HJORTH'S TURBULENCE THEOREM

In this note, we give a short and self-contained exposition of Larson-Zapletal's beautiful proof of Hjorth's turbulence theorem [LZ, Section 3]. A good reference for the original "classical proof" of Hjorth is [K]. Larson-Zapletal's proof uses forcing over models of ZFC. The motivation for using models of set theory is that turbulence is really about absolute maps to hereditarily countable sets. It is easier to simply deal with these sets themselves instead of coding them into elements of some Polish space. By doing this, we bypass issues such as the fact that there is no "canonical" code for each hereditarily countable set; there are many different codes for each single set. Thanks to Dino Rossegger, Assaf Shani, Ted Slaman, and Jindrich Zapletal for helpful conversations about this proof. These notes were prepared for the Berkeley computability seminar in 2022.

Before we jump into the details, we give a quick sketch of the proof. We will recall the all the definitions we use below in the next few sections. Suppose for a contradiction that G is a Polish group, X is a Polish space, $G \curvearrowright X$ is turbulent continuous action, L is a countable language, and $f\colon X \to X_L$ is a Borel reduction from the orbit equivalence relation of $G \curvearrowright X$ to the isomorphism relation $\cong$ on the space of L-structures. Let $\mathbb{P}_X$ be Cohen forcing in X and let $\mathbb{P}_G$ be Cohen forcing in G. Let (g, x) be $\mathbb{P}_G \times \mathbb{P}_X$-generic. The key step of the proof uses the properties of turbulence to show that $V[x] \cap V[g \cdot x] = V$. (This type of result–that the intersection of two different generic extensions is V–is common in the theory of forcing. For example, if x and y are mutually Cohen generic, then $V[x] \cap V[y] = V$). To conclude the argument, note that since the canonical Scott Sentence of two isomorphic structures is the same, and the definition of the Scott Sentence is absolute, $\mathrm{css}(f(x))^{V[x]} = \mathrm{css}(f(g \cdot x))^{V[g \cdot x]}$. Calling this Scott Sentence $\varphi = \mathrm{css}(f(x))$, we therefore have $\varphi \in V$ since $V[x] \cap V[g \cdot x] = V$. Since $V[x] \vDash \mathrm{css}(f(x)) = \varphi$ and $\varphi \in V$, there must be a condition $p \in \mathbb{P}_X$ forcing $p \Vdash \mathrm{css}(f(x)) = \check{\varphi}$. Hence, all generic x that extend p are all assigned the same isomorphism class by f. This is a nonmeager set and so this contradicts that f is a Borel reduction since orbits of the action $G \curvearrowright X$ are meager.

1. Turbulence

Recall that a **Polish group** G is a topological group with a Polish topology where the inverse $g \mapsto g^{-1}$ and the group operation $(g, h) \mapsto gh$ are continuous functions. For example, $(\mathbb{R}, +)$ and $(2^\omega, \triangle)$ are Polish groups where $\triangle$ is the symmetric difference operation on subsets of ω.

Suppose $G \curvearrowright X$ is a continuous action of a Polish group G on a Polish space X, and $U \subseteq X$ and $V \subseteq G$. If $x, y \in U$, a U, V**-walk** from x to y is a finite sequence of points $x_0, x_1, \ldots, x_n \in U$ where $x = x_0$ and $y = x_n$ such that there exists a sequence of group elements $h_0, \ldots, h_{n-1} \in V$ such that for all $i < n$, $h_i \cdot x_i = x_{i+1}$. The U, V-local orbit of $x \in U$ is the set of $y \in U$ such that there exists a U, V-walk from x to y. The U, V-local orbit of x is denoted $\mathcal{O}(U, V, x)$. We will use the following simple lemma about walks later, which uses continuity of the action.

Date: December 12, 2022.

