A constructive solution to Tarski's circle squaring problem

Andrew Marks (UCLA), joint with Spencer Unger (UCLA)

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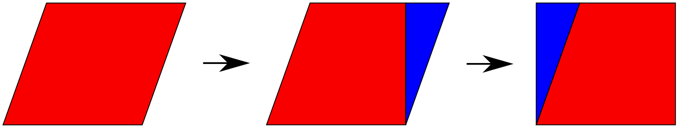
I. History

Dissection congruence

Two polygons are **dissection congruent** if we can cut the first into finitely many polygons which we can rearrange to get the second (ignoring boundaries). This idea dates back to Euclid.



Area$=bh$

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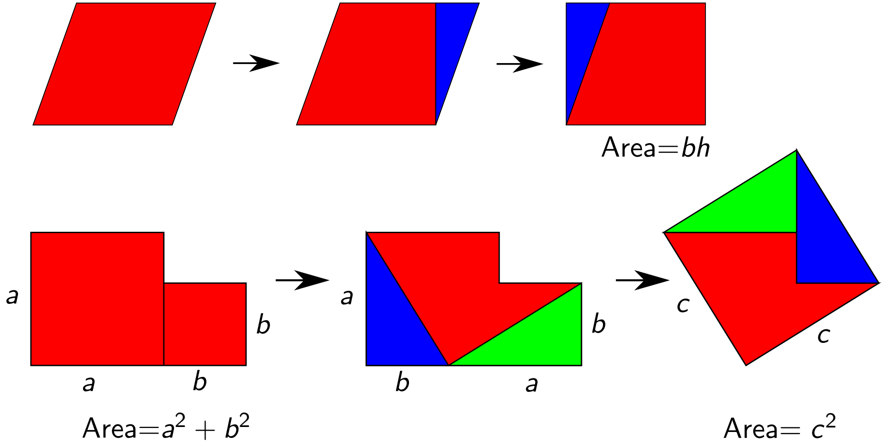

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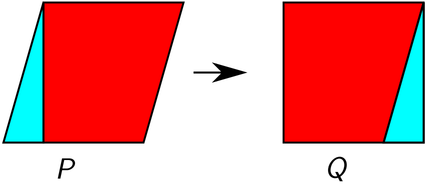

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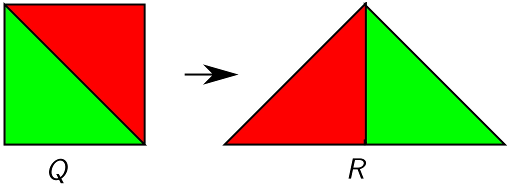

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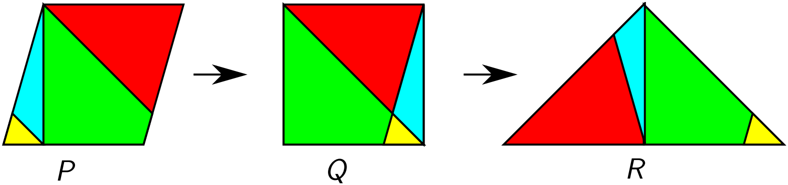

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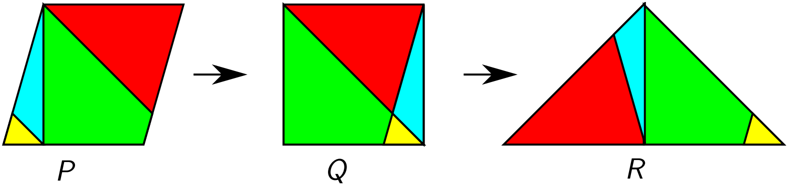
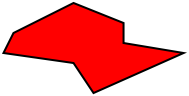


So it is enough to show that any polygon is dissection congruent to a square of the same area.

Proving the Wallace-Bolyai-Gerwein theorem

To show a polygon is dissection congruent to a square:

Chop the polygon into triangles.



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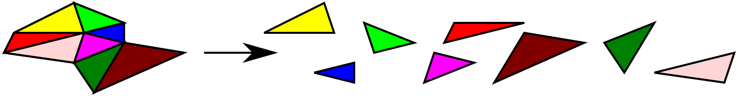

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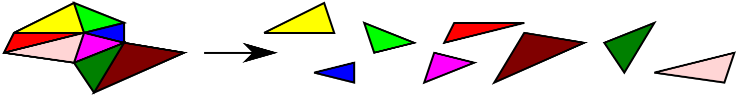
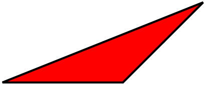


Dissect each triangle into a parallelogram



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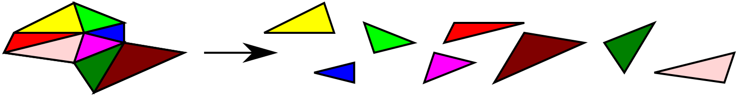

