# 6c Lecture 3 & 4: April 8 & 10, 2014

## 3.1 Graphs and trees

We begin by recalling some basic definitions from graph theory.

**Definition 3.1.** A *(undirected, simple) graph* consists of a set of vertices $V$ and a set $E \subseteq V \times V$ of *edges* with the property that for every $x, y \in V$ we have:

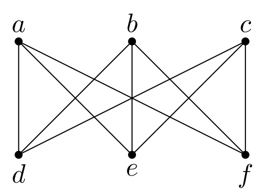
1. $(x, x) \notin E$.
2. $(x, y) \in E$ iff $(y, x) \in E$.

If $(x, y) \in E$, then we say that $x$ and $y$ are *adjacent*

Graphically, we represent graphs by drawing points to represent the vertices, and lines between them to represent the edges. For example, the graph whose vertices are $\{a, b, c, d, e, f\}$, and whose edges are

$$\{(a,d),(d,a),(a,e),(e,a),(a,f),(f,a),(b,d),(d,b),(b,e),(e,b),\\ (b,f),(f,b),(c,d),(d,c),(c,e),(e,c),(c,f),(f,c)\}$$

we can represent using the following picture:




**Definition 3.2.** A *path* from $x$ to $y$ in a graph is a finite sequence of distinct vertices $x_0, x_1, \ldots, x_n$ where $x_0 = x$, $x_n = y$, and for each $i < n$, $(x_i, x_{i+1})$ is an edge.

**Definition 3.3.** A graph is *connected* if for each two vertices $x, y$ in the graph, there is a path from $x$ to $y$.

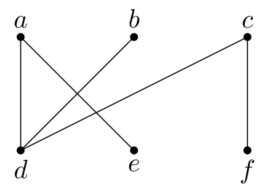
**Definition 3.4.** A *cycle* in a graph is a finite sequence of vertices $x_0, x_1, x_2, \ldots, x_n$, where $n \geq 3$, $x_0 = x_n$, $(x_i, x_{i+1})$ is an edge for every $i < n$, and $x_i \neq x_j$ for all $i, j < n$.

For example, the sequence $a, d, b, f, a$ is a cycle in the above graph, while $a, d, b, a$ is not (since $(b, a)$ is not an edge), and neither is $a, d, e, c$ (since $a$ and $c$ are not equal) or $a, d, a$ (since the length must be at least 3).

Note that if $x_0, x_1, x_2, \ldots, x_n$ is a cycle, then the sequence $x_0, \ldots, x_{n-1}$ must be a path in the graph.

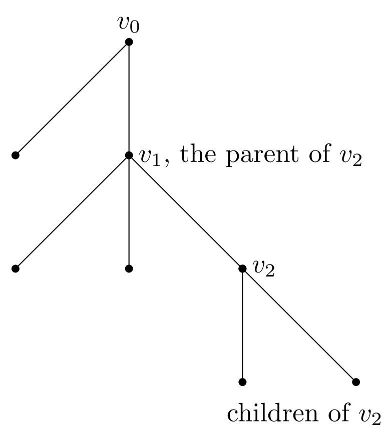
**Definition 3.5.** A *tree* is a connected graph having no cycles. Equivalently, a tree is a graph such that for every two vertices $x, y$ in the graph, there is a unique path from $x$ to $y$.

So for example, the graph we have drawn above is not a tree. However, the following graph is:




**Definition 3.6.** A *rooted tree* is a tree with a distinguished vertex called the root.

We will generally use $v_0$ to denote the root of a tree. Note that in a rooted tree, for every vertex $v$, there is a unique path $v_0, v_1, \ldots, v_n = v$ from the root to $v$. We say that a vertex is on *level* $n$ when the length of this path is $n$. Give a vertex $v \neq v_0$ in a rooted tree, its *parent* is the vertex $v_{n-1}$ such that $v_0, \ldots, v_{n-1}, v_n = v$ is the unique path from $v_0$ to $v$. The *children* of a vertex $v$ is the set of all vertices having $v$ as their parent.




## 3.2 König's lemma and compactness

**Definition 3.7.** An infinite branch of a rooted tree $T$ is an infinite sequence starting at the root $v_0, v_1, v_2, \ldots$ such that for every $i$, $v_{i+1}$ is a child of $v_i$.

**Definition 3.8.** A rooted tree is said to be *finitely splitting* if each node has only finitely many children.

**Theorem 3.9** (König's lemma)**.** *Suppose $T$ is a finitely splitting tree with infinitely many vertices. Then $T$ has an infinite branch.*

Recall the pigeon-hole principle for infinite sets. If $S_0 \cup S_1 \cup \ldots \cup S_n$ is infinite, then some $S_i$ must be infinite. (This is the contrapositive of the obvious fact that a finite union of finite sets is finite). The key to proving König's lemma is to repeatedly use the pigeon-hole principle to find a sequence $v_0, v_1, \ldots$ of vertices such that each $v_i$ has infinitely many vertices below it in the tree.

*Proof of König's lemma.* Given in class. □

Note that we need the condition that the graph is finitely splitting for König's lemma to be true. For example, consider the rooted tree whose root has infinitely many children labeled $\{1, 2, 3, \ldots\}$ but has no other vertices.

