitioned incoming orders according to whether or not they received an immediate full matching† of size ω_x at time t_x. The resulting functional form of $\Phi^m(\omega_x)$ was different in the two cases.

- For incoming orders that only partially matched upon arrival, $\Phi^m(|\omega_x|)$ was constant for all $|\omega_x| < 10000$ shares; it then increased for larger values of $|\omega_x|$.
- For incoming orders that fully matched upon arrival, $\Phi^m(|\omega_x|)$ followed the power law $\Phi^m(|\omega_x|) \approx A\,|\omega_x|^\alpha$, where A is a stock-specific constant. Among buy orders, the mean value of α was 0.66 ± 0.05; among sell orders, the mean value of α was 0.69 ± 0.06.

After applying the change of variables

$$\Phi'^m(|\omega_x|) := \frac{\Phi^m(|\omega_x|)}{\langle \Phi^m \rangle}, \qquad \omega_x' := \frac{\omega_x}{\langle |\omega_x| \rangle}, \tag{20}$$

where the angle brackets $\langle\cdot\rangle$ denote the mean value taken across all incoming market orders in the data, Zhou concluded that the $\Phi'^m(\omega_x')$ curves for all stocks that they studied collapsed onto a single curve for incoming orders that were fully matched upon arrival and onto a different single curve for incoming orders that were only partially matched upon arrival. The asymmetry between the bid side and the ask side was no longer present after the rescaling.

In a study of the Paris Bourse and NASDAQ, Potters and Bouchaud (2003) reported that a logarithmic functional form provided a better fit to ϕ^m than did a power-law relationship. Furthermore, Farmer and Lillo (2004) concluded that power-law relationships overestimated the mean instantaneous mid-price impact of very large market orders on both the LSE and the NYSE.

4.6.2. Permanent price impact. As discussed above, it is impossible to quantify exactly the permanent price impact of a market event. However, to gain some insight into the longer term effects of market events, several empirical studies have compared changes in $b(t)$ and $a(t)$ over specified time intervals with measures of *trade imbalance.*

Definition *The* trade imbalance count during time interval $T = [t_1, t_2]$*, denoted* $\Omega^c(T)$*, is the difference between the total number of incoming buy market orders and the total number of incoming sell market orders that arrive during time interval* T.

Definition *The* trade imbalance size during time interval $T = [t_1, t_2]$*, denoted* $\Omega^\omega(T)$*, is the difference between the total absolute size of all incoming buy market orders and the total size of all incoming sell market orders that arrive during time interval* T.

Evans and Lyons (2002) reported a statistically significant, positive, linear relationship between the daily trade imbalance count and the ask-price logarithmic return for successive trading days in FX markets. For German Stock Index futures, Kempf and Korn (1999) reported that the mean mid-price logarithmic return in a 5-min window was a concave function of the trade imbalance count during that window. For the largest 100 stocks on the NYSE, Gabaix *et al.* (2006) reported that the mean mid-price logarithmic return followed the relationship $\Omega^\omega(T)^{0.5}$ for time intervals of length $T = 15$ min. Using a variety of different time interval lengths for the 116 most liquid stocks in the US in 1994–1995, Plerou *et al.* (2002) reported that the mean change in mid-price over the interval was a concave function of $\Omega^\omega(T)$. Furthermore, for small values of $\Omega^\omega(T)$, the mean change in mid-price over the interval was well-approximated by $\Omega^\omega(T)^\alpha$, where the value of α depended on the length of T. The values of α ranged from $\alpha \approx 1/3$ for intervals of length 5 min to $\alpha \approx 1$ for intervals of length 195 min. Similarly, Bouchaud *et al.* (2009) reported that as the length of T increased, the mean mid-price logarithmic return of the AstraZeneca stock on the LSE was approximated more closely by a linear function of the length T.

Cont *et al.* (2011) recently proposed that price impact in LOBs should be studied as a function of the difference between aggregate order flow on the bid and ask sides, rather than of $\Omega^\omega(T)$. They thereby acknowledged that cancellations can also have price impact. Using data for 50 stocks traded on the NYSE, they performed (separately for each stock) an ordinary least-squares regression of the mean change in mid price over a time window of length 10 s onto the order flow imbalance over the same time window. For 43 of the stocks studied, the slope of the regression line was significantly different from 0 at the 95% level and was larger on average for those stocks with smaller mean values of $\left|n^b(b(t), t)\right|$ and $n^a(a(t), t)$. Cont *et al.* (2011) noted that their ordinary least-squares regressions provided a strong fit across all stocks, despite the nuances of how the individual stocks were traded. Regressions using $\Omega^\omega(T)$ rather than order flow imbalance as the independent variable produced significantly worse fits to the data. Cont *et al.* (2011) conjectured that any observable relationship between price impact and $\Omega^\omega(T)$ was actually a byproduct of the correlation between $\Omega^\omega(T)$ and order flow imbalance.

4.6.3. Market impact. In contrast to the wealth of empirical studies on price impact, almost no publications address market impact. To our knowledge, the sole exception is the study of how order arrivals affected the state of the LOBs $\mathcal{L}(t)$ for 30 stocks on Euronext by Hautsch and Huang (2011). Limit orders placed with negative relative price had a significant market impact, and limit orders placed with price $p \leq b(t)$ (respectively, $p \geq a(t)$) caused a significant permanent

†Incoming orders that are fully matched upon arrival always have a strictly smaller instantaneous mid-price impact than orders that are not.

increase in $b(t)$ (respectively, decrease in $a(t)$). On average, the market impact of a market order was about four times greater than that of a limit order of the same size, and limit orders with relative prices of π or 2π affected $b(t)$ and $a(t)$ about 20% less than limit orders placed at $b(t)$ and $a(t)$. Limit orders that arrived with non-negative relative price had no immediate market impact but significant permanent market impact. This impact materialized more quickly for limit orders that arrived at $b(t)$ and $a(t)$ than it did for limit orders that arrived with positive relative prices. By contrast, the market impact of limit orders placed inside of the bid-ask spread was largely instantaneous,† with little permanent impact.

Hautsch and Huang (2011) reported similar results for all stocks studied, but they reported asymmetries between the bid side and the ask side of $\mathcal{L}(t)$, much like Kempf and Korn (1999) reported for price impact. Hautsch and Huang (2011) conjectured that the impact that they observed was due partly to arriving orders triggering an instantaneous imbalance in supply and demand and partly to other traders interpreting order arrivals as containing information, which thereby caused them to adjust their own future actions and led to permanent market impact. This observation provides a possible explanation as to why so many traders choose to place iceberg orders: placing an iceberg order is an effective way to hide the true size of limit orders from the market and thus to minimize market impact. They also reported that traders interpreted the arrivals of market orders as particularly strong information signals.

4.7. Stylized facts

Several nontrivial statistical regularities exist in empirical data from a wide range of different markets. Such regularities are known as the *stylized facts* of markets (Buchanan 2011), and they can provide interesting insights into the behaviour of traders (Cont 2001) and the structure of markets themselves (Bouchaud *et al.* 2009). Stylized facts are also useful from a modelling perspective, because a model's inability to reproduce one or more stylized facts can be used as an indicator for how it needs to be improved or as a reason to rule it out altogether. For example, the existence of volatility clustering eliminates the simple random walk as a model for the temporal evolution of the mid price $m(t)$, as the existence of volatility clustering in real mid-price time series implies that large price variations are more likely to follow large price variations than they are to occur unconditionally (Lo and MacKinlay 2001).

Reproduction of stylized facts remains a serious challenge for LOB models (Chakraborti *et al.* 2011a, Chakraborti *et al.* 2011b, Chen *et al.* 2012). This is particularly true for those based on zero-intelligence assumptions, which have tended to produce more volatile price series than empirical observations suggest is appropriate (Chakraborti *et al.* 2011a). This may imply that the strategic behaviour of real traders somehow stabilizes prices and is therefore an important ingredient in real LOB trading.

Cont (2001) and Chen *et al.* (2012) both reviewed a wide range of stylized facts; we will survey a small subset that we consider to be the most relevant from an LOB perspective. These stylized facts are of particular theoretical interest as they suggest that non-equilibrium behaviour plays an important role in LOBs. A result from statistical mechanics is that systems in equilibrium yield distributions from the exponential family (Mike and Farmer 2008), whereas distributions describing several aspects of LOB behaviour have been reported to exhibit power-law tails, which highlights the possibility that LOBs might always be in a transient state.

†A buy (respectively, sell) limit order placed inside the bid-ask spread necessarily affects $b(t)$ (respectively, $a(t)$) immediately.

4.7.1. Heavy-tailed return distributions. Over all timescales ranging from seconds to days, unconditional distributions of mid-price returns have been reported to display tails that are heavier than a normal distribution (i.e. they have positive excess kurtosis). Understanding heavy tails is central to risk management of investment strategies, because large price movements are more likely than they would be if returns were normally distributed. Heavy tails have been observed on Euronext (Chakraborti *et al.* 2011b), the Paris Bourse (Plerou and Stanley 2008), the S&P 500 index (Gallant *et al.* 1992, Gopikrishnan *et al.* 1999, Cont 2001), FX markets (Guillaume *et al.* 1997), the NYSE (Gopikrishnan *et al.* 1998), the American Stock Exchange (Gopikrishnan *et al.* 1998, Plerou and Stanley 2008), NASDAQ (Gopikrishnan *et al.* 1998), the LSE (Plerou and Stanley 2008), and the Shenzhen Stock Exchange (Gu *et al.* 2008a). However, the exact form of the distribution varied with the timescale used. Across a wide range of different markets (see, e.g. Gopikrishnan *et al.* (1998) and Gu *et al.* (2008a)), the tails of the distribution at the shortest timescales were reported to be well-approximated by a power law with exponent $\alpha \approx 3$, thereby earning the monicker 'the inverse cubic law of returns'. Stanley *et al.* (2008) conjectured that such a universal tail might be a consequence of power-law tails in both the distribution of market order sizes and the instantaneous mid-price logarithmic return impact function. However, Mu and Zhou (2010) reported that this relationship did not hold in emerging markets. Drożdż *et al.* (2007) reported that the tails were less heavy (i.e. $\alpha > 3$) in high-frequency market data for German, American, and Polish indices from 2004 to 2006, highlighting that the quantitative form of stylized facts might themselves have changed over time as trading styles evolved. Several authors have reported that at longer timescales, distributions of returns became increasingly well-approximated by a normal distribution. This behaviour is often called *aggregational Gaussianity* (Gopikrishnan *et al.* 1999, Cont 2001, Zhao 2010).