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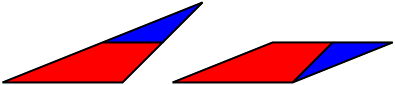

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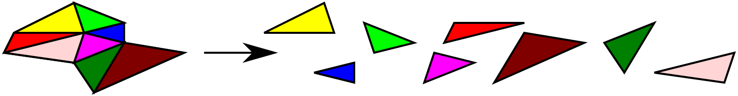

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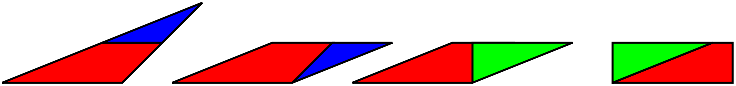

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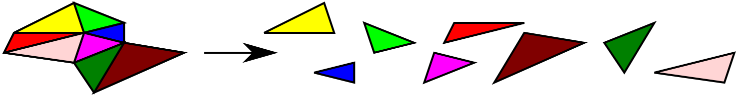

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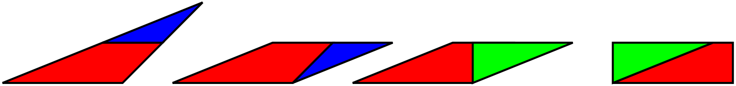

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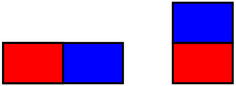

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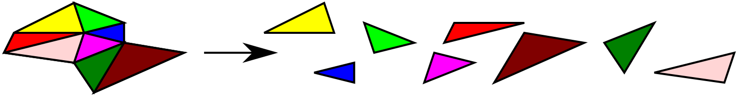

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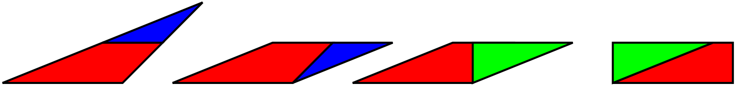

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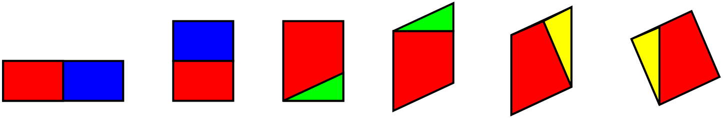

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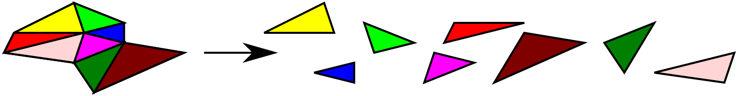

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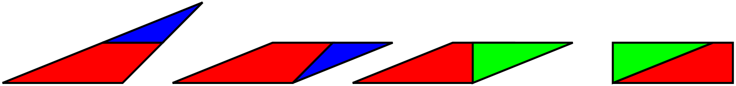

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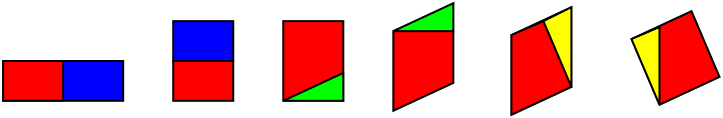

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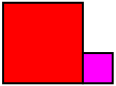

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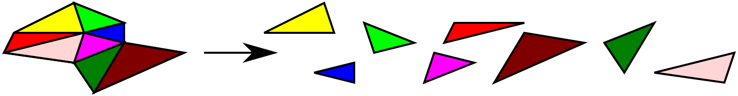

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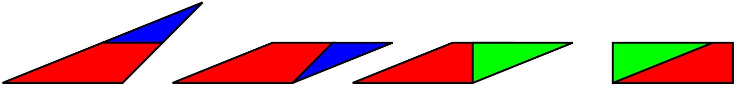

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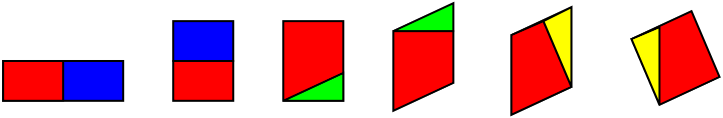

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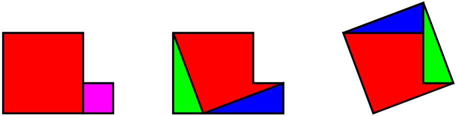

$\square$

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Indeed, if P is a polyhedron with edge lengths ℓ_i and edge dihedral angles θ_i, then the **Dehn invariant**

$$\sum_i \ell_i \otimes \theta_i$$

(taking values in the tensor product $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/2\pi\mathbb{Z}$) is an invariant of dissection congruence.

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Theorem (Sydler, 1965)

Two polyhedra are dissection congruent if and only if they have the same volume and Dehn invariant.