As a sidenote for those who know some topology, König's lemma is very closely related to compactness. For example, it is a good exercise to show that from König's lemma one can easily prove the Heine-Borel theorem that the unit interval is compact. That is, if $[0, 1]$ is covered by infinitely many open intervals of the form $(a_i, b_i)$, then there is a finite subset of these open intervals which still covers $[0, 1]$. König's lemma also is easily seen to be a simple reformulation of the compactness of Cantor space.

We are now ready to give some applications of König's lemma. As an abstract principle, König's lemma excels at taking a collection of finite objects, and converting them into a single coherent infinite object.

## 3.3 Graph colorings

**Definition 3.10.** A *$k$-coloring of a graph* is a function assigning one of the numbers $\{1, \ldots, k\}$ to each vertex of the graph such that adjacent vertices are assigned different numbers.

Recall also that given a set $S$ of the vertices of a graph $G$, the induced subgraph of $G$ on the set of vertices $S$ is the graph $G'$ whose vertices consist just of the vertices of $S$, and where there is an edge between two vertices in $G'$ iff there is an edge between those vertices in $G$.

**Theorem 3.11.** *Suppose that $G$ is an infinite graph on the set of vertices $\{x_1, x_2, \ldots\}$. Then $G$ has a $k$-coloring iff every finite induced subgraph of $G$ has a $k$-coloring.*

*Proof.* The direction $\rightarrow$ is easy, since a $k$-coloring of $G$ obviously gives a $k$-coloring of all of its induced subgraphs, simply by restricting the coloring to any smaller set of vertices. To prove the direction $\leftarrow$, we will use König's lemma to combine colorings of finite induced subgraphs of $G$ to yield a single coloring of all of $G$.

Consider the tree $T$ whose $n$th level consists of $k$-colorings of the induced subgraph of $G$ on the set $\{x_1, \ldots, x_n\}$, and where such a coloring $c$ on $\{x_1, \ldots, x_n\}$ is a child of a coloring $c'$ on $\{x_1, \ldots, x_{n-1}\}$ iff $c'$ assigns the same colors as $c$ to the vertices $x_1, \ldots, x_{n-1}$. It is easy to check that $T$ is finitely splitting, since there are only finitely many possible colorings of the vertices $\{x_1, \ldots, x_n\}$, and

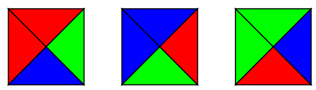
$T$ is also infinite, since for every $n$, by assumption there is at least one $k$-coloring of the induced subgraph on $\{x_1, \ldots, x_n\}$.

Hence, by König's lemma there is an infinite branch in the tree $T$. Two colorings on this infinite branch always agree on what color is assigned to a vertex whenever it occurs in both their domains by definition. Hence, by combining all these colorings, we obtain a single coloring of all of $G$. $\square$

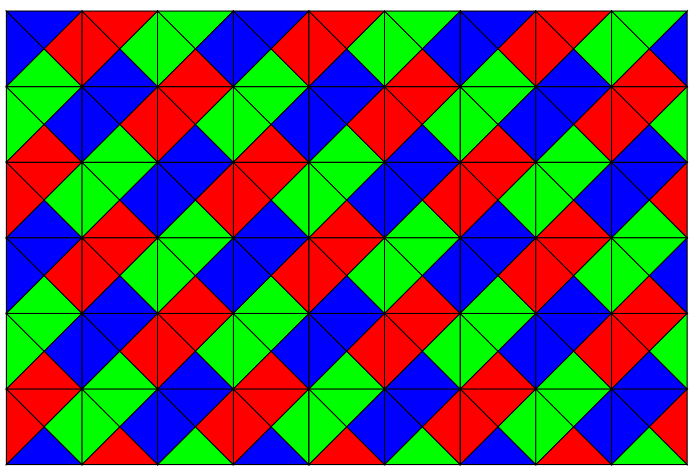
For those who know somthing about topology and cardinality, it is a good exercise to show that the above theorem is true also for uncountable graphs $G$.

## 3.4 Tiling problems

A Wang tile is a square tile whose edges have each been assigned a color. A *tileset* is a finite set of Wang tiles. For example, here is a picture of a tileset of size 3:



A *tiling* using a tileset is an arrangement of these tiles in a grid, where edges of adjacent tiles match each other. In a tiling we may repeat any tile as many times as we like, however, each tile may only be translated horizontally and vertically (and not reflected or rotated). Here is a picture of a tiling using the above three tiles:



Note that by repeating the above pattern over and over, we can tile the entire infinite plane. This gives an example of a *periodic tiling*, a tiling such that there is some $m \times n$ rectangle such that the tiling consists entirely of this rectangle repeatedly translated.

