Homework 8: Due Tuesday May 27, 1pm

1. (10 pts) Consider the theory of linear orders:

   $T = \{\forall x \forall y(x < y \to \neg(y < x)), \forall x \forall y \forall z(x < y \wedge y < z \to x < z), \forall x(\neg(x < x))\}$

   Show that there is no theory $T' \supseteq T$ whose models are exactly the wellorders. Recall that a wellorder is a linear order such that there is no infinite descending sequence $a_1 > a_2 > a_3 > \ldots$). [Hint: add infinitely many constants $c_1, c_2, c_3, \ldots$ to the language. Then for each n, construct a sentence ϕ_n starting that the first n of these constants $c_1, c_2, \ldots, c_n$ form a descending sequence. Then use compactness.]

2. (10 pts, no collab) Consider the theory T of graphs. Show that there is no theory $T' \supseteq T$ whose models are exactly the disconnected graphs. Recall that a graph is disconnected if there exist two vertices in the graph that do not have a path between them.

3. (10 pts) Recall our two versions of the compactness theorem.

   - Version 1: Suppose L is a language, T is a theory in L, and ϕ is a sentence in L. Then if $T \vDash \phi$ there is a finite subset $T_0 \subseteq T$ such that $T_0 \vDash \phi$.
   - Version 2: Suppose L is a language, T is a theory in L. Then T is satisfiable iff every finite subset $T_0 \subseteq T$ is satisfiable.

   Without assuming the completeness theorem or any of its consequences we have derived, use version 2 of the compactness theorem to prove version 1 of the compactness theorem.

4. (10 pts) Suppose $\delta > 0$ is a positive infinitesimal number in a nonstandard model of the reals $^*\mathcal{R}$. Show δ^2 is also a nonzero infinitesimal and that for every positive standard real number r, we have $0 < \delta^2 < r\delta$. (Here we say that δ^2 is infinitesimally smaller than δ).

   (homework continues on the next page...)

5. (20 pts) Suppose that $p(x)$ and $q(x)$ are nonzero polynomials and $\langle p_i(x)\rangle_{0\le i\le n}$ is the sequence

$$
\begin{aligned}
p_0(x) &= p(x)\\
p_1(x) &= p'(x)q(x)\\
p_2(x) &= -\text{remainder}(p_0(x), p_1(x))\\
p_3(x) &= -\text{remainder}(p_1(x), p_2(x))\\
&\vdots\\
p_n(x) &= -\text{remainder}(p_{n-1}(x), p_{n-2}(x))
\end{aligned}
$$

so that $p_n(x)$ is nonzero, but $p_n(x)$ divides into $p_{n-1}(x)$ with a remainder of 0. Let $s(-\infty)$ be the sequence giving the sign of each $p_i(x)$ as $x \to -\infty$ and $s(\infty)$ be the sequence giving the sign of each $p_i(x)$ as $x \to \infty$.

(a) Suppose $p_n(x)$ is constant, so $p(x)$ does not share any roots with $p'(x)q(x)$. Then show the number of sign changes in the sequence $s(-\infty)$ minus the number of sign changes in the sequence $s(\infty)$ is equal to the number of roots of $p(x)$ where $q(x) > 0$ minus the number of roots of $p(x)$ where $q(x) < 0$.

(b) Now show that the above is true for all $p(x)$ and $q(x)$ even when $p_n(x)$ is not constant. [Hint: let $g(x)$ be the gcd of $p(x)$ and $p'(x)q(x)$. Then divide the sequence $p_0(x), p_1(x), \ldots$ by $g(x)$.]