The foundations of measure theory

The existence of Vitali sets implies that for all $n \geq 1$, there is no extension of Lebesgue measure to the full powerset $P(\mathbb{R}^n)$ which is

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(The difference hinges on the fact that if $n \geq 3$, the isometry group of $\mathbb{R}^n$ contains a free group on two generators. If $n \leq 2$ it does not.)

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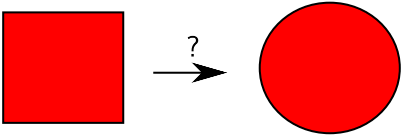
Question (Tarski's circle squaring problem, 1925)

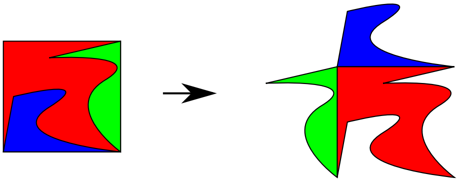
Are a disc and square in $\mathbb{R}^2$ (necessarily of the same area) equidecomposable?



The disc and square must have the same area because of the existence of Banach measures.

A square and disc are not scissors congruent

A and B are **scissors congruent** if A can be cut into finitely pieces–each of which is homeomorphic to a disc and bounded by a curve of finite length–which can be rearranged to form B (ignoring boundaries).



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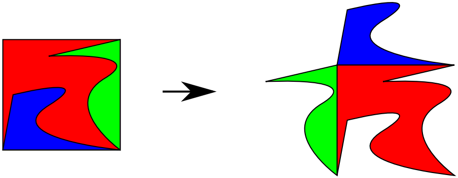

In scissors congruence, any time a section of convex circular perimeter is created or destroyed it cancels with a corresponding pieces of concave circular perimeter. So

$$\text{convex circular perimeter} - \text{concave circular perimeter}$$

is an invariant of scissors congruence.

Corollary (Dubins-Hirsch-Karush, 1964)

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II. Laczkovich's solution

Laczkovich's circle squaring

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More generally,

Theorem (Laczkovich, 1992 (AC))

If $A, B \subseteq \mathbb{R}^k$ are bounded sets with the same positive Lebesgue measure whose boundaries have upper Minkowski dimension less than k, then A and B are equidecomposable.

Laczkovich's proof

First idea: **Work in the torus**
Fix sets A, B. Scale and translate A and B so that they lie in $[0, 1)^k$ which we identify with the k-torus $\mathbb{T}^k = (\mathbb{R}/\mathbb{Z})^k$. Then A and B are equidecomposable by translations as subsets of $\mathbb{T}^k$ iff they are equidecomposable by translations in $\mathbb{R}^k$.
(Though perhaps using more pieces).

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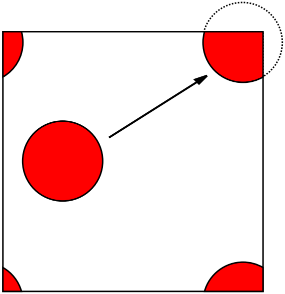
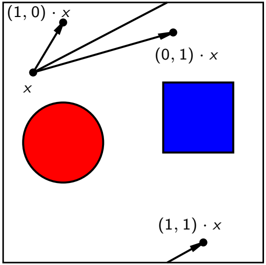
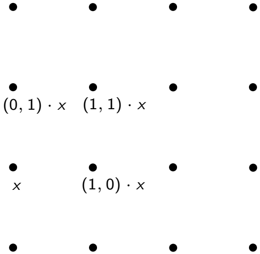


Use random translations



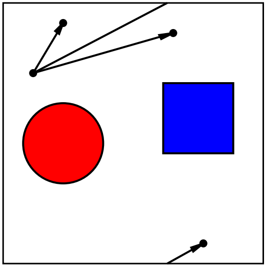


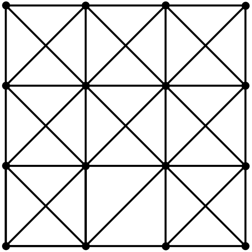
Fix a sufficiently large d, and random $u_1, \ldots, u_d \in \mathbb{T}^k$. Obtain a random action of $\mathbb{Z}^d$ on $\mathbb{T}^k$ by translations:

$$(n_1, \ldots, n_d) \cdot x = n_1 u_1 + \ldots + n_d u_d + x$$

This action is almost surely free. We can visualize each orbit as a copy of $\mathbb{Z}^d$.

Use random translations





Let G be the graph with vertex set $\mathbb{T}^k$ where $x, y \in \mathbb{T}^k$ are adjacent if there is $g \in \mathbb{Z}^d$ such that $g \cdot x = y$ where $|g|_\infty = 1$.