Lemma 1.1. *Suppose $G \curvearrowright X$ is a continuous action, $D \subseteq G$ is dense, $V \subseteq G$ is open, and $U' \subseteq U \subseteq X$ are open. If there is a U, V-walk from $x \in U$ to some $y \in U'$, then there is a U, V-walk from $x \in U$ to some $z \in U'$ using group elements from $D \cap V$.*

Proof. Suppose $x_0, x_1, \ldots, x_n \in U$ is a U, V-walk where $x = x_0$ and $y = x_n$ via the group elements $h_i \in V$ where $h_i \cdot x_i = x_{i+1}$. Then by continuity of the action, if we take sufficiently close approximations $h'_i \in D \cap V$ to the h_i, the walk $x'_0, \ldots, x'_n$ defined by $x'_0 = x_0$ and $x'_{i+1} = h'_i \cdot x'_i$ is a walk from x to a point which can be made arbitrarily close to y and hence may end at some $z \in U'$. $\square$

We're ready to define turbulence:

Definition 1.2. We say that an action $G \curvearrowright X$ of a Polish group G on a Polish space X is **turbulent** if:

(1) Every orbit $G \cdot x$ is dense and meager.
(2) For all open $U \subseteq X$ and $V \subseteq G$ with $1 \in V$, the U, V-local orbit $\mathcal{O}(U, V, x)$ is somewhere dense. (That is, there is an open $U' \subseteq U$ such that $\mathcal{O}(U, V, x)$ is dense in U').

There are many simple examples of turbulent actions, such as the action of $\ell^1(\mathbb{R})$ on $\mathbb{R}^\omega$.

Example 1.3. Let $\ell^1(\mathbb{R})$ be the Polish group of sequences $(g_n)_{n\in\omega} \in \mathbb{R}^\omega$ that are summable: $\sum_{n\in\omega} |x_n| < \infty$. The Polish topology on $\ell^1(\mathbb{R})$ is generated by the ℓ^1 metric $|x - y|_1 = \sum_{n\in\omega} |x_n - y_n|$ and the group is equipped with the operation of pointwise addition. $\ell^1(\mathbb{R})$ acts on the Polish space $X = \mathbb{R}^\omega$ by coordinatewise addition. This is a continuous action. We claim this action is turbulent.

Every orbit of this action is dense since given any $x, y \in \mathbb{R}^\omega$ and N there is $g \in \ell^1(\mathbb{R})$ so that $(g + x)(n) = y(n)$ for every $n < N$. Every orbit is also meager since given any x, for every N $\{y \in \mathbb{R}^\omega \colon \sum_n |x(n) - y(n)| > N\}$ is dense open, and hence $\{y \in \mathbb{R}^\omega \colon \sum_n |x(n) - y(n)| = \infty\}$ is comeager. Finally, every local orbit is somewhere dense. Given any open $U \subseteq X$, pick $x \in U$, and let $U' \subseteq U$ be any basic open set containing x, where $U' = (a_0, b_0) \times \ldots \times (a_n, b_n) \times \mathbb{R} \times \mathbb{R} \times \ldots$. We claim $\mathcal{O}(U, V, x)$ is dense in U'. Given any $y \in U'$, we can choose a large N so that $y' \in U'$ defined by $y'(n) = y(n)$ for $n < N$, $y'(n) = x(n)$ for $n \geq N$ is arbitrarily close to y. Then consider $g = x - y'$. $g \in \ell^1(\mathbb{R})$ since g is eventually 0. Given any neighborhood V of the identity in $\ell^1(\mathbb{R})$, $g/k \in V$ for some sufficiently large k. So we can make a U', V-walk from x to y' by letting $x_0 = x$, and $x_i = i/kg + x$, where we are use the group elements $h_i = g/k$. Then $x_k = y'$, so this is a U, V-walk from x to y' so the local orbit $\mathcal{O}(U, V, x)$ is dense in U'.

2. Proof of the Turbulence theorem

We begin with a simple lemma characterizing when two generic extensions of V have intersection equal to V.

Lemma 2.1 (ZFC)**.** *Suppose $V[x]$ and $V[y]$ are two generic extensions of V. Then $V[x] \cap V[y] = V$ iff every set of ordinals in $V[x] \cap V[y]$ is in V.*

Proof. We prove $\Leftarrow$ by contraposition. Suppose there is a set $a \in V[x] \cap V[y]$ which is not in V. We may assume a is an $\in$-minimal such set. Then every element of a is both in $V[x]$ and $V[y]$, and hence in V. Thus, $a \subseteq V$. Let $b \in V$ be a set such that $a \subseteq b$. As V satisfies choice, there is a bijection $f \colon \kappa \to b$ in V for some cardinal κ. So $f^{-1}[a] \in V[x]$ and $f^{-1}[a] \in V[y]$ and this is a set of ordinals which is not in V. $\square$