4.7.2. Volatility clustering. Time series of absolute or square mid-price returns have been reported to display long memory (see Section 3.10.2) over timescales of weeks or even months (Liu *et al.* 1997, Cont 2001, Stanley *et al.* 2008). For example, the square mid-price returns for S&P 500 index futures (Cont 2001), the NYSE (Cont 2005), the USD/JPY currency pair (Cont *et al.* 1997), and crude oil futures (Zhao 2010) have all been reported to exhibit long memory at intra-day timescales, as have absolute mid-price returns on the Paris Bourse (Chakraborti *et al.* 2011b) and the Shenzhen Stock Exchange (Gu and Zhou 2009a). The values of the Hurst exponent H varied from $H \approx 0.8$ on the Paris Bourse and $H \approx 0.815$ for the USD/JPY currency pair to $H \approx 0.58$ on the Shenzhen Stock Exchange. The long memory of absolute or square mid-price returns is often called *volatility clustering* because it indicates that large price changes tend to follow other large price changes. There are several possible explanations for volatility clustering, including the arrival of external news and the strategic splitting of orders by traders (Bouchaud *et al.* 2009).

4.7.3. Long memory in order flow. Using data from the LSE, Lillo and Farmer (2004) reported that the time series $n^b(b(t), t)$ and $n^a(a(t), t)$ exhibited long memory, and Zovko and Farmer (2002) reported that the time series of relative prices of limit orders exhibited long memory with Hurst exponent $H \approx 0.8$. Gu and Zhou (2009a) reported similar long memory in the relative prices of limit orders on the Shenzhen Stock Exchange, with $H \approx 0.78$. The time series constructed by assigning the value $+1$ to incoming buy orders and -1 to incoming sell orders has been reported to exhibit long memory on the Paris Bourse (Bouchaud *et al.* 2004), the NYSE (Lillo and Farmer 2004) and the Shenzhen Stock Exchange (Gu and Zhou 2009a). In studies of the LSE, Bouchaud *et al.* (2009), Lillo and Farmer (2004), and Mike and Farmer (2008) reported that similar results held for market order arrivals, limit order arrivals, and active order cancellations, with statistically significant differences between the estimated values of H for different stocks. However, Axioglou and Skouras (2011) also studied the series of arriving market orders on the LSE and concluded that the apparent long memory reported by Lillo and Farmer (2004) was actually an artifact caused by market participants changing trading strategies once per day.†

4.7.4. Autocorrelation and long memory of returns. Several studies have reported that return series lacked significant autocorrelation, except for weakly negative autocorrelation on very short timescales (Cont 2005, Stanley *et al.* 2008, Chakraborti *et al.* 2011b). This well-established empirical fact has been observed in a very large number of markets, including the NYSE (Cont 2005, Aït-Sahalia *et al.* 2011), Euronext (Chakraborti *et al.* 2011b), FX markets (Cont *et al.* 1997, Bouchaud and Potters 2003), the S&P 500 index (Gopikrishnan *et al.* 1999, Bouchaud and Potters 2003), German interest rates futures contracts (Bouchaud and Potters 2003), and crude oil futures (Zhao 2010). The absence of autocorrelation in returns can be explained using perfect-rationality arguments (Cont 2001). If returns were indeed autocorrelated, rational traders would employ simple strategies that used this fact to generate positive expected earnings. Such actions would themselves reduce the level of autocorrelation, so autocorrelation would not persist.

It appears that the negative autocorrelation present on the shortest timescales disappears more quickly in more recent market data than it does in older data, which indicates that the exact quantitative details of this stylized fact may have changed over time. Using data from the S&P 500 index, Gopikrishnan *et al.* (1999) reported negative autocorrelation in mid-price returns on timescales of up to about 20 min during 1984–1996 but only on timescales of up to 10 min during 1991–2001. During 1991–1995, Bouchaud and Potters (2003) reported that negative autocorrelation persisted up to timescales of 20–30 min for the GBP/USD currency pair and for German interest rate futures contracts, but did not persist for timescales longer than 30 min. On the NYSE, Cont (2005) reported that negative autocorrelation persisted on 5-min timescales but not on 10-min timescales, but did not report an exact date of when the data itself were collected. Using data from Euronext during 2007–2008, Chakraborti *et al.* (2011b) found no significant autocorrelation over time windows of 1 min. Furthermore, using NYSE data from 2010, Cont *et al.* (2011) found no significant autocorrelation over any timescales of 20 s or longer. For crude oil futures contracts traded in 2005, Zhao (2010) reported that negative autocorrelation persisted for only 10–15 s.

The various forms of long memory in order flow (see Section 4.7.3) might be expected to lead to long memory in return series. However, studies of the Hurst exponent for return series on the LSE (Lillo and Farmer 2004), the Paris Bourse (Bouchaud *et al.* 2004), the Deutsche Bourse (Carbone *et al.* 2004), and in FX markets (Gould *et al.* 2013c) have all reported $H \approx 0.5$ (i.e. no long memory) on all but the shortest timescales.‡ Bouchaud *et al.* (2004) conjectured that this was because the long memory in price changes caused by the long memory in the arrival of market orders was negatively correlated to the long memory in price changes caused by the long memory in the arrival and cancellation of limit orders. However, Lillo and Farmer (2004) found no evidence to support this hypothesis using data from the LSE. Instead, they concluded that the long memory in the arrival of market orders was offset by the long memory in $n^b(b(t), t)$ and $n^a(a(t), t)$. When predictability of market order arrivals was high, the probability that a buy (respectively, sell) market order caused a change in $m(t)$ was low, because $\left|n^b(b(t), t)\right|$ and $n^a(a(t), t)$ were large. Therefore, the long memory in the arrival of market orders did not cause long memory in price changes.

5. Modelling LOBs

In recent years, the economics and physics communities have both made substantial progress with LOB modelling (Parlour and Seppi 2008, Chakraborti *et al.* 2011a). However, work by the two communities has remained largely independent (Farmer *et al.* 2005). Economists have tended to be trader-centric, using perfect-rationality frameworks to derive optimal trading strategies given certain market conditions. The LOB models produced by economists have generally treated order flow as static. By contrast, models from physicists have tended to be conceptual toy models of the evolution of $\mathcal{L}(t)$. By relating changes in order flow to properties of $\mathcal{L}(t)$, these models treat order flow as dynamic (Farmer *et al.* 2005). The two

†Stochastic processes that undergo regime switching are known to cause several estimators to report a Hurst exponent $H \neq \frac{1}{2}$ even in the absence of long memory.

‡There is no clear agreement about the long-memory properties of return series at the shortest timescales. This is unsurprising, however, because microstructure effects (which vary greatly from market to market) play a prominent role in the statistical properties of return series at the shortest timescales, and estimation of H is extremely sensitive to such differences in data.

approaches have different strengths: an understanding of trading strategies is crucial for traders and regulators (Seppi 1997, Almgren and Chriss 2001, Sandås 2001, Evans and Lyons 2002, Foucault *et al.* 2005, Goettler *et al.* 2006, Hall and Hautsch 2006, Hollifield *et al.* 2006, Cao *et al.* 2008, Wyart *et al.* 2008, Alfonsi *et al.* 2010, Gatheral 2010, Roşu 2010); an understanding of the state of $\mathcal{L}(t)$ and order flow provides insight into the origins of statistical regularities, including whether they are a consequence of market microstructure or of traders' strategic behaviour (Smith *et al.* 2003, Farmer *et al.* 2005, Mike and Farmer 2008, Bouchaud *et al.* 2009, Gu and Zhou 2009a).

In this section, we assess existing LOB models in terms of their ability to accurately mimic the trading mechanism and to reproduce empirical facts (see Section 4). We also highlight the main modelling difficulties that are yet to be resolved.

5.1. *Perfect-rationality approaches*

In the traditional economics approach, rational investors faced with straightforward buy or sell possibilities choose portfolio strategies of holdings to maximize personal utility, subject to budget constraints (Parlour and Seppi 2008). However, LOBs provide a substantially more complicated scenario. Rather than submitting orders for exact quantities at exact prices, an investor may attempt to construct an ideal portfolio using both limit orders and market orders. The inherent uncertainty of execution of limit orders thereby creates uncertainty about the state of the portfolio at a given time. When deciding whether to submit a given limit order, a trader must estimate its fill probability, which depends endogenously on both $\mathcal{L}(t)$ and future order flow.

5.1.1. Cut-off strategies.

Many early perfect-rationality models aimed to address traders' decision-making via the use of a *cut-off strategy*.