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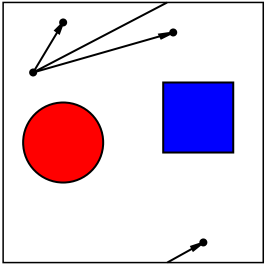
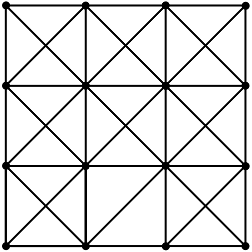

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To show A and B are equidecomposable, it suffices to find a Borel bijection $f \colon A \to B$ of bounded distance in G. (For some fixed N, for all $x \in A$, $d_G(x, f(x)) \leq N$).

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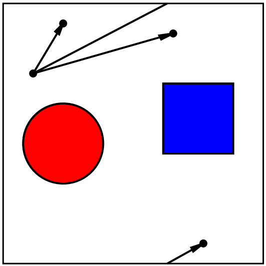
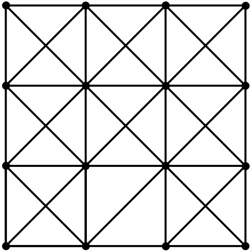

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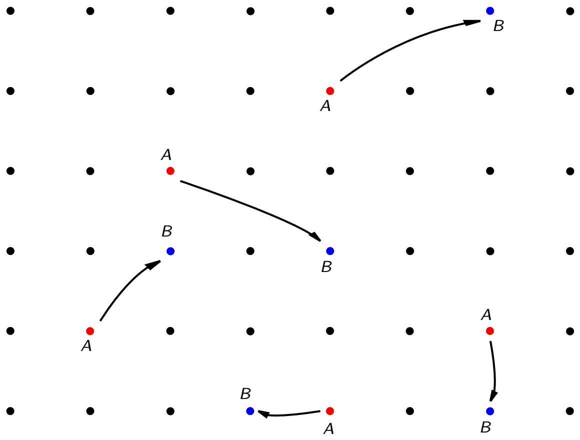
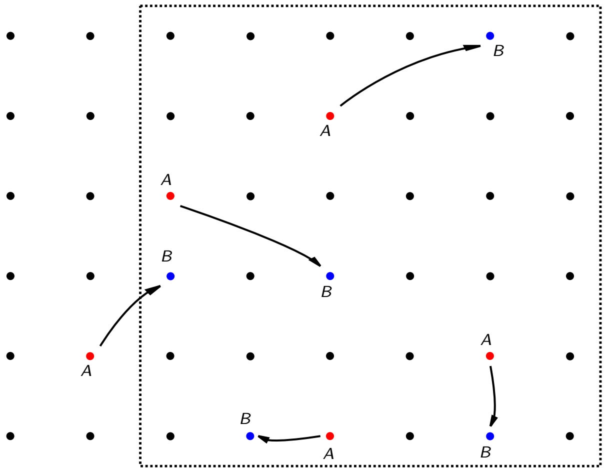
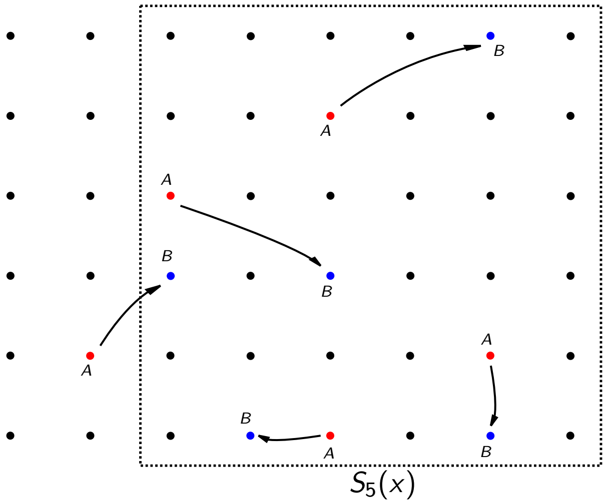
Then if $A_g = \{x : f(x) = g \cdot x\}$, the sets $\{A_g\}_{|g|_\infty \leq N}$ partition A, and the sets $\{g \cdot A_g\}_{|g|_\infty \leq N}$ will partition B.



A picture of an equidecomposition viewed inside a single orbit of the action.



For an equidecomposition to exist, any sufficiently large "square" $S_N(x) = \{(n_1, \ldots, n_d) \cdot x \in \mathbb{Z}^d : 0 \leq n_i < N\}$ in the orbit must contain roughly the same number of elements of A and B.



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By the ergodic theorem, we would expect $|S_N(x) \cap A| \approx \lambda(A) N^d$.

Laczkovich's key lemma

The key to Laczkovich's proof is a strong quantitative refinement of the ergodic theorem for translation actions, using ideas from Diophantine approximation and discrepancy theory.