If X is a Polish space, we let $\mathbb{P}_X$ be Cohen forcing in X. So conditions in X are basic open subsets of X under inclusion. Similarly, if G is a Polish group, let $\mathbb{P}_G$ be Cohen forcing with basic open sets just using the Polish structure. Finally, note that if $G \curvearrowright X$ is a continuous action and $h \in G^V$, then x is $\mathbb{P}_X$-generic if and only if $h \cdot x$ is $\mathbb{P}_X$-generic. This is since the map $x \mapsto h \cdot x$ is continuous with a continuous inverse, and so it maps open dense sets to open dense sets. Similarly, (g, x) is $\mathbb{P}_G \times \mathbb{P}_X$-generic iff $(gh_0, h_1 \cdot x)$ is $\mathbb{P}_G \times \mathbb{P}_X$-generic for any $h_0, h_1 \in G^V$.

Lemma 2.2. *Suppose $G \curvearrowright X$ is a continuous turbulent action of a Polish group G on a Polish space X. Let (g, x) be $\mathbb{P}_G \times \mathbb{P}_X$ generic. Then $V[x] \cap V[g \cdot x] = V$.*

Proof. Work in $V[g, x]$ which contains both $V[x]$ and $V[g \cdot x]$. Suppose $a \in V[x] \cap V[g \cdot x]$ is a set of ordinals. We will show that $a \in V$, which suffices by Lemma 2.1.

There are names σ, τ for elements of $V[g \cdot x]$ and $V[x]$ respectively so that $\sigma[g \cdot x] = a = \tau[x]$, where $\tau[x]$ is the value of the name τ with respect to the generic x. Since $V[g, x] \vDash \sigma[g \cdot x] = a = \tau[x]$, there must be a condition $(p, q) \in \mathbb{P}_G \times \mathbb{P}_X$ such that

$$(*) \qquad (p, q) \Vdash \tau[x] \text{ is a set of ordinals and } \tau[x] = \sigma[g \cdot x].$$

Since p is a basic open set in $\mathbb{P}_G$ and $g \in p$, we can find an open neighborhood $W \subseteq G$ of the identity so that $gW^{-1} \subseteq p$ since the function $h \mapsto gh^{-1}$ on G is continuous. By the definition of turbulence, let $q' \subseteq q$ be so that the local orbit $\mathcal{O}(q, W, x)$ is dense in q'. We claim that $q' \Vdash \tau[x] = a$, and hence by the definability of forcing, $a \in V$.

To show that $q' \Vdash \tau[x] = a$, it suffices to show that the set of $\mathbb{P}_X$-generic y extending q' such that $\tau[y] = \tau[x]$ is dense in q'. If this is true, then for each ordinal $\beta \in \tau[x]$, since the set of $\mathbb{P}_X$-generic y with $\tau[y] = \tau[x]$ is dense in q', the set of conditions q'' extending q' that force $q'' \Vdash \beta \in \tau[y]$ is dense in q', so $q' \Vdash \beta \in \tau[x]$. Similarly, for each ordinal $\beta \notin \tau[x]$, since the set of $\mathbb{P}_X$-generic y with $\tau[y] = \tau[x]$ is dense in q', the set of conditions q'' extending q' that force $q'' \Vdash \beta \notin \tau[y]$ is dense in q', so $q' \Vdash \beta \notin \tau[x]$. Hence, $q' \Vdash \tau[x] = a$.

So we show now that for all q'' extending q', there is some $\mathbb{P}_X$ generic $y \in q''$ such that $\tau[y] = \tau[x]$. Let $D \subseteq G$ be a dense set in V. Note that by Π^1_1 absoluteness D is also dense in the generic extension. Given any $q'' \subseteq q'$, since the local orbit $\mathcal{O}(q, W, x)$ is dense in q', let $x = x_0, \dots, x_n$ be a q, W-walk from x to some $y = x_n \in q''$. By Lemma 1.1 we may assume that the group elements h_i are from D. Let $g_i = gh_i^{-1}$ so

$$(**) \qquad g_i \cdot x_{i+1} = gh_i^{-1} h_i \cdot x_i = g \cdot x_i.$$

Now all the x_i are P_X-generic by our remark before the lemma, and all the pairs (g, x_i) and (g_i, x_{i+1}) are similarly $\mathbb{P}_G \times \mathbb{P}_X$-generic. Thus, since $gW^{-1} \subseteq p$ and all the x_i are in q, and by using (*) and (**), we have

$$\tau[x_0] = \sigma[g \cdot x_0] = \sigma[g_0 \cdot x_1] = \tau[x_1] = \sigma[g \cdot x_1] = \sigma[g_1 \cdot x_2] = \tau[x_2] \ldots = \tau[x_n]$$