Definition *When choosing between decision D_1 and decision D_2 at time t, an individual employing a* cut-off strategy *compares the value of a statistic $Z(t)$ with a cut-off point z and makes the decision*

$$\begin{array}{ll} D_1, & \text{if } Z \le z, \\ D_2, & \text{otherwise.} \end{array} \tag{21}$$

A cut-off strategy is analogous to a hypothesis test in statistical inference. The statistic $Z(t)$ can be any statistic related to $\mathcal{L}(t)$, current or recent order flow, the actions of other traders, and so on. For example, a trader who wishes to place a buy order at time t might decide to submit a buy market order if $s(t)$ is smaller than 5π or to submit a buy limit order otherwise. Cut-off strategies often appear in perfect-rationality models because they drastically reduce the dimensionality of the decision space available to traders. This is very appealing from the standpoint of tractability.

To our knowledge, the first model that addressed endogenous decision-making between limit orders and market orders in a setting that resembles an LOB was the single-period model of Chakravarty and Holden (1995). First, a market maker arrives and set quotes. All other traders then arrive simultaneously and choose between submitting limit or market orders using a cut-off strategy based on the difference between their private valuations of the asset and the quotes set by the market maker. Finally, all trades occur simultaneously using pro-rata priority.† This model demonstrated that optimal strategies for informed traders can involve submitting either limit orders or market orders, depending on how the market maker acts. In turn, this highlighted endogenous order choice for traders as a crucial feature of a successful LOB model. However, the inclusion of the designated market maker and the assumption that the market operates for only a single time period poorly reflects trading in real LOBs.

Foucault (1999) extended the work of Chakravarty and Holden (1995) by modelling LOB trading as a multi-step game in which traders arrive sequentially. Limit orders remain active for only one period; if the next arriving trader does not submit a market order to match to an existing limit order, then it expires. Upon arrival, each trader chooses between placing a limit order or a market order and then leaves the market forever. After each such departure, the game ends with some fixed probability; otherwise, a new trader arrives and the process repeats. Foucault showed that each trader's optimal strategy in this game is a cut-off strategy based on his/her private valuation of the asset and the price of the existing limit order (if one exists).

Foucault's model highlighted that an active order's probability of matching depends explicitly on future traders' actions (which themselves are endogenous) and that traders must actively consider other traders' strategies. However, Foucault's model contains several assumptions that poorly mimic important aspects of real LOBs—e.g. that limit orders remain active for only a single period and that a random, exogenous stopping time governs trading. These assumptions restrict the model's ability to make realistic predictions about order flow dynamics and how traders estimate order fill probabilities.

Parlour (1998) studied a multi-step game in an LOB that only allows limit orders to be submitted at a single, specific price. Traders arrive sequentially and choose between submitting a limit order at this price or submitting a market order. Unlike in the model proposed by Foucault (1999), limit orders do not expire. Parlour (1998) identified explicit links between traders' strategies and $\mathcal{L}(t)$. In particular, she demonstrated that the optimal decision between submitting a limit order or a market order should be made by employing a cut-off strategy that assesses both sides of $\mathcal{L}(t)$ to estimate the fill probability for a limit order. If the estimated fill probability is sufficiently high, then the trader should submit a limit order; otherwise, he/she should submit a market order. Parlour argued that limit orders become less attractive later in a trading day due to their lower fill probabilities before the end of trading. However, by disallowing cancellations of active orders and by restricting the pricing grid to a single value, Parlour's model is an over simplification of the decision-making process facing traders in real LOBs (Hollifield *et al.* 2006).

Hollifield *et al.* (2004) tested the hypothesis that cut-off strategies such as those discussed above could explain the actions of traders trading the Ericsson stock on the Stockholm Stock Exchange. Working at the 1% level, they accepted their hypothesis for the bid side or the ask side of $\mathcal{L}(t)$ in isolation but

†There is no concept of time priority in a single-period framework.

rejected it when they considered both sides of $\mathcal{L}(t)$ together due to the existence of several limit orders with extremely low fill probabilities and insufficiently high payoffs. Hollifield *et al.* (2004) concluded that cancellations, which are absent from the models discussed above, must play an important role in real LOBs.

Hollifield *et al.* (2006) studied a model in which cancellations are endogenous (i.e. traders can choose when to cancel their orders). By comparing predictions made by the model to data from the Vancouver Stock Exchange, they concluded that real traders do not make decisions using a common cut-off strategy.

5.1.2. Fundamental values and informed traders. Some perfect-rationality models centre around the idea that a subset of traders are *informed traders* who know the 'fundamental' or 'true' value of the traded asset, whereas everyone else is *uninformed* and does not know this true value (see, e.g. Copeland and Galai (1983); Glosten and Milgrom (1985); Glosten (1994); Kyle (1985)). Bouchaud *et al.* (2009) noted that many researchers now reject the idea that assets have fundamental values, but such models can still provide insight into price formation in markets with asymmetric information.

In the classic Kyle (1985) model, uninformed traders place limit orders and market orders in an LOB. At the same time, informed traders observe this LOB and, if an uninformed trader posts a buy limit order with a price above (respectively, sell limit order with a price below) the fundamental value, then an informed trader submits a market order that matches to the mispriced limit order and thereby makes a profit. However, more recent models (Chakravarty and Holden 1995, Roşu 2010) have noted several reasons that informed traders should sometimes choose to submit limit orders rather than market orders—e.g. to avoid detection by other traders who would mimic their actions if they believed that they were informed (Roşu 2010).

Goettler *et al.* (2006) studied a model in which traders arrive at an LOB following a Poisson process. Upon arrival, a trader submits any desired orders, choosing freely among prices. He/she then leaves the market and rearrives following an independent Poisson process. Upon rearrival, a trader can cancel or modify his/her active orders. When a trader performs a trade, he/she leaves the market forever. Additionally, any trader can, at any time, pay a fee to become informed about the fundamental value of the asset. Such traders remain informed until they eventually trade and leave the market. Goettler *et al.* concluded that a trader's willingness to purchase the information should decrease as his/her desire to trade increases. They concluded that speculators, who trade purely for profit, should buy the information most often, that the value of the information increases with volatility, and that the optimal strategy for an informed trader includes submissions of both limit orders and market orders. However, as Parlour and Seppi (2008) discussed, Goettler *et al.*'s step forward in realism comes at the cost of discarding analytical tractability and relying solely on numerical computations.

Roşu (2010) also investigated how informed traders should choose between limit orders and market orders in a model that allows cancellations. He showed that if an informed trader observes a mispricing that is sufficiently beneficial, then he/she should submit a market order to capitalize on the opportunity before anyone else. If the mispricing is below some threshold (but still in the informed trader's favour), then he/she should instead submit a limit order, to gain a better price for the trade if it matches. Roşu also concluded that the price impact of a single informed trader's order submissions are insufficient to reset $b(t)$ and $a(t)$ to their fundamental levels, so subsequent informed traders who arrive at the market with the same information are able to perform similar actions to make a profit. He argued that this is a possible explanation for the empirically observed phenomenon of event clustering (see Section 4.5.3).

Roşu (2009) replaced the idea that traders who select different prices for their orders must do so because of asymmetric information (Glosten and Milgrom 1985, Kyle 1985) with the notion that different traders might select different prices for their orders because of differences in how they value the immediacy of trading. For example, in real markets, some traders need to trade immediately and therefore submit market orders; others do not and can submit limit orders in the hope of eventually trading at a better price. In Roşu's model, traders can modify and cancel their active orders in real time, making it the first perfect-rationality LOB model to reflect the full range of actions that are available in real LOBs. Roşu demonstrated that limit order cancellations simplify the decision-making problem. He proved the existence of a unique Markov-perfect equilibrium in the game and derived the optimal strategy for a newly arriving trader. He also showed that a hump-shaped depth profile emerges in an LOB that is populated by traders who follow such a strategy, in agreement with empirical findings from several different markets (see Section 4.4).

5.1.3. Minimizing market impact. As discussed in Section 4.6, determining how to minimize the market impact of an order is a key consideration for traders. Several perfect-rationality models have suggested that the event clustering found in empirical data (see Section 4.5.3) may be a signature of traders attempting to minimize their market impact when executing large orders (Bouchaud *et al.* 2009). Lillo *et al.* (2005) showed that the power-law decaying autocorrelation function exhibited by order flows in empirical data can be reproduced by a model in which traders who wish to buy or sell large quantities of an asset do so by submitting a collection of smaller orders sequentially through time.

Using a discrete-time framework, Bertsimas and Lo (1998) derived an optimal trading strategy for a trader who seeks to minimize expected trading costs, including those due to market impact, when processing a very large order that has to be completed in the next k time steps. They showed that if prices follow an arithmetic random walk, then the trader should split the original order into k equal blocks and submit the blocks uniformly through time. They also showed that if prices include the effects of exogenous information, then the optimal strategy involves dynamically adjusting trade quantities at each step. Almgren and Chriss (2001) derived a similar strategy for traders who maximize the utility of trading revenues (including a penalty for uncertainty) when executing a large order. However, both assumptions about the

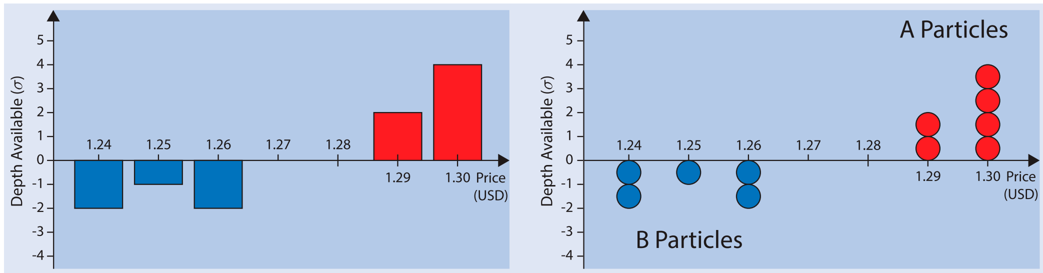

Figure 5. An LOB and its corresponding representation as a system of particles on a one-dimensional pricing lattice.

behaviour of prices in these models poorly mimic the structure of empirically observed price series (Lo and MacKinlay 2001).