Lemma (Laczkovich 1992 after Schmidt, Niederreiter-Wills)

For A, B and the action as above, $\exists \epsilon > 0$ and M such that for every x and N,

$$\left| S_N(x) \cap A - \lambda(A) N^d \right| \leq M N^{d-1-\epsilon}$$

and

$$\left| S_N(x) \cap B - \lambda(B) N^d \right| \leq M N^{d-1-\epsilon}$$

Roughly, every square $S_N(x)$ contains very close to $\lambda(A)N^d$ many elements of A and B.

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Laczkovich combines this estimate with compactness and Hall's matching theorem to find an equidecomposition.

III. A constructive solution

A constructive solution

Question (Wagon, 1986)

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So for sets whose boundaries aren't wildly fractal, having the same measure is *equivalent* to having an explicit equidecomposition. This gives a "Borel solution" to Hilbert's third problem.

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But after spending a couple years on the problem thinking just in terms of definable matchings, we were still quite far from a solution.

Ingredients in our solution

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  - There are well known combinatorial equivalences between flows and matchings. (E.g. Hall's theorem can be proved using max-flow min-cut).
- Recent progress in ergodic theory and descriptive set theory on hyperfiniteness of actions of abelian groups. We use a detailed descriptive-set-theoretic analysis of the translation action on the torus.

Flows in graphs

Suppose G is a graph (symmetric irreflexive relation) on a vertex set X. If $f : X \to \mathbb{R}$ is a function, then an f-**flow** of G is a function ϕ: $G \to \mathbb{R}$ such that

- For every edge $(x, y) \in G$, $\phi(x, y) = -\phi(y, x)$, and
- For every vertex $x \in X$:

$$f(x) = \sum_{(x,y) \in G} \phi(x, y) \quad \text{(Kirchoff's law).}$$

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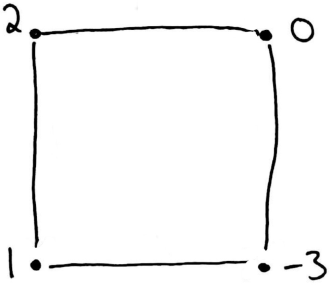
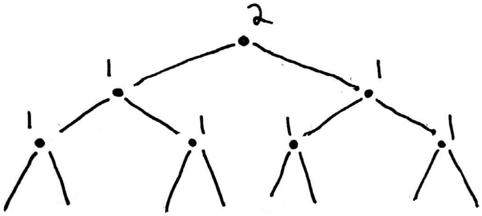
Examples of flows





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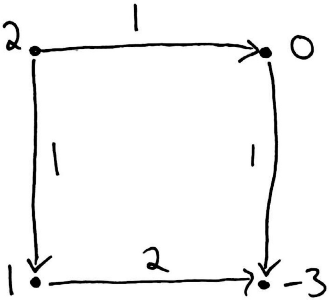

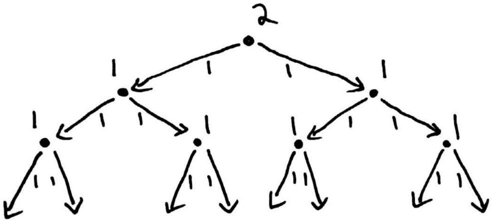

The first step of our proof

As in Laczkovich's proof, fix a sufficiently large d and a random translation action of $\mathbb{Z}^d$ on $\mathbb{T}^k$. Once again let G be the graph with vertex set $\mathbb{T}^k$ where $x, y \in \mathbb{T}^k$ are adjacent if there is $g \in \mathbb{Z}^d$ such that $g \cdot x = y$ where $|g|_\infty = 1$.

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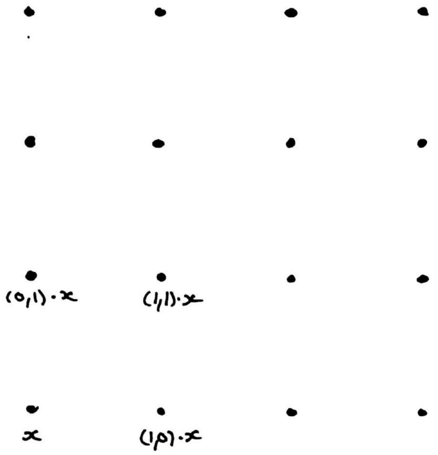
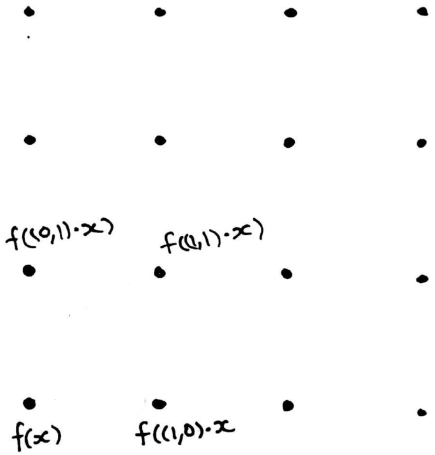
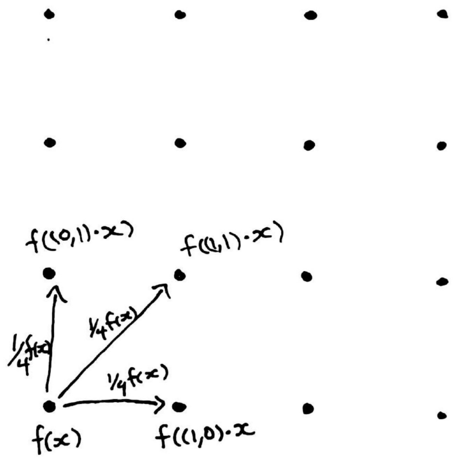
(We can interpret such a flow as a "continuous" equidecomposition from A to B. That is, each point of A has charge 1, and this charge can be split into finitely many pieces. After rearranging, we must obtain a charge of 1 at every point of B.)



