Homework 5: Due Tuesday May 5, 1pm

1. (5 pts) Write a sentence φ in the language whose only relation is $=$ such that φ is true in a structure M iff M has a universe with exactly 6 elements.

2. (5 pts) Find a sentence which is true in the structure $\langle \mathbb{Q}; < \rangle$ but not in the structure $\langle \mathbb{Z}; < \rangle$, where $\mathbb{Z}$ is the set of integers, and $\mathbb{Q}$ is the set of rational numbers.

3. (5 pts) Show that there is no structure M with a finite universe such that exactly one element of the universe of M is not definable.

4. (No collab. 10 pts) Let $S \colon \mathbb{N} \to \mathbb{N}$ be the successor function: $S(n) = n+1$. Call a set $X \subseteq \mathbb{N}$ eventually periodic if there is a p (a *period*) and an n_0 such that for all $n \geq n_0$, we have

$$n \in X \leftrightarrow n + p \in X.$$

   Show that every eventually periodic set is definable in the structure $\langle \mathbb{N}, 0, S, + \rangle$.

5. (10 pts) Use the automorphism method to show that the set of rational numbers is not definable in $\langle \mathbb{R}; 0, 1, \cdot \rangle$.

6. (15 pts) Use the automorphism method to show that the function $+$ is not definable in the structure $\langle \mathbb{N}, \cdot \rangle$.

7. (15 pts) Prove that there are no nontrivial automorphisms of the structure $\langle \mathbb{R}; 0, 1, +, \cdot \rangle$. (Recall that an automorphism is called nontrivial if it is not equal to the identity function $\pi(x) = x$).

8. (8 pts) Find equivalent formulas to the following that are in prenex normal form:

   (a) $\neg \forall x [\neg \big(\exists y (P(y)) \vee P(x) \vee \neg \forall z (R(x, z)) \big)]$

   (b) $\neg \forall x (P(x) \to \neg (\exists y Q(x, y) \vee \exists y R(x, y)))$.

   (continued...)

9. (A small amount of extra credit) Prove there is two player game of perfect information which lasts infinitely many moves, and has no winning strategy for either player. [Hint: First, for each $X \subseteq \mathbb{Z}$, define $X + m = \{n + m : n \in X\}$. Say that X is *periodic* if there is a nonzero $m \in \mathbb{Z}$ such that $X + m = X$. Let $V = \{X \subseteq Z : X \text{ is not periodic}\}$, and consider the graph G whose vertices are the elements of V, and where there is an edge between vertices $X, Y \in V$ if $X + 1 = Y$ or $Y + 1 = X$.

   Show that G has a 2-coloring c. Now fix such a 2-coloring c and define a two-player game as follows. The players I and II alternate playing pairs (X_i, n_i) as follows:

$$\begin{array}{cc} \text{I} & \text{II} \\ (X_0, n_0) & \\ & (X_1, n_1) \\ (X_2, n_2) & \\ & \vdots \end{array}$$

   subject to the requirements that each X_i is a finite subset of $\mathbb{Z}$, n_i is a positive integer, and for each i, $X_i \subseteq \{-n_0, \ldots, n_0\}$, $n_{i+1} \geq n_i$, and $X_{i+1} \cap \{-n_0, \ldots, n_0\} = X_i$. After infinitely many moves, once for each $i \in \mathbb{N}$, the game is over, and we let $X = \bigcup_{i \in \mathbb{N}} X_i$. Now we declare the winner as follows: if X is periodic, I wins. If X is not periodic, then I wins iff X is assigned the first color used by c.

   Assuming that one of the two players has a winning strategy, show that there must be an X (which is obtained using this strategy) which is aperiodic, and such that there is an odd number k so that X and $X + k$ are assigned the same color by c. [Further hint: both X and $X + k$ will be the result of games played using this strategy]]