□

Suppose L is a countable language and let X_L be the space of all L-structures with universe $\mathbb{N}$. Then the map sending each structure $A \in X_L$ to its canonical Scott Sentence $\mathrm{css}(A)$ is Σ_1 definable in the language of set theory.[1] Furthermore if A and B are isomorphic

[1] The formula defining $\varphi = \mathrm{css}(A)$ says there exists an ordinal α, and a map from $\alpha + 1$ to the back-and-forth relations $\sim_\alpha$ on the structure and the fragments of the Scott sentence, and α is least such that $\sim_\alpha = \sim_{\alpha+1}$ so α is the Scott Rank of A, and $\mathrm{css}(A)$ is the conjunction of all the fragments of the Scott sentence. A detailed description of this and the definability of $\mathrm{css}(A)$ can be found in Barwise's book [B, Chapter VII]. See also [URL, Fact 3.2.3]

L-structures, then $\mathrm{css}(A) = \mathrm{css}(B)$; they are equal as sets. Note here that we are viewing the Scott Sentence here as a set, and we are not coding it by representing it via something like an infinite labeled tree. So for example, each infinite conjunction $\bigwedge_i \phi_i$ is represented by the pair $(\text{"}\bigwedge\text{"}, \{\phi_i : i \in \omega\})$ with no ordering on the formulas ϕ_i.

Now we are ready to prove that turbulent actions cannot be classified by countable structures.

Theorem 2.3 (Hjorth's turbulence theorem, [H00, Cor. 3.19])**.** *Suppose $G \curvearrowright X$ is a turbulent action of a Polish group G on a Polish space X. Let E_G be the equivalence relation where $x \mathrel{E_G} y$ iff $(\exists g \in G) g \cdot x = y$. Let L be a countable language, and $f\colon X \to X_L$ be a Borel function so that $x \mathrel{E_G} y \to f(x) \cong f(y)$. Then there is an L-structure A and a comeager set of x such that $f(x) \cong A$. Hence, there are $x \not\mathrel{E_G} y$ so that $f(x) \cong f(y)$, since by the definition of turbulence orbits are meager.*

Proof. Suppose $f\colon X \to X_L$ is a Borel classification by countable structures. Let M be a countable transitive model of (a sufficiently large finite fragment of) ZFC such that L, and (codes for) G, X, and the action $G \curvearrowright X$ are in M. Let (g, x) be $\mathbb{P}_G \times \mathbb{P}_X$-generic. Then since $\varphi = \mathrm{css}(f(x))$ is Σ_1 definable, by Σ_1 upwards absoluteness, $\mathrm{css}(f(x))^{M[x]} = \mathrm{css}(f(x)) = \mathrm{css}(f(g \cdot x)) = \mathrm{css}(f(g \cdot x))^{V[g \cdot x]}$. Calling this Scott Sentence $\varphi = \mathrm{css}(f(x))$, we therefore have $\varphi \in V$. Since $M[x] \vDash \mathrm{css}(f(x)) = \varphi$ and $\varphi \in M$, there must be a condition $p \in \mathbb{P}_X$ forcing $p \Vdash \mathrm{css}(f(x)) = \check{\varphi}$. So by absoluteness of $\mathrm{css}(f(x))$ all M-generic x extending p are mapped to structures with Scott sentence φ. Since M only contains countably many open dense sets, this is a comeager set of x in p. □

3. Concluding remarks

The truly remarkable and deep fact about turbulence is Hjorth's turbulence dichotomy below. It says that turbulence is the precise obstruction to having a classification by countable structures. Theorem 2.3 above says that options (1) and (2) are incompatible in the dichotomy:

Theorem 3.1 ([H02], Hjorth's turbulence dichotomy)**.** *Suppose $G \curvearrowright X$ is a continuous action of a Polish group G on a Polish space X whose orbit equivalence relation E_G is Borel. Then exactly one of the following holds:*