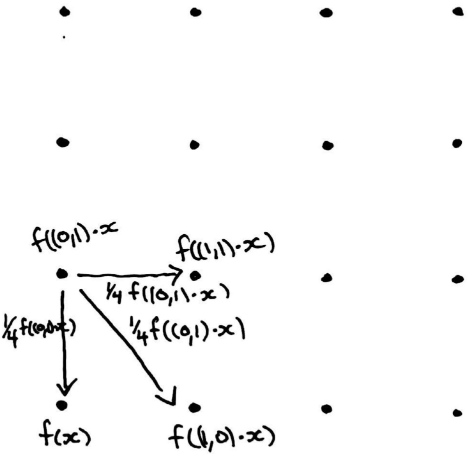
We'll describe an algorithm for constructing a real-valued f-flow in the connected component of some $x \in \mathbb{T}^k$. We draw pictures with $d = 2$.



Our flow will be constructed in ω many steps in which the error will approach 0.



Step 1: The idea is to spread out the error in the flow evenly over each 2×2 square. Each point contributes $1/4$ of its f-value to the other 3 points.



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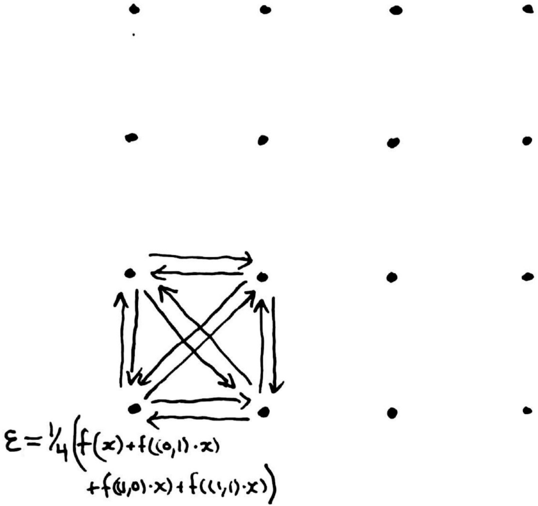
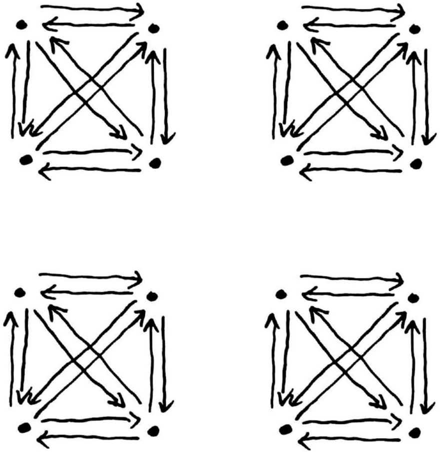
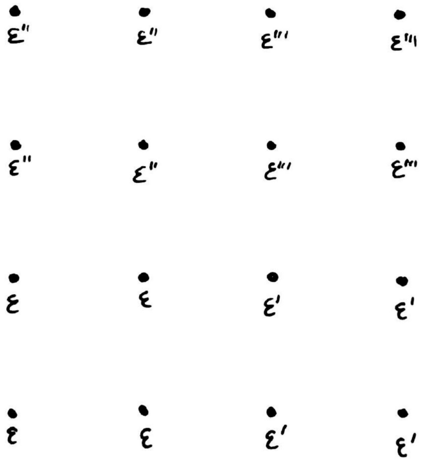


The error in the flow after step 1 is the average of f over the 2×2 square.



We do this for every 2×2 square in the orbit.