Obizhaeva and Wang (2013) considered the above optimal-execution problem in continuous time. In a continuous-time set-up, it is also necessary to choose optimal times, in addition to optimal sizes, at which to submit orders. The authors showed that considering the limit $k \to \infty$ of a k-period, discrete-time model does not provide a valid solution to the problem, as it leads to a degenerate situation in which execution costs are strategy-independent. By making several strong assumptions—including that, after the arrival of a market order, the depth profile undergoes exponential recovery in time† to a neutral uniform state—Obizhaeva and Wang derived explicit optimal execution strategies and concluded that the theoretical optimum requires the submission of uncountably many orders during a finite time period. Alfonsi *et al.* (2010) developed the model further by removing the assumption that the neutral state of the depth profile must be uniform, although they still assumed that recovery to the neutral state is exponential. They showed that in continuous time, the optimal execution strategy involves initially submitting a large market order to stimulate new limit order submissions, then submitting small, equal-sized market orders at a fixed rate, and finally submitting another large market order at the end.

5.2. Zero-intelligence approaches

As noted above, most perfect-rationality models rely on a series of auxiliary assumptions to quantify unobservable parameters. Such assumptions often make it difficult to relate perfect-rationality models to real LOBs. By contrast, zero-intelligence models assume that order arrivals and cancellations are directly governed by stochastic processes. The parameters of such stochastic processes can be estimated directly from historical data, and the statistical properties of the models' outputs can be compared to those of real data. In this way, falsifiable hypotheses can be formulated and tested empirically. Furthermore, the predictive power of models can be measured by training them on a subset of available data in-sample and then evaluating them out-of-sample.

5.2.1. Model framework. Most zero-intelligence LOB models use the framework introduced by Bak *et al.* (1997) to model the evolution of $\mathcal{L}(t)$. Orders are modelled as particles on a one-dimensional lattice whose locations correspond to price. Sell orders are represented as a particle of type A and buy orders are represented as a particle of type B (see figure 5). Each particle corresponds to an order of size σ, so an order of size $k\sigma$ is represented by k separate particles. When two orders of opposite type occupy the same point on the pricing grid, an annihilation $A + B \to \emptyset$ occurs.

5.2.2. Diffusion models. Bak *et al.* (1997) introduced the earliest class of zero-intelligence LOB models involving particles diffusing along a price lattice. Given an initial LOB state with all A particles to the right of all B particles, they modelled the movement of each particle along the price lattice using a random walk. Several authors studied such models analytically and via Monte Carlo simulation (Bak *et al.* 1997, Eliezer and Kogan 1998, Tang and Tian 1999, Chan *et al.* 2001). Such work produced several possible explanations for empirical regularities observed in real LOB data, such as the hump-shaped depth profile (see Section 4.4). However, the Bak *et al.* (1997) model has since been rejected because the diffusion of active orders across different prices is not observed in empirical data (Challet and Stinchcombe 2001, Farmer *et al.* 2005, Chakraborti *et al.* 2011a). Nonetheless, these models sparked the idea that empirical regularities in LOB data that were previously thought to be a direct consequence of traders' strategic actions could be reproduced in a zero-intelligence framework. This has subsequently become a central theme of zero-intelligence models throughout the literature (see, e.g. Bouchaud *et al.* (2009), Farmer *et al.* (2005), Farmer and Foley (2009), Smith *et al.* (2003)).

5.2.3. Discrete-time models. Maslov (2000) introduced a model that bears a stronger resemblance to real LOBs than the price diffusion models discussed above. In Maslov's model, a single trader arrives at each discrete time step. With probability 1/2, this trader is a buyer; otherwise, he/she is a seller. Independently, with probability $1 - r$, the trader submits a market order; otherwise, he/she submits a limit order $x = (p_x, \sigma, t_x)$ with

$$p_x = \begin{cases} p' - K, & \text{if the trader is a buyer,} \\ p' + K, & \text{if the trader is a seller,} \end{cases} \tag{22}$$

where p' is the most recent price at which a matching occurred and K is a random variable with a specified distribution. The model disallows cancellations and modifications

†Discussion about such recovery of the depth profile, often known as its *resiliency*, has appeared in both the empirical (Biais *et al.* 1995, Potters and Bouchaud 2003, Bouchaud *et al.* 2004) and modelling literatures (Foucault *et al.* 2005, Roşu 2009).

to active orders. Even with only 1000 iterations and in very simple set-ups (such as $r = 1/2$ and $K = 1$ with probability 1; or $r = 1/2$ and $K \sim \text{Uniform}\,\{1, 2, 3, 4\}$), the return series generated by the model exhibits heavy tails and negative autocorrelation at low lags on event-by-event timescales. Slanina (2001) implemented a mean-field approximation to replace the tracking of prices of individual limit orders with a mean value that increases when a limit order arrives and decreases when a market order arrives. Under this approximation, the return distribution is still heavy-tailed and the autocorrelation is still negative at low lags. However, this model generates mid-price returns with a Hurst exponent of $H \approx 0.25$ on all timescales. By contrast, as discussed in Section 4.7, LOB data exhibits no long memory (i.e. $H \approx 0.5$) in mid-price returns on all but the shortest timescales (Lillo and Farmer 2004).

Challet and Stinchcombe (2001) refined Maslov's model by allowing multiple particles to be deposited on the pricing grid during a single time step. They also allowed existing particles to evaporate, corresponding to the cancellation of an active order, although they assumed that such evaporations occur exogenously and independently for each particle. Challet and Stinchcombe's model exhibits a heavy-tailed return distribution and volatility clustering, and the Hurst exponent of the mid-price return series at large timescales is $H \approx 0.5$. The authors conjectured that the evaporations in their model (which are absent in the model of Maslov (2000)) ensure that the Hurst exponent at large timescales matches that of empirical data.

5.2.4. Continuous-time models. The first zero-intelligence model in continuous time was introduced by Daniels *et al.* (2003), who produced a master equation for $\mathcal{L}(t)$ under the assumptions that market order arrivals, limit order arrivals, and cancellations are all governed by independent Poisson processes, and that incoming limit orders arrive at the same rate at each relative price in the semi-infinite interval $(-s(t), \infty)$. Smith *et al.* (2003) solved the master equation in the limit of infinitesimal tick size $\pi \to 0$ using a mean-field approximation that the depths available at neighbouring prices are independent. Guided by dimensional analysis, they constructed simple, closed-form estimators for a variety of LOB properties—such as the mean spread, mean depth available at a given price, and mid-price diffusion—in terms of only the lot size σ and the Poisson processes' arrival rates. Their Monte Carlo simulations produced similar results. Their model also provides possible explanations for why some empirical properties of LOBs vary between different markets (see Section 4). In particular, the lot size σ appears explicitly in many of their closed-form estimators, and there are phase transitions between different types of market behaviour as σ is varied.

Many of the assumptions made by Daniels *et al.* (2003) and Smith *et al.* (2003) to maintain analytical tractability provide poor resemblance to some aspects of real LOBs. For example, in the limit $\pi \to 0$, the only possible numbers of limit orders that can reside at a given price p are 0 and 1. This destroys the notion of limit orders queueing at given prices and thereby removes a primary consideration for traders: when to submit an order at the back of an existing priority queue versus when to start a new queue at a worse price (see Section 3.7). Despite the simplifications in the above model, Farmer *et al.* (2005) reported that it performed well when tested against some aspects of empirical data. In particular, they made predictions of the mean spread and a measure of price diffusion† for 11 stocks traded on the LSE by calibrating the model's parameters using historical data and then compared these predictions to the real data using an ordinary least-squares regression:

$$Z_{\text{emp}}(i) = z Z_{\text{mod}}(i) + c, \tag{23}$$

where $Z_{\text{emp}}(i)$ and $Z_{\text{mod}}(i)$ are the mean empirical and model output values of statistic Z for stock i. Using this set-up, $z = 1$ and $c = 0$ correspond to a perfect fit of the model to the data. For the mean spread, the ordinary least-squares estimates of the parameters were $z \approx 0.99 \pm 0.10$ and $c \approx 0.06 \pm 0.29$. For the price diffusion, the ordinary least-squares estimates of the parameters were $z \approx 1.33 \pm 0.25$ and $c \approx 2.43 \pm 1.75$. Farmer *et al.* (2005) used bootstrap resampling to estimate the standard errors of the regression coefficients, because serial correlations within the data invalidate the assumptions required to use the standard estimators (see Section 3.10.2). However, the distribution of mid-price returns generated by the model does not exhibit heavy tails, and Tóth *et al.* (2011) reported that time series of logarithmic mid-price returns generated by the model have a Hurst exponent of $H < \frac{1}{2}$ when the model's parameters are chosen to mimic realistic market conditions. Both of these facts are contrary to findings in empirical data (see Section 4.7).

Cont *et al.* (2010) recently introduced a variant of the Daniels *et al.* (2003) and Smith *et al.* (2003) model to understand how the occurrence frequency of certain events is conditional on $\mathcal{L}(t)$. The model does not assume that $\pi \to 0$ and thereby ensures that priority queues form at discrete points on the price lattice. Cont *et al.* (2010) also removed the assumption of Daniels *et al.* (2003) and Smith *et al.* (2003) that the relative prices of limit orders are drawn from a uniform distribution, and replaced it with a power–law distribution to fit observations from empirical data more closely (Bouchaud *et al.* 2002, Zovko and Farmer 2002, Potters and Bouchaud 2003, Cont *et al.* 2010). Simulations of the Cont *et al.* (2010) model exhibit the hump-shaped depth profile that is commonly reported in empirical data (see Section 4). Using Laplace transforms, the authors computed conditional probability distributions for the matching of limit orders in particular situations.

