Homework 9: Due Tuesday June 2, 1pm

1. (10 pts) Construct a Turing machine so that if the head of the machine is positioned starting at the left of a positive integer n written in binary, the machine halts outputting $n - 1$ in binary.

2. (no collab, 10 pts) A *finite tape* Turing machines has a tape with a fixed finite number of cells l. If the machine tries to move left at the left edge of the tape or right at the right edge of the tape, the head stays in the same place.

   Give an algorithm which given a finite tape Turing machine program and an input written on its tape, decides whether the Turing machine will eventually halt or not.

   [Hint: A *configuration* of a Turing machine consists of what state the Turing machine program is in, and what is written on its tape. First show that there are only finitely many configurations that a given finite tape Turing machine can have. Then show that if a Turing machine ever reaches the same configuration twice, it will never halt]

3. (10 pts) Let T_n be the partial function from $\mathbb{N} \to \mathbb{N}$ computed by the nth Turing machine. In class we showed that there is an incomputable function $f \colon \mathbb{N} \to \mathbb{N}$. That is, for every Turing machine T_n, there is at least one value k which f and T_n differ. (That is, $T_n(k)$ is undefined, or $T_n(k)$ is defined and not equal to $f(k)$).

   Show that if $f \colon \mathbb{N} \to \mathbb{N}$ is incomputable, then for every n, T_n differs from f at infinitely many values.

4. (Review problem, 20 pts) In class, we proved that the natural numbers $\mathbb{N}$ are not definable in the structure $\langle \mathbb{R}; 0, 1, +, \cdot; < \rangle$. In this problem, we prove a strengthened version of this fact:

   (a) Recall $a \in \mathbb{R}$ is said to be an *algebraic number* if there is a polynomial $p(x)$ with integer coefficients so that a is a root of p. An *algebraic interval* is a subset of $\mathbb{R}$ of the form (a, ∞), $(-\infty, a)$ or (a, b), where a and b are algebraic numbers. Prove that every set which is a finite union of algebraic numbers and algebraic intervals is definable in the structure $\langle \mathbb{R}; 0, 1, +, \cdot; < \rangle$.

   (b) Use elimination of quantifiers to prove that every definable set in the structure $\langle \mathbb{R}; 0, 1, +, \cdot; < \rangle$ is a finite union of algebraic numbers and algebraic intervals.