1. *There is a countable language L and a Borel reduction of E_G to the isomorphism relation on L-structures.*
2. *There is a turbulent continuous action $G \curvearrowright Y$ of G on a Polish space Y and a continuous embedding of this action into the action $G \curvearrowright X$.*

The proof of this dichotomy is beyond the scope of this note. It uses Gandy-Harrington forcing and a version of a Scott analysis for arbitrary Polish group actions. See [A] for a streamlined proof of Hjorth's result. These kinds of Scott analyses continue to be an important part of our understanding of Polish group actions. See for example [H00, Chapter 6], [D], and [Sol].

The proof of the turbulence theorem above crucially relies on the fact that $V[x] \cap V[g \cdot x] = V$. More generally, Larson and Zapletal have a beautiful characterization of when $V[x] \cap V[y] = V$ in the case when the generics x and y are generated in the following way: there is a single generic z and Borel functions f_0, f_1 such that $f_0(z) = x$ and $f_1(z) = y$. This includes turbulence as a special case: where $z = (g, x)$ is a $\mathbb{P}_G \times \mathbb{P}_X$ generic and the two functions are $f_0(g, x) = g \cdot x$ and $f_1(g, x) = x$. See Theorem 3.1.5 in [LZ].

The only property of the Scott Sentence that we used is that it is an absolute complete classification in the following sense. (Below, we ignore some metamathematical issues for brevity and we write "model of ZFC" instead of "model of a sufficiently large finite fragment of ZFC".)

Definition 3.2. Suppose E is a definable equivalence relation on a Polish space X, and $c\colon X \to V$ is a definable function in ZFC. That is, there is a formula φ in the language of set theory so that $\mathsf{ZFC} \vdash (\forall x \in X)(\exists! y)\varphi(x,y)$, and $c(x) = y \leftrightarrow \varphi(x,y)$. We say that c is a **complete classification for** E if $\mathsf{ZFC} \vdash (\forall x, y \in X)[x \mathrel{E} y \leftrightarrow c(x) = c(y)]$. Note here that c is a function to the whole universe of sets. Say that c is **absolute** if whenever M, N are transitive models of ZFC, then if $x, y \in X$ and $x \in M$ and $y \in N$, then $c(x)^N = c(y)^M$.

See [H00, Chapter 9] and [Sha] for more about these sort of absolute complete classifications.

Interesting absolute complete classification will typically be functions to sets that are more complicated than reals; if there is an absolute complete classification for E whose range is contained in the reals, then the equivalence relation is smooth. However, absolute complete classifications must always be to hereditarily countable sets. This is since if $c\colon X \to V$ is an absolute complete classification, then if we take a countable transitive model M of ZFC which contains x, then $c(x)^M$ must be a hereditarily countable set, since every element of M is a hereditary countable set. So for example, the map $x \mapsto [x]_E$ is a complete classification, but it will not be an absolute complete classification if E has uncountable equivalence classes.

Besides the Scott Sentence of a countable structure, another often considered absolute complete classification comes from Friedman-Stanley jumps. If E is an equivalence relation on X, E^+ is the equivalence relation on X^ω where $(x_n)_{n\in\omega} \mathrel{E^+} (y_m)_{m\in\omega}$ if $(\forall n)(\exists m) x_n \mathrel{E} y_m$ and $(\forall m)(\exists n) y_m \mathrel{E} x_n$. E^+ is called the Friedman-Stanley jump of E, and it is a theorem that $E <_B E^+$ if E is a Borel equivalence relation. We can iterate this jump operation along countable ordinals and at limit λ we define E^λ to be the disjoint union of E^α for $\alpha < \lambda$. It is easy to check that if $c\colon X \to V$ is an absolute complete classification for E, then $(x_n)_{n\in\omega} \mapsto \{c(x_n) \colon n \in \omega\}$ is an absolute complete classification for E^+. So all the equivalence relations $=^\alpha$ have absolute complete classifications. Of course, Friedman-Stanley jumps are closely related to classification by countable structures and S_∞ actions. See e.g. [HKL].

Finally, we note that an equivalence relation having an absolute complete classifications is equivalent to having an absolute Δ^1_2 classification by countable structures. The notion of absolute Δ^1_2 reductions between equivalence relations is well-studied and appears in many papers. See for example Chapter 9 of Hjorth's book [H00].