Zhao (2010) and Toke (2011) recently extended the Cont *et al.* (2010) model by revising the assumed arrival structure of market events. Based on an empirical study of crude oil futures traded at the International Petroleum Exchange, Zhao (2010) rejected the assumption that the inter-arrival times of market events are independent draws from an exponential distribution and thereby rejected the use of independent Poisson processes to model market event arrivals. Zhao replaced the independent Poisson processes with a Hawkes process‡

†Farmer *et al.* (2005) studied price diffusion by calculating the variance v_τ of the set $\{(m(t_i + \tau) - m(t_i)) \mid i = 1, \ldots, k\}$ for various values of τ, where $\{t_i \mid i = 1, \ldots, k\}$ is the set of times at which the mid-price changed. They then performed an ordinary least-squares regression to estimate d in the expression $v_\tau = d\tau$.

‡A *Hawkes process* is a point process with time-varying intensity parameter $\lambda(t) = \lambda_0(t) + \sum_{t_i < t} \sum_j C_j e^{-D_j(t-t_i)}$, where t_i denotes the time of the i^{th} previous arrival and C_j and D_j are parameters that control the intensity of arrivals.

(Bauwens and Hautsch 2009) that describes the arrival rate of all market events as a function of recent order arrival rates and the number of recent order arrivals. When an arrival occurs, its type (e.g. market order arrival, limit order cancellation, etc.) is determined exogenously. This produces order flows in which periods of high arrival rates cluster in time and in which periods of low arrival rates cluster in time. This agrees with empirical data (Ellul *et al.* 2003, Hall and Hautsch 2006). Zhao demonstrated that this improves the fit of the model output to the empirically observed mean relative depth profile. Toke (2011) similarly replaced the Poisson processes in the Cont *et al.* (2010) model with Hawkes processes. Unlike Zhao, however, Toke used multiple mutually exciting Hawkes processes (one for each type of market event). By studying empirical data from several different asset classes, Toke observed that when a market order arrived, the mean time until the next limit order arrival was less than the corresponding unconditional mean time. Their simulated order flow and spread dynamics matched their empirical observations more closely than those produced by a Poisson-process model.

Cont and de Larrard (2011) recently introduced a model that tracks only $n^b(b(t), t)$ and $n^a(a(t), t)$ rather than the whole depth profile. When either becomes zero, the model assumes that the depth available at the next best price is a random variable drawn from a distribution f. The state space of this model is $\mathbb{N}^2$ rather than $\mathbb{Z}^P$ (which is used in most other recent LOB models). The authors' justification for such a simplified set-up was that many traders can only view the depths available at the best prices and not the entire depth profile (although this is becoming increasingly less common as electronic trading platforms deliver ever more up-to-date information in real time (Boehmer *et al.* 2005, Bortoli *et al.* 2006)). Independent Poisson processes govern market order arrivals, limit order arrivals, and limit order cancellations. Using only the Poisson processes' rate parameters and the distribution f, the authors derived analytical estimates for several market properties—including volatility, the distribution of time until the next change in $m(t)$, the distribution and autocorrelation of price changes, and the conditional probability that $m(t)$ moves in a specified direction given $n^b(b(t), t)$ and $n^a(a(t), t)$. Different levels of autocorrelation of the mid-price series emerge at different sampling frequencies, in agreement with empirical observations (Zhou 1996, Cont 2001).

5.2.5. Beyond zero intelligence.

Tóth *et al.* (2011) extended the Daniels *et al.* (2003) and Smith *et al.* (2003) model by using a long-memory process to determine whether arriving orders are buy or sell orders. They also introduced an extra parameter to relate the size of arriving buy (respectively, sell) market orders to $n^a(a(t), t)$ (respectively, $n^b(b(t), t)$). This extra parameter makes it possible to control the strength of long memory in the logarithmic mid-price return series generated by the model, thereby addressing an issue with the original model.

Based on an empirical study of data from the LSE, Mike and Farmer (2008) incorporated the empirically observed long memory of order flow (see Section 4.7.3) into their model of the evolution of $\mathcal{L}(t)$. They used a Student's t distribution to model the relative prices of incoming orders, and they closely matched cancellation rates for active orders to empirical data. For stocks with small tick size and low volatility, they found that their model exhibits negative autocorrelation of logarithmic mid-price returns on short timescales, in agreement with empirical data. Furthermore, they found that it makes good predictions of the distribution of mid-price returns (including heavy tails) and the distribution of $s(t)$. However, it is less successful for stocks other than those with both small tick size and low volatility.

Gu and Zhou (2009a) simulated the Mike and Farmer (2008) model and performed a DFA*m* (see Section 3.10.2) on the output mid-price return and volatility series. They found that neither the mid-price return series nor the volatility series exhibits long memory. The former agrees with empirical data, whereas the latter disagrees with the widely observed stylized fact of volatility clustering (see Section 4.7). Gu and Zhou then proposed an extension to the model in which the relative prices of orders are not drawn independently, but instead are simulated with an imposed long memory. This modification causes long memory to emerge in the volatility series and preserves all of the model's other results.

Gu and Zhou (2009b) replaced several of the stochastic processes governing order flow in the Mike and Farmer (2008) model with other distributions to examine how this affects the output. They concluded that a power-law tail in the mid-price return distribution only appears in the model's output when the distribution from which positive relative prices are drawn has heavy tails, irrespective of whether the distribution from which negative relative prices are drawn has heavy tails.

Although Tóth *et al.* (2011) and Mike and Farmer (2008) did not directly assume that traders are rational, the conditional structure of random variables in their models can be construed as consequences of rational decision-making. For example, the dependence of market order sizes on $n^a(a(t), t)$ and $n^b(b(t), t)$ in the Tóth *et al.* (2011) model can be interpreted as traders attempting to minimize their market impact, and the lower rate of cancellation among active orders with larger relative prices in the Mike and Farmer (2008) model can be construed to reflect how traders would not submit such orders unless they were willing to wait for them to be matched in the future.

5.3. Agent-based models

An *agent-based model (ABM)* is a model in which a large number of possibly heterogeneous agents interact in a specified way (Gilbert 2007). A key advantage of ABMs is the ability to incorporate heterogeneity between different traders (Buchanan 2008, Chakraborti *et al.* 2011a). Such models can provide insight into both the performance of individual agents and the aggregate effect of all agents' interactions. By allowing each individual agent's behaviour to be specified without any explicit requirements regarding rationality, ABMs lie between the two extremes of zero-intelligence and perfect-rationality models. However, ABMs of LOBs also have significant drawbacks. Due to the large number of interacting components in an LOB, it is difficult to track explicitly how a specified input parameter affects the output of an ABM. It is also very difficult to encode a quantitative set of rules to describe traders' complex and interacting strategies, and finding a set of agent rules that produces a specific behaviour from an ABM provides no guarantee that such a set of rules is the only one to do so (Preis *et al.* 2007). Abergel and Jedidi (2011) attempted to address these issues by studying systems of stochastic differential equations that describe price dynamics in terms of some ABMs' input parameters, thereby deriving exact links between the two approaches. For example, they demonstrated that a very simple ABM can result in Gaussian process dynamics, with a diffusion coefficient that depends on the model's input parameters.

Early ABMs of LOBs assumed that agents arrive sequentially (Foucault 1999) and that LOBs empty at the end of each time step. Such set-ups fail to acknowledge an LOB's key function of storing supply and demand for later consumption by other traders (Smith *et al.* 2003). However, more recent ABMs have more closely mimicked real LOBs and have successfully reproduced a wide range of empirical features present in empirical data (Cont and Bouchaud 2000, Chiarella and Iori 2002, Challet and Stinchcombe 2003, Preis *et al.* 2006).

Cont and Bouchaud (2000) showed that when agents in a simple market imitate each other, the resulting output exhibits a heavy-tailed return distribution, clustered volatility, and aggregational Gaussianity (see Section 4.7).

Chiarella and Iori (2002) studied an ABM in which all agents share a common valuation for the asset traded in a given LOB. They noted that the realized volatility produced by their model is too low compared to empirical data and that there is no volatility clustering. They thereby argued that substantial heterogeneity must exist between traders in real LOBs for the highly nontrivial properties of volatility to emerge (see Section 3.6). Cont (2005) noted that differences in agents' levels of impatience can be a source of such heterogeneity in real markets.

Preis *et al.* (2006) reproduced the main findings of Smith *et al.* (2003) using an ABM rather than independent Poisson processes. By fine-tuning agents' trading strategies, their model reproduces the heavy-tailed distribution of mid-price returns, the diffusivity of mid-price returns over long timescales, and the negative autocorrelation of $m(t)$ on an event-by-event timescale, Preis *et al.* (2007) studied the performance of individual agents in the model. They found that the Hurst exponent H of the mid-price return series depends on the number of agents in the model, and that the best fit of H against values calculated from empirical data occurred with 150 to 500 liquidity-provider (i.e. limit order placing) agents and 150 to 500 liquidity-taker (i.e. market order placing) agents.

Challet and Stinchcombe (2003) studied how allowing the parameters of a simple ABM of an LOB to vary in time affects traded price series. They concluded that such time-dependence results in the emergence of a heavy-tailed distribution of mid-price changes, autocorrelated mid-price returns, and volatility clustering. They noted that many LOB models assume that parameter values remain constant in time, and they conjectured that several stylized facts (see Section 4.7) might be caused by real traders changing their actions over time.

