Homework 1: Due Tuesday April 8, 1pm

1. (10 pts) Write each of the following sentences as propositional formulas, using the propositional variables given.

   (a) It is sunny except if there are clouds.
   Let p: "it is sunny", q: "there are clouds".

   (b) It is raining or there are clouds, but it is not raining if there are not clouds.
   Let p: "it is raining", q: "there are clouds".

   (c) Assuming it rains, then there will be enough water.
   Let p: "it rains" , q: "there will be enough water".

   (d) It is cold outside, but the sun is shining and the sky is clear.
   Let p: "it is cold outside", q: "the sun is shining", r: "the sky is clear".

   (e) Your car is clean provided if it rained, you parked in the garage.
   Let p: "your car is clean", q: "it rained", r: "you parked in the garage".

2. (8 pts) Determine whether each of the following is a tautology, a contradiction or neither. Justify your answers.

   (a) $((p \to q) \to p) \to p$

   (b) $(q \wedge (p \to q)) \to p$

   (c) $p \wedge (p \leftrightarrow q) \wedge \neg q$

   (d) $((p \to q) \to r) \leftrightarrow (p \to (q \to r))$

3. (10 pts) Recall that a set S of formulas is called satisfiable if there is a single valuation which makes all the formulas in S true. For each number $n = 2, 3, 4, \ldots$ find a set $S = \{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ of n formulas that is not satisfiable, but so that every proper nonempty subset S' of S is satisfiable.

4. (10 pts)

   (a) Prove that the set $\{\neg, \vee\}$ is functionally complete.

   (b) Prove that the set $\{\to, \vee\}$ is not functionally complete.

(Homework continued on the next page)

5. (15 pts) Let φ be a formula containing only connectives from the set $\{\neg, \wedge, \vee\}$, and let φ^* be the formula obtained from φ by replacing each instance of $\wedge$ with $\vee$, each instance of $\vee$ with $\wedge$, and each of its propositional variables p_i with $(\neg p_i)$. (For instance, if we have $\varphi = (p_1 \vee (\neg p_2)) \wedge p_3$, then we obtain $\varphi^* = ((\neg p_1) \wedge (\neg(\neg p_2))) \vee (\neg p_3))$.

   Prove using structural induction that $\neg\varphi$ is equivalent to φ^*.

6. (20 pts) (No collaboration) A set S of formulas is said to be *independent* if for any formula $\varphi \in S$, φ is not implied logically by the rest of the formulas in S. (So for example, the empty set $\emptyset$ is independent, and a set $S = \{\varphi\}$ consisting of a single formula is independent iff φ is not a tautology).

   (a) Which of the following sets are independent? Justify your answers.

      i. $\{p \to q, q \to r, r \to q\}$

      ii. $\{p \vee q, p \to q, p \leftrightarrow q\}$

      iii. $\{p \to q, q \to r, p \to r\}$

   (b) Two sets of formulas S and R are called equivalent if S logically implies every formula $\varphi \in R$ and R logically implies every formula $\psi \in S$. Prove that for any *finite* set S of formulas, there is a subset $S' \subseteq S$ which is independent and S' is equivalent to S.