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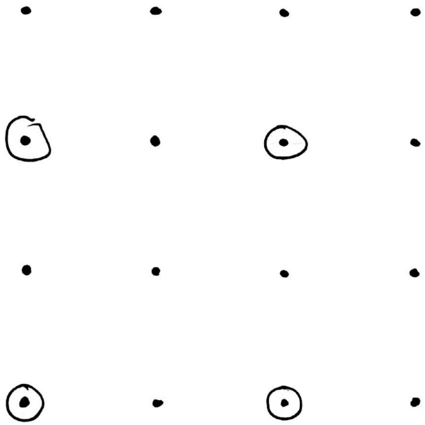
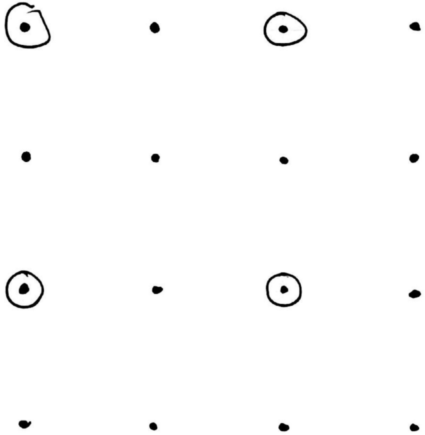


Now we use roughly the same idea in each 4×4 square, but dealing with 4 points at a time in the way given above.



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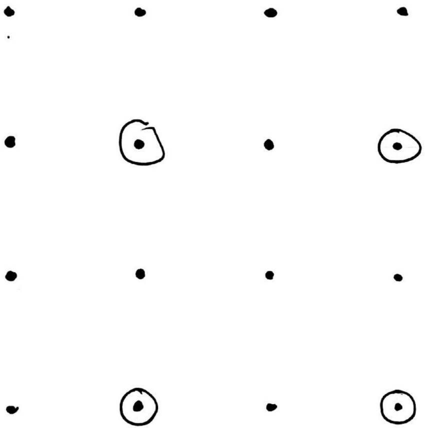

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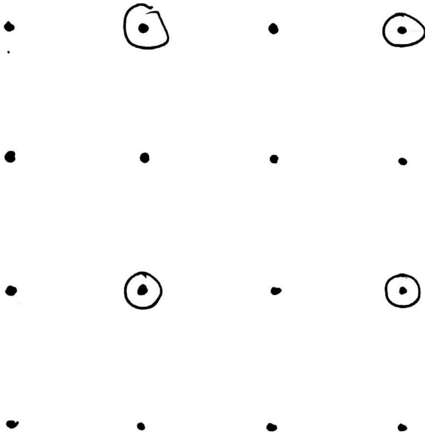

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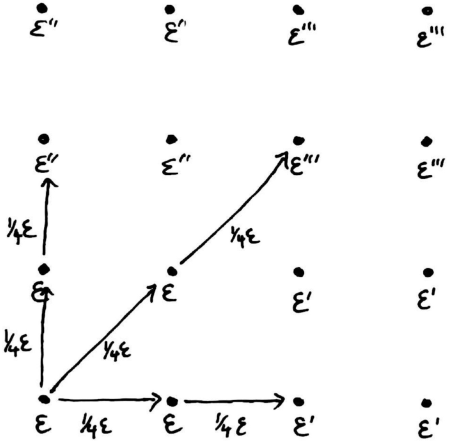
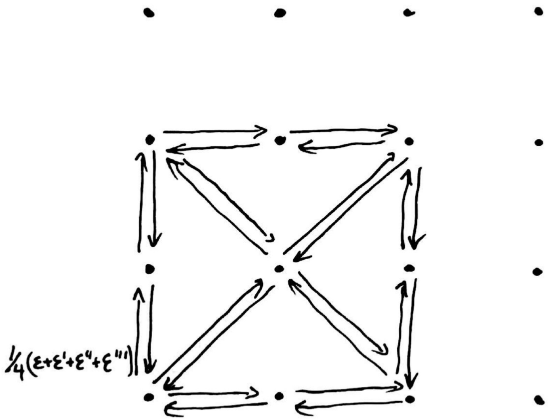
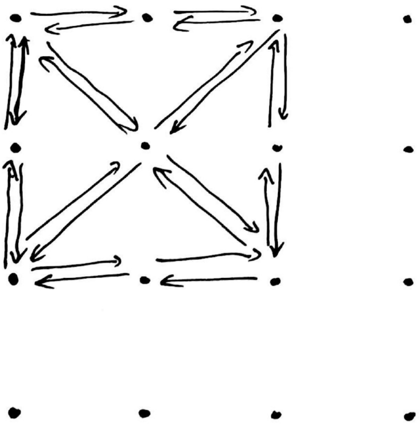


We add to the flow already constructed at the previous step. Once again, each point contributes $1/4$ of its error to the other 3 points.



After this second step, the error at each point will be the average of f over its 4×4 square.