Lillo (2007) showed how an ABM can explain the empirically observed power-law distribution of relative prices of incoming orders (see Section 4.2). In particular, he solved a utility maximization problem to show that if mid-price movements are assumed to follow a Brownian motion, then each perfectly rational agent should choose the relative price of his/her submitted orders to be

$$\delta^{x*} = \sqrt{2T}\, g^{-1}(\alpha) V, \tag{24}$$

where $g(\alpha)$ describes the agent's risk aversion, T is the agent's maximum time horizon (i.e. the maximum length of time that the agent is willing to wait before performing the trade), and V is the market volatility. He then studied how empirically observed homogeneity in g and T and fluctuations in V affect the price choices of interacting agents with different risk aversions g and different maximum time horizons T. He concluded that heterogeneity in T is the most likely source of the power-law tails in the distribution of δ^{x*} and that the homogeneity in g and fluctuations in V that have been observed empirically in a wide range of markets are unlikely to lead to a power-law tail in the distribution of δ^{x*}.

6. Key unresolved problems

In this section, we discuss key unresolved problems currently facing researchers of LOBs.

- **Understanding statistical regularities:** As discussed in Section 4, several empirical regularities appear in data from a wide range of different markets. Some such statistical regularities describe features of order flow or LOB state; others describe stylized facts relevant to price formation and market stability. Many authors (see, e.g. Gu and Zhou (2009a), Lillo (2007), Stanley *et al.* (2008) and Tóth *et al.* (2011)) agree that one of the main challenges facing researchers of LOBs is to gain a better understanding the origins of these statistical regularities. LOB models can help to achieve this, and some progress has been made. However, no single model has yet been capable of simultaneously reproducing all of the statistical regularities, and there is no clear picture about how the stylized facts emerge as a consequence of the actions of many heterogeneous traders.
- **Understanding recent data:** A great deal of effort has been invested in empirical study of LOB data. Figure 6 shows the approximate number of days' data per year that studies discussed in this article have examined. Although the breadth of such empirical work is substantial, the overwhelming picture painted by figure 6 is that the data studied is old. It is also often of poor quality, so extensive auxiliary assumptions are required before any statistical analysis can even begin. Strong assertions have been made by empirical studies based on single stocks over very short time periods. Many LOB models are built upon statistical regularities observed in old data, but traders' strategies and the rules governing trade change over time, so empirical observations from more than a decade ago may not accurately describe current LOB activity. However, recent advances in computational and storage capabilities have made it feasible to record data detailing all order flows at all prices, and tools have been developed to assist researchers with reconstructing the full LOB in certain markets (Huang and Polak 2011). By studying recent, high-quality data, researchers will be able to assess whether the existing foundations for LOB modelling accurately reflect today's markets.

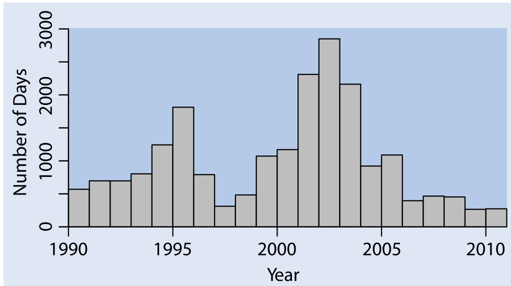

Figure 6. Approximate total number of days' data per year that has been examined by empirical studies discussed in this article.

- **Non-stationary behaviour:** Although precisely what is meant by 'equilibrium' depends upon context, almost all LOB models to date have focused on some form of equilibrium, such as a Markov-perfect equilibrium in sequential-game models or a state-space equilibrium in reaction-diffusion models. However, empirical evidence strongly suggests that LOBs are subject to frequent shocks in order flow that cause them to display non-stationary behaviour, so they may never settle into equilibrium (Buchanan 2009). Preliminary work on non-equilibrium models has hinted at promising results, such as quantitative replication of some of the stylized facts (Challet and Stinchcombe 2003), but there is very little progress in this direction.
- **Volatility:** Price changes and volatility are among the most hotly debated topics in the literature (Hasbrouck 1991, Almgren and Chriss 2001, Potters and Bouchaud 2003, Bouchaud *et al.* 2009, Tóth *et al.* 2011). How can estimates of volatility be designed to incorporate information about the entire state of $\mathcal{L}(t)$? What causes volatility to vary over time? Why should periods of high activity cluster together? Why should price fluctuations be so frequent and so large on intra-day timescales, given that external news events occur so rarely (Maslov 2000)? It is not even agreed whether the number of market orders (Jones *et al.* 1994), the size of market orders (Gallant *et al.* 1992), or the fluctuation of liquidity (Bouchaud *et al.* 2009) plays the dominant role in determining volatility. It seems likely that the answers to such questions will not be found in isolation, but rather that there is an intricate interplay between the many pieces of the volatility puzzle. Recent work has attempted to tie together some of these ideas. For example, Bouchaud *et al.* (2009) and Wyart *et al.* (2008) conjectured that volatility might be understood better by considering the need for traders to minimize their market impact.
- **Algorithmic trading:** Electronic trading algorithms are able to process vast quantities of LOB data to interpret market conditions and submit or cancel orders in a small fraction of the time that it would take a human to perform the same task. The use of electronic trading algorithms has increased rapidly in recent years, but empirical research in this area is extremely difficult due to a lack of data in which algorithmic trades are identified (Chaboud *et al.* 2011). To date, the published literature on algorithmic trading consists of only a handful of empirical studies and models, yet there is fierce debate about whether such algorithms are beneficial or detrimental to markets. Different studies have drawn contradictory conclusions. Chaboud *et al.* (2011) and Hendershott *et al.* (2011) reported that electronic trading algorithms narrow spreads, reduce adverse selection, speed up price discovery, increase liquidity, and improve the informativeness of $b(t)$ and $a(t)$. However, Biais *et al.* (2011) and Kirilenko *et al.* (2011) reported that electronic trading algorithms increase adverse selection, create an unfair advantage for wealthier traders, decrease liquidity, and exacerbate volatility during stressed market scenarios. From a regulatory standpoint, it is crucial to understand how electronic trading algorithms affect market stability, yet almost nothing concrete is currently known.
- **Liquidity fragmentation:** In recent years, it has become increasingly common for assets to be traded on several different electronic trading platforms simultaneously (Bennett and Wei 2006). The resulting competition between exchanges has stimulated technological innovation and driven down the fees incurred by traders, but it has also caused a fragmentation of liquidity because limit orders for a given asset are now spread between several different LOBs. This poses a problem for empirical research, as the study of any individual LOB in isolation no longer provides a snapshot of the whole market for an asset. Furthermore, differences between different trading platforms' matching rules and transaction costs complicate comparisons between different LOBs for the same asset. Cont *et al.* (2011) reported similarities between different LOBs that traded the same asset simultaneously, but there is no reason that this must hold in general. The development of robust methods for assimilating data across multiple platforms will be of paramount importance to understand the implications of liquidity fragmentation on market stability and price formation.

7. Conclusion

The literature on LOBs has grown rapidly, and both empirical and theoretical work has deepened understanding of the LOB trading process. LOBs are a rich and exciting testing ground for theories, and have provided new insight into long-standing economic questions regarding market efficiency, price formation, and the rationality of traders. However, despite the progress made on specific aspects of limit order trading, it remains unclear how the various pieces of the puzzle fit together. For example, models that capture the dynamics of event-by-event price changes poorly reproduce price dynamics on longer timescales. Similarly, models that explain price dynamics on inter-day timescales offer little understanding of how they emerge as the aggregate effect of individual trades.

There are substantial challenges associated with studying historical LOB data, and several empirical studies contain systematic errors in their calculations. Moreover, performing quantitative comparisons between different empirical studies is very difficult for two reasons. First, it is unclear whether differences in the findings of such studies are caused by differences in different markets, or are simply a result of differences in methodology. Sampling frequency, choice of asset class, LOB resolution parameters, specific trade-matching nuances, and many other factors all influence empirical findings, but so too do the choice of statistical estimators and the details of their implementation. This makes it difficult to assess the influence of specific LOB factors on trade. Second, LOB platforms, LOB rules, and trading strategies have all changed over time, so the date range over which data was collected may itself play a role in the values of the statistics reported. This issue is particularly important given the recent surge in popularity of electronic trading algorithms. Studies of recent, high-quality LOB data that are conducted with stringent awareness of potential statistical pitfalls are needed to understand better the LOBs of today.

It is clear from empirical studies how poorly the data supports the very strong assumptions made by many LOB models. Although every model must make assumptions to facilitate computation, many LOB models depend on elaborate and inaccurate assumptions that make it almost impossible to relate their output to real markets. ABMs appear to offer some compromise between the extremes of zero-intelligence and perfect-rationality models; they also provide an explicit way to remove the inherent homogeneity associated with many existing approaches (Lux and Westerhoff 2009, Zhao 2010, Toke 2011). Furthermore, the level of game-theoretic considerations involved in agents' decision-making can be controlled by specifying how strongly agents react to each other and forecast each other's actions. Therefore, ABMs have the potential to provide a rich toolbox for investigating LOBs without requiring extreme modelling assumptions. However, it remains unclear whether the ABMs studied to date offer a deeper understanding of market dynamics or merely amount to curve-fitting exercises in which parameters are varied until some form of nontrivial behaviour emerges. Recently, statistical techniques from econometrics have enabled consistent estimation of ABMs' parameters from market data (Chen *et al.* 2012). It will be interesting to see whether the use of such techniques in an LOB context paves the way for new, quantitative explanations of LOB phenomena.