Proposition 3.3 (Folklore)**.** *Suppose E is a definable equivalence relation on a Polish space X. Then there exists an absolute complete classification $c\colon X \to V$ for E iff E has an absolute Δ^1_2 classification by countable structures. That is, there is a countable language L and a Δ^1_2 function $f\colon X \to X_L$ so that $x \mathrel{E} y \leftrightarrow f(x) \cong f(y)$ where $\cong$ is the isomorphism relation on L-structures, and the definition of f is absolute to all transitive models of set theory.*

Proof. $\Rightarrow$ is clear. We prove $\Leftarrow$. Suppose a is any hereditarily countable set. Then consider the language $L_{E,a}$ with the a binary relation E and a constant symbol a. We can map each set a to the countable structure $S_a = (\mathrm{TC}(\{a\}); \in, a)$ whose universe is $\mathrm{TC}(\{a\})$ – the transitive closure of $\{a\}$, and where we interpret E^{S_a} to be the $\in$ relation on $\mathrm{TC}(\{a\})$, and we interpret the constant symbol a^{S_a} to be the set a. Then S_a is a countable structure with a finite language and it is easy to check that for sets a and b, $a = b \leftrightarrow S_a \cong S_b$.

Now suppose E is an equivalence relation on X and $c\colon X \to V$ is an absolute complete classification. Let $L_{E,a}$ be the language described above and let $X_{L_{E,a}}$ be the space of L-structures on ω. Let $f\colon X \to X_{L_{E,a}}$ map x to the L-least $y = (\omega, E^y) \in X_L$ so that $\pi_{(\omega,E_y)}(a^y) = c(x)$ where π is the Mostowski collapse. Then $f(x) = y$ iff there exists a wellfounded countable model M of $\mathsf{ZFC}+V=L$ so that $M \vDash f(x) = y$ iff for all wellfounded countable models M of $\mathsf{ZFC}+V=L$ so that $M \vDash f(x) = y$. Here we're using the fact that countable well-founded models are correct about the L-predecessors of the reals that they can see, and our assumption that $c(x)$ is absolute to all wellfounded models of ZFC to see that these Σ^1_2 and Π^1_2 definitions are equivalent and absolute.

$\square$

Note that we can relativize the above theorem and the notion of an absolute complete classification to a real parameter, so an equivalence relation E definable relative to a real parameter z has an absolute complete classification relative to z if and only if it has an absolute Δ^1_2 classification relative to z.

References

[A] S. Allison, *Dichotomy results for orbit equivalence relations and the Gandy-Harrington topology*, unpublished notes.

[B] J. Barwise, *Admissible sets and structures*, Perspectives in Logic 7, (1975) Springer-Verlag.

[D] O. Drucker, *A boundedness principle for the Hjorth rank*. Arch. Math. Logic 61, 223–232 (2022).

[H00] G. Hjorth, *Classification and Orbit Equivalence Relations*, Mathematics Surveys and Monographs 75 (2000), American Mathematical Society, Providence, RI.

[H02] G. Hjorth, *A dichotomy theorem for turbulence*, J. Symb. Log. Vol. 67, No. 4 (2002) 1520–1540

[HKL] G. Hjorth, A. S. Kechris, and A. Louveau, *Borel equivalence relations induced by actions of the symmetric group*, Ann. of Pure and Applied Logic Vol 92 No. 1 (1998), 63–112.

[K] A. S. Kechris, *Actions of Polish Groups and Classification Problems*, Ann. of Pure and Applied Logic Vol 92 No. 1 (1998), 63–112.

[LZ] P. B. Larson and J. Zapletal, *Geometric set theory.*, Mathematical Surveys and Monographs Vol. 248, American Math Society, 2020.

[Sol] S. Solecki, *Transfinite sequences of topologies, descriptive complexity, and approximating equivalence relations*, Israel J. Math., 242 (2021), 933–953.

[Sha] A. Shani, *Classifying invariants for E_1: A tail of a generic real*, preprint (2021), `https://arxiv.org/abs/2112.12881`.

[URL] D. Ulrich, R. Rast and C. Laskowski, *Borel complexity and potential canonical Scott sentences*, Fund. Math. 239, 2 (2017) 101–147.