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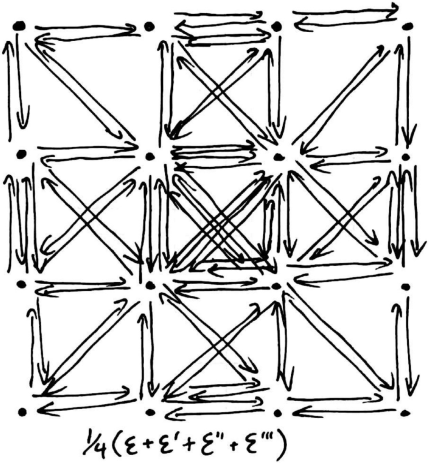

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Constructing a flow from A to B

After step n, the error in our flow at each point will be the average value of f over the $2^n \times 2^n$ square containing the point. Since $f = \chi_A - \chi_B$, and each $2^n \times 2^n$ square contains nearly the same number of points of A and B, this error is very small.

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However, we cannot pick a single x in each orbit to be a "starting point" for this construction (since this would be a nonmeasurable Vitali set).

To fix this problem, we use an averaging trick (the average of flows is a flow!).

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Essentially, we average this construction over every possible way of choosing 2×2 grids, 4×4 grids, 8×8 grids, etc. that fit inside each other. The result is invariant of our starting point.

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Formally, for every $i > 0$, let $\pi_i \colon \mathbb{Z}^d/(2^i\mathbb{Z})^d \to \mathbb{Z}^d/(2^{i-1}\mathbb{Z})^d$ be the canonical homomorphism. This yields the inverse limit

$$\hat{\mathbb{Z}}^d = \varprojlim_{i \geq 0} \mathbb{Z}^d/(2^i\mathbb{Z})^d$$

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For each $x \in \mathbb{T}^k$ and $h \in \hat{\mathbb{Z}}^d$, our above construction yields a flow $\phi_{(x,h)}$ of the connected component of x, using the grids given by h. The construction is such that if $g \in \mathbb{Z}^d$, then $\phi_{(x,h)} = \phi_{(g \cdot x, -g+h)}$. Hence, the average value of this construction is invariant of our starting point ($h \mapsto -g + h$ is measure preserving):

$$\int_h \phi_{(x,h)} \, \mathrm{d}\mu(h) = \int_h \phi_{(\gamma \cdot x, -\gamma + h)} \, \mathrm{d}\mu(h) = \int_h \phi_{(\gamma \cdot x, h)} \, \mathrm{d}\mu(h)$$

This average value is our real-valued Borel $\chi_A - \chi_B$ flow!

Finishing the proof

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This definition has become extremely important in dynamics. Amenability forms a crucial dividing line in the subject; amenable actions are often tame, classifiable, etc. whereas nonamenable actions are wild, unclassifiable, paradoxical, etc.

Hyperfiniteness

A major program in modern descriptive set theory is to understand the complexity of Borel actions of countable groups from the perspective of Borel reducibility.

Definition

A Borel action of a group G on a set X is **hyperfinite** if there are Borel equivalence relations $F_0 \subseteq F_1 \subseteq \ldots$ all of whose classes are finite such that their union $\bigcup_i F_i$ is the orbit equivalence relation of the action.

By the Glimm-Effros dichotomy of Harrington-Kechris-Louveau (1990), the simplest nontrivial actions are the hyperfinite ones.

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Is every Borel action of a countable amenable group hyperfinite?

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Our proof uses a refinement of Gao-Jackson due to Gao-Jackson-Krohne-Seward (2015); special types of witnesses to the hyperfiniteness of actions of $\mathbb{Z}^d$ which are well suited to meshing with combinatorial constructions.

Finishing the proof

We use the hyperfiniteness of the translation action on the torus to run a "local" version of the Ford-Fulkerson algorithm to convert our real-valued flow to be integer. Our proof also relies on work of Timár (2013) on the connectivity of boundaries of finite regions in $\mathbb{Z}^d$ for $d \geq 2$.

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After converting the flow to be real-valued we use machinery of Gao-Jackson to find a Borel tiling of the action. Finally, we use the integer valued flow between A and B to compute how many points of A to move to points of B inside each tile.

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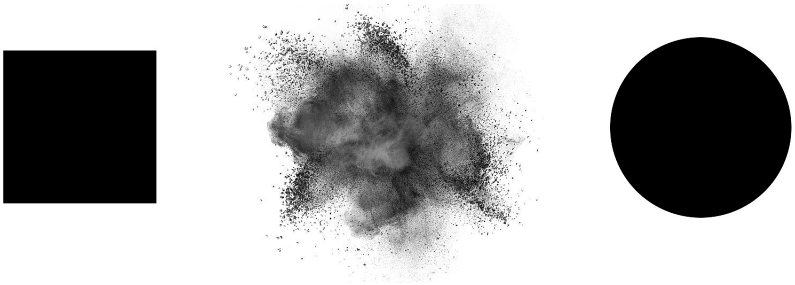

Thanks!