Price impact and market impact also continue to be active areas of research. A deeper understanding of these notions is very desirable, as they form a conceptual bridge between the microeconomic mechanics of order matchings and the macroeconomic concepts of price formation. Considerations about price impact and market impact could also help to explain the actions of traders in certain situations. However, despite the striking regularities that have been observed in empirical studies, little is understood about why price impact functions have the forms that they do and almost nothing is understood about market impact.

LOBs have revolutionized trading by providing traders the freedom to evaluate their own need for immediate liquidity. Their study has hitherto been hampered by their inherent complexity, with all the associated technical difficulties, and above all by wholly inadequate empirical data. However, our growth in understanding allied to massive improvements in data and in computational power suggest that answers to many important open questions will not be long in coming.

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Appendix A: Table Empirical Studies

Reference	Assets studied	Date range	Data type	Main points studied
Aït-Sahalia *et al.* (2011)	The 30 Dow Jones Industrial Average stocks	19–23 and 26–30 Apr 2004	$b(t)$, $a(t)$, $n^b(b(t), t)$, $n^a(a(t), t)$, and all market orders	Volatility and long-range dependence in order flows
Anand *et al.* (2005)	144 stocks traded on the NYSE	Nov 1990–Jan 1991	All order flows at all prices	Decision between using limit orders or market orders for informed traders
Bandi and Russell (2006)	All stocks in the S&P 100 index	Feb 2002	$b(t)$, $a(t)$, $n^b(b(t), t)$, $n^a(a(t), t)$, and all market orders	Volatility
Bennett and Wei (2006)	39 stocks that voluntarily switched their listing from NASDAQ to the NYSE	Jan 2002–Mar 2003	Total size of arriving market orders, daily returns, $s(t)$, and several summary statistics describing individual assets	How liquidity fragmentation affects markets
Biais *et al.* (1995)	The CAC 40, traded on the Paris Bourse	6 trading days in Jun/Jul 1991 and 19 trading days in Oct/Nov 1991	First 5 levels of bid-side relative depth profile and ask-side relative depth profile (updated every time the depth available at one of the displayed levels changed)	Returns, percentage of market orders that matched to hidden liquidity, mean total depth available, and $s(t)$ (both unconditionally and dependent on time of day), order flow (both unconditionally and dependent on recent order flow and time of day) and state of $\mathcal{L}(t)$
Biais *et al.* (1999)	The CAC 40, traded on the Paris Bourse	19 trading days in Oct/Nov 1991, 26 trading days in 1993, and 234 trading days in 1995	$b(t)$ and $a(t)$ (sampled once per minute)	Whether the evolution of the price process indicates that traders learn during the daily opening auction
Boehmer *et al.* (2005)	400 stocks traded on the NYSE	7–18 Jan, 4–15 Feb, 4–15 Mar, 1–12 Apr, 6–17 May (all in 2002)	All order flows at all prices in the electronic LOB and information about the handling of both electronic and manual (broker-handled) orders	How the introduction of an electronic LOB on the NYSE affected traders' behaviour
Bortoli *et al.* (2006)	The 4 most actively traded futures contracts on the Sydney Futures Exchange	15 Sep 2000–19 Jun 2001	Every matching, change in $b(t)$ or $a(t)$, and change in depth available at the best prices (respectively, best three prices) prior to (respectively, after) the change in disseminated market information (timestamped to the nearest second)	Whether order flow and $\mathcal{L}(t)$ changed when the Sydney Futures Exchange increased the real-time information disseminated to traders
Bouchaud *et al.* (2002)	France Telecom, Vivendi, and Total stocks (traded on the Paris Bourse)	Feb 2001	All order arrivals at all prices along with their time of arrival and a list of all orders that were cancelled (but not the time at which they were cancelled)	Mean depths available, distribution of relative prices, ω_x, $n^b(b(t), t)$, and $n^a(a(t), t)$
Bouchaud *et al.* (2004)	France Telecom Stock, traded on the Paris Bourse (with similar results reported for other unnamed liquid French and British stocks)	Jan 2001–Dec 2002	$b(t)$ and $a(t)$ (recorded once every time either of them changed) and all market orders (timestamped to the nearest second)	How order flow affects prices
Cao *et al.* (2008)	100 largest stocks traded on the Australian Stock Exchange	Mar 2000	All order arrivals and cancellations at all prices (timestamped to the nearest 0.01 s)	How the state of $\mathcal{L}(t)$ affects order flow
Chaboud *et al.* (2011)	EUR/USD, USD/JPY, and EUR/JPY currency pairs on EBS	2003–2007	$b(t)$ and $a(t)$ (sampled once per second) and total size of arriving market orders (sampled once per minute)	How electronic trading algorithms affect markets
Chakraborti *et al.* (2011b)	Four stocks traded on the Paris Bourse	All trading days, 1 Oct 2007–30 May 2008	All market orders and the five highest-priority active orders on each side of the LOB	Whether the traditional stylized facts are present in the data

(*continued*)

Appendix A. Continued.

Challet and Stinchcombe (2001)	Four stocks traded on the Island ECN (on NASDAQ)	Not specified	15 highest-priority active orders on each side of the LOB (updated every time the list changed)	Order flow rates, autocorrelation of order flow rates, diffusion of active orders (i.e. cancellation of an active order immediately followed by resubmission at a neighbouring price), instantaneous price impact, distribution of ω_x, lifetime of limit orders, and δ^x for incoming orders
Cont *et al.* (2010)	Sky Perfect Communications stock, traded on the Tokyo Stock Exchange	Not specified	$N^b(p,t)$ and $N^a(p,t)$ for the five smallest relative prices with non-zero depth available (updated whenever either changed) and all market orders	Arrival rates of market orders and arrival and cancellation rates of limit orders
Cont *et al.* (2011)	50 stocks from the S&P 500, traded on the NYSE	All 21 trading days in Apr 2010	$n^b(b(t),t)$ and $n^a(a(t),t)$ (updated whenever either changed and timestamped to the nearest second) and all market orders	Relationship between order flow imbalance and price impact
Dufour and Engle (2000)	18 of the most frequently traded stocks on the NYSE	62 trading days, 1 Nov 1990–31 Jan 1991	$b(t)$ and $a(t)$(updated whenever either changed) and all market orders	Relationship between market order inter-arrival times and price impact
Eisler *et al.* (2012)	14 stocks traded on NASDAQ	3 Mar 2008–19 May 2008	$b(t)$, $a(t)$, $n^b(b(t),t)$, and $n^a(a(t),t)$(updated whenever any of them changed)	Price impact of market order submissions, and limit order submissions and cancellations
Ellul *et al.* (2003)	The 50 most actively traded stocks and 98 other stocks on the NYSE	30 Apr 2001–5 May 2001	All market order submissions and all limit order submissions and cancellations (timestamped to the nearest second)	Which factors traders assess when choosing the price of an order
Engle and Patton (2004)	100 stocks traded on the NYSE	18 months of data, no date range specified	$b(t)$ and $a(t)$ (updated whenever either changed) and all market orders	$s(t)$ and how price impact varies according to how frequently trades occur for a specific stock
Farmer and Lillo (2004)	3 stocks traded on the LSE and 3 stocks traded on the NYSE	May 2000–Dec 2002 for the LSE stocks and 1995–1996 for the NYSE stocks	All order flows for the LSE; $b(t)$ and $a(t)$ (updated whenever either changed) and all market orders for the NYSE	Price impact of individual market orders and distribution of ω_x for market orders
Farmer *et al.* (2005)	11 stocks traded on the LSE	1 Aug 1998–30 Apr 2000	All market order submissions and all limit order submissions and cancellations	Goodness-of-fit of the predictions regarding mean spread and price diffusion of the Smith *et al.* (2003) model and mean instantaneous mid-price logarithmic return impact as a function of market order size
Field and Large (2008)	Short Sterling, Euribor, EUR/USD, and 2-Year US Treasury Note futures	23 Nov–11 Dec 2006 and 16–20 Apr 2007	$b(t)$, $a(t)$, $n^b(b(t),t)$, and $n^a(a(t),t)$ (updated whenever any of them changed)	Order flow rates and $n^b(b(t),t)$ and $n^a(a(t),t)$ in markets in which $s(t)=\pi$
Gode and Sunder (1993)	Laboratory experiment with human beings and computerized zero-intelligence traders	N/A	All order flows at all prices	Relative applicability of perfect-rationality and zero-intelligence assumptions, and emergence of seemingly rational behaviour when aggregating across irrational individuals
Gopikrishnan *et al.* (2000)	1000 largest stocks traded in the US	1994–1995	$b(t)$, $a(t)$, and all market orders	Price impact as a function of trade imbalance count and trade imbalance size, and distribution and autocorrelation of trade imbalance count and trade imbalance size
Gu *et al.* (2008a)	23 stocks traded on the Shenzhen Stock Exchange	All of 2003	All order flows at all prices	Distribution of mid-price returns on various τ second timescales and various event-by-event timescales
Gu *et al.* (2008b)	23 stocks traded on the Shenzhen Stock Exchange	All of 2003	All order flows at all prices	Distribution of relative prices of incoming orders and whether this is conditional on $s(t)$ or volatility

(*continued*)

Appendix A. Continued.

