Homework 6: Due Tuesday May 13, 1pm

1. (15 pts) Let L be a first order language, and ϕ a sentence of L. Then the *spectrum* of ϕ is the set of possible sizes of universes of *finite* models of ϕ. That is,

$$\text{spectrum}(\phi) = \{n \geq 1 : \text{ there is a model } M \text{ of } \phi \\ \text{so that the universe of } M \text{ has exactly } n \text{ elements}\}$$

   For each of the sets X below find a language L and sentence ϕ such that spectrum$(\phi) = X$.

   (a) $X = \emptyset$

   (b) $X = \{1, 2, 3, \ldots\}$

   (c) $X = \{n \geq 1 : \ n \text{ is even}\}$.

   (d) $X = \{n \geq 1 : \ n \text{ is a square}\}$.

2. (No collab 10 pts) Suppose S is a set of first order formulas, and ϕ and ψ are first order formulas. Then show:

   (a) (The deduction theorem) $S \cup \{\phi\} \vdash \psi$ iff $S \vdash \phi \to \psi$.

   (b) (Proof by contradiction) If $S \cup \{\phi\} \vdash \psi$ and $S \cup \{\phi\} \vdash \neg\psi$, then $S \vdash \neg\phi$.

3. (15 pts) Show using our formal proof system that:

   (a) $\vdash \forall x(P(x) \wedge Q(x)) \leftrightarrow (\forall x P(x) \wedge \forall x Q(x))$.

   (b) $\{P(x), \forall y(P(y) \to (\forall z Q(z)))\} \vdash \forall x Q(x)\}$.

   (c) $\vdash \exists x \forall y R(x, y) \to \forall y \exists x R(x, y)$

4. (20 pts) Let L be the language containing a single binary relation $<$. Let T be the theory of dense linear orderings without endpoints

$$T = \{\forall x(\neg x < x), \forall x \forall y(x < y \to \neg y < x), \forall x \forall y(x = y \vee x < y \vee x > y), \\ \forall x \forall y(x < y \to \exists z(x < z \wedge z < y)), \forall x \exists y(y > x), \forall x \exists y(y < x)\}$$

(a) Suppose ϕ is a formula with no quantifiers in the variables $x_1, \ldots, x_k$. Show that ϕ is equivalent to a formula of the form:

$$(\theta_{1,1} \wedge \ldots \wedge \theta_{1,n_1}) \vee (\theta_{2,1} \wedge \ldots \wedge \theta_{2,n_2}) \vee \ldots \vee (\theta_{m,1} \wedge \ldots \wedge \theta_{m,n_m})$$

where each $\theta_{n,m}$ is of one of the forms: $x_i < x_j$ or $\neg x_i < x_j$ or $x_i = x_j$ or $\neg x_i = x_j$ for some $i, j \leq k$.

(b) Show that for each formula in the language L of the form $\exists x_i(\phi)$ where ϕ has no quantifiers, there is a formula ϕ^* with no quantifiers such that T logically implies $(\exists x_i \phi) \leftrightarrow \phi^*$. [Hint: Assume ϕ is in the form described above. Then show either $\exists x_i \phi$ is unsatisfiable, (hence equivalent to a formula which is always false), or we can obtain ϕ^* from ϕ by removing every $\theta_{n,m}$ which uses the variable x_i].

(c) Use this to show for each sentence ψ in the language L, either T logically implies ψ or T logically implies $\neg\psi$. (That is, T is complete).