Gu *et al.* (2008c)	23 stocks traded on the Shenzhen Stock Exchange	All of 2003	All order flows at all prices	$\overline{N}^b(p)$, $\overline{N}^a(p)$, and changes in relative depth profiles through time
Gu and Zhou (2009a)	23 stocks traded on the Shenzhen Stock Exchange	All of 2003	All order flows at all prices	Autocorrelation of δ^x for incoming orders
Hall and Hautsch (2006)	The 5 most liquid stocks traded on the Australian Stock Exchange	Jul–Aug 2002	All order flows at all prices	Whether the distribution of δ^x for incoming orders is conditional on $\mathcal{L}(t)$, volatility, and recent order flows
Harris and Hasbrouck (1996)	144 stocks traded on the NYSE	Nov 1990–Jan 1991	All order flows at all prices	Analysis of performance measures aiding decision-making between limit orders vs. market orders
Hasbrouck and Saar (2002)	The 300 largest equities on NASDAQ, traded on Island ECN	1 Oct–31 Dec 1999	All order flows at all prices	How volatility is related to order flow and $\mathcal{L}(t)$, and how order fill probabilities and mean time to execution vary with volatility
Hautsch and Huang (2011)	The 30 most frequently traded stocks on Euronext Amsterdam	All trading days between 1 Aug and 30 Sep 2008	First 2 levels of bid-side relative depth profile and ask-side relative depth profile (updated whenever either changed) and a record of all trades (timestamped to the nearest millisecond)	Market impact of incoming limit orders
Hendershott and Jones (2005)	3 exchange-traded funds on Island ECN	16 Aug–31 Oct 2002	For activity on Island: for first part of data, $b(t)$, $a(t)$, $n^b(b(t),t)$, and $n^a(a(t),t)$ (updated whenever any of them changed), and all market orders; for second part of data, only market orders; for activity not on Island, $b(t)$, $a(t)$, $n^b(b(t),t)$, and $n^a(a(t),t)$ (updated whenever any of them changed), and all market orders for entire data period	How showing traders $\mathcal{L}(t)$ affects price series
Hendershott *et al.* (2011)	943 stocks traded on the NYSE	Feb 2001–Dec 2005	$b(t)$, $a(t)$, $n^b(b(t),t)$, and $n^a(a(t),t)$ (updated whenever any of them changed)	How algorithmic trading affects $\mathcal{L}(t)$
Hollifield *et al.* (2004)	The Ericsson stock, traded on the Stockholm Stock Exchange	59 trading days, 3 Dec 1991–2 Mar 1992	All order flows at all prices	Whether traders' actions can be explained by a cut-off strategy based on their private valuation of the traded asset
Hollifield *et al.* (2006)	3 stocks traded on the Vancouver Stock Exchange	May 1990–Nov 1993	All order flows at all prices	Distribution of traders' personal valuations (inferred from their actions)
Kempf and Korn (1999)	DAX futures contracts, traded on the German Futures and Options Exchange	17 Sep 1993–15 Sep 1994	$b(t)$, $a(t)$, and all market orders	Permanent price impact, as a function of several measures of trade imbalance, over 1-min time horizons
Kirilenko *et al.* (2011)	E-mini S&P 500 index futures contracts, traded on the Chicago Mercantile Exchange	6 May 2010	All order flows at all prices, including details of which traders submitted which orders	Possible causes of the 'Flash Crash'
Lillo and Farmer (2004)	20 stocks traded on the LSE	1999–2002	All order flows at all prices	Autocorrelations of ω_x series, $m(t)$, $n^b(b(t),t)$, $n^a(a(t),t)$, and order type (buy or sell) for arriving LOs, arriving MOs, and cancelled LOs
Lillo *et al.* (2005)	20 stocks traded on the LSE	May 2000–Dec 2002	All LOB order flows and all off-book trades for the same stocks	Effects of order splitting and hidden liquidity on observed order flows

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Appendix A. Continued.

Lillo (2007)	Astrazeneca Stock, traded on the LSE	May 2000–Dec 2002	Order arrivals, partitioned by who submitted them	Distribution of δ^x for incoming limit orders from specified traders
Lo and Sapp (2010)	DEM/USD and USD/CAD currency pairs, traded on Reuters	5 Oct–10 Oct 1997 for DEM/USD; 1 May–30 Jun 2005 for USD/CAD	All order flows at all prices	How traders choose ω_x and δ^x for their orders
Madhavan *et al.* (2005)	109 stocks traded via LOBs and 240 stocks traded by floor traders on the Toronto Stock Exchange	Mar and May 1990	For March: $b(t)$, $a(t)$, $n^a(a(t),t)$, and $n^b(b(t),t)$; for May: $b(t)$, $a(t)$, first 5 levels of bid-side relative depth profile and ask-side relative depth profile; for both months: all market orders all floor-trader trades	How real-time disclosure of more information about the depth profile affects traders' behaviour
Maskawa (2007)	13 stocks traded on the LSE	Jul–Dec 2004	All order flows at all prices	Distribution of δ^x for incoming limit orders, and whether this distribution is affected by the state of $\mathcal{L}(t)$
Maslov and Mills (2001)	Cisco Systems, Broadcom Corporation, and JDS Uniphase Corporation stocks (traded on NASDAQ)	30 Jun 2000 for Cisco Systems; 3 Jul for Broadcom Corporation; and 5, 6, and 11 Jul for JDS Uniphase Corporation	$b(t)$, $a(t)$, first 4 levels of bid-side relative depth profile and ask-side relative depth profile, and all market orders	Distribution of ω_x, $n^b(b(t),t)$, $n^a(a(t),t)$, depth profiles, and instantaneous price impact
Mike and Farmer (2008)	25 stocks traded on the LSE	May 2000–Dec 2002	All order flows at all prices	δ^x for incoming orders, autocorrelation of order type in order flows, and order cancellations
Mizrach (2008)	The 4 largest stocks on NASDAQ; 95 of the NASDAQ 100 stocks; and 87 other smaller NASDAQ stocks	Dec 2002	All order flows at all prices	How $\mathcal{L}(t)$ affects the next change in $b(t)$ or $a(t)$
Mu *et al.* (2009)	22 stocks traded on the Shenzhen Stock Exchange	All of 2003	All order flows at all prices	Distribution of ω_x for market orders
Mu and Zhou (2010)	978 stocks traded on the Shenzhen Stock Exchange	Jan 2004–Jun 2006	$b(t)$ and $a(t)$ (updated once every 6–8 s)	Distribution of mid-price logarithmic returns for stocks in emerging markets, and how this varies with time window and market capitalization of the studied asset
Plerou *et al.* (2002)	The 116 most frequently traded US stocks	1994–1995	$b(t)$, $a(t)$, $n^b(b(t),t)$, $n^a(a(t),t)$, and all market orders	Price impact as a function of trade imbalance count and trade imbalance size over a variety of time horizons
Plerou and Stanley (2008)	1000 major US stocks; 85 of the FTSE 100 stocks, traded on the LSE; 13 of the CAC 40 stocks, traded on the Paris Bourse; 422 stocks from the Center for Research in Security Prices (CRSP)	1994–1995 for US stocks; 2001–2002 for LSE stocks; 3 Jan 1995–22 Oct 1999 for Paris Bourse stocks; Jan 1962–Dec 1996 for CRSP database stocks	All market orders	Distribution of mid-price returns and number of arriving market orders, and whether they vary according to market capitalization or industry sector, on various τ-second timescales
Potters and Bouchaud (2003)	Exchange-traded funds that track NASDAQ and the S&P 500, and the Microsoft stock	1 Jun–15 Jul, 2002	All order flows at all prices	Distribution of δ^x for arriving orders, relative depth profiles, arrival and cancellation rates, and instantaneous price impact

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Appendix A. Continued.

Ranaldo (2004)	15 stocks traded on the Swiss Stock Exchange	Mar and Apr 1997	$b(t)$, $a(t)$, $n^b(b(t), t)$, $n^a(a(t), t)$, and all market orders	How volatility, recent order flow, and the state of $\mathcal{L}(t)$ affect order flow, intra-day patterns in $s(t)$, and volatility, symmetry between the buy side and sell side
Sandås (2001)	10 stocks traded on the Stockholm Stock Exchange	3 Dec 1991–2 Mar 1992	All order flows at all prices	Whether the depth profile supports hypotheses about how traders make decisions related to order submissions and cancellations
Toke (2011)	3 stocks from the CAC 40, 3 month Euribor futures, and FTSE 100 futures	10 Sep 2009–30 Sep 2009	First 5 levels of bid-side relative depth profile and ask-side relative depth profile (updated whenever any of them changed and timestamped to the nearest millisecond)	Whether Hawkes processes provide a better explanation of order flows than do Poisson processes
Tóth *et al.* (2011)	500,000 trades on a variety of futures contracts	Jun 2007–Dec 2010	Changes in $b(t)$ and $a(t)$	Price impact
Wyart *et al.* (2008)	The 68 most liquid stocks on the Paris Bourse, small-tick index futures contracts, and the 155 most actively traded stocks on the NYSE	2002 for the Paris Bourse and 2005 for the small tick futures and NYSE stocks	$b(t), a(t), n^b(b(t), t)$ and $n^a(a(t), t)$, and all market orders	Price impact and how the profit of a market maker trading in an LOB depends on $s(t)$
Zhao (2010)	Crude oil futures contracts, traded on the International Petroleum Exchange	17 Oct 2005	First 5 levels of bid-side relative depth profile and ask-side relative depth profile (updated whenever either changed) and all market orders (timestamped to the nearest second)	Order flow rates
Zhou (1996)	DEM/USD, USD/JPY and DEM/JPY currency pairs, traded on Reuters	1 Oct 1992–30 Sep 1993	$b(t)$	Volatility
Zhou (2012)	23 stocks traded on the Shenzhen Stock Exchange (although 1 is later removed, as its price was reported to be manipulated in the data)	All of 2003	All order flows at all prices	Instantaneous price impact of individual orders
Zovko and Farmer (2002)	50 stocks traded on the LSE	1 Aug 1998–31 Apr 2000	δ^x for incoming limit orders	δ^x for incoming limit orders, autocorrelation of order type in order flows, and volatility