Now using König's lemma, we can prove the following theorem:

**Theorem 3.12.** *A finite set of Wang tiles can tile the infinite plane iff it can tile every $n \times n$ square.*

*Proof.* Given in class. $\square$

In 1961 Hao Wang conjectured the following:

**Conjecture 3.13** (Wang's conjecture)**.** *A finite tileset can tile the infinite plane iff it has a periodic tiling.*

If Wang's conjecture were true, it would have the following nice consequence:

**Proposition 3.14.** *If Wang's conjecture is true, then there is an algorithm for checking (in a finite time, always outputting the correct answer) whether a finite tileset can tile the infinite plane.*

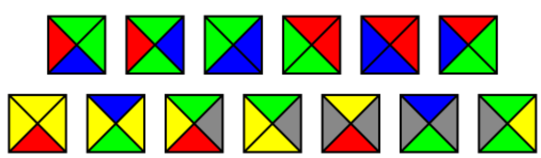
*Proof.* The algorithm goes as follows. For each $n$ in order, check first whether there is an $n \times m$ rectangle for any $m < n$ which can periodically tile the plane (if so output that there is a tiling of the plane). Then check if there no tiling at all of an $n \times n$ square (if there is none, then output that there is no tiling of the plane).

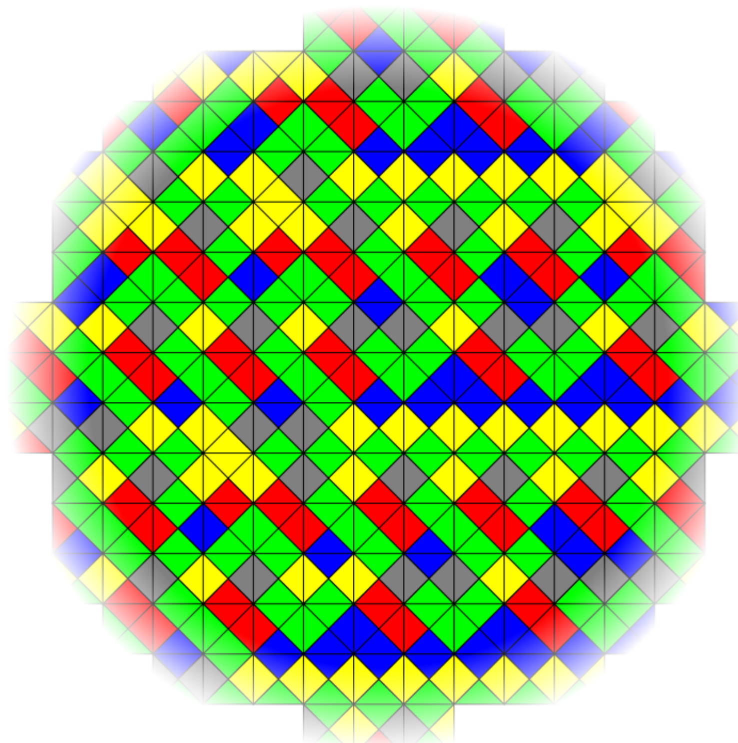
This algorithm will always halt assuming Wang's conjecture, since either there is a tiling of the plane (and hence a periodic tiling by Wang's conjecture which we will eventually find), or there is no tiling of the plane (and hence by our above theorem no tiling of some $n \times n$ rectangle, which we will eventually find). $\square$

In 1966, Berger proved a startling result refuting Wang's conjecture in a strong way.

**Theorem 3.15** (Berger)**.** *There is no algorithm for checking in a finite amount of time whether a given finite tileset can tile the infinite plane.*

We will discuss the proof of Berger's theorem later in the class. However, note that Berger's theorem implies that Wang's conjecture is false (since we have shown that it would give such an algorithm). In fact, by the contrapositive of Proposition 3.14, it implies that there is a finite set of tiles which can tile the infinite plane, but only aperiodically. An example of such a set of tiles is shown below, taken from `http://en.wikipedia.org/wiki/File:Wang_tesselation.svg`





## 3.5 The compactness theorem for propositional logic

We now use König's lemma to prove the compactness theorem for propositional logic. We will give two versions of the compactness theorem. The first is as follows:

**Theorem 3.16** (The compactness theorem for propositional logic, I)**.** *If $S = \{\phi_1, \phi_2, \ldots\}$ is a set of formulas in the propositional variables $\{p_1, p_2, \ldots\}$, then $S$ is satisfiable iff every finite subset of $S$ is satisfiable.*

*Proof.* The proof was given in class, and was quite similar in spirit to Theorem 3.11. The direction $\rightarrow$ is trivial. For the direction $\leftarrow$, we made a tree whose $n$th level consists of valuations of the variables $\{p_1, \ldots, p_n\}$ that do not make the formulas $\phi_1, \ldots, \phi_n$ false, arranged by compatibility. Then we showed that an infinite branch gave a valuation of $\{p_1, p_2, \ldots\}$ making all the formulas of $S$ true. $\square$

We next give another version of the compactness theorem. However, first we will need the following lemma:

**Lemma 3.17.** *Suppose $\phi$ is a formula and $S$ is a set of formulas. Then $S$ implies $\phi$ iff $S \cup \{\neg\phi\}$ is unsatisfiable.*

*Proof.* If $S$ implies $\phi$, then every valuation making the formulas of $S$ true make $\phi$ true. Hence, there is no valuation making all the formulas of $S$ true and $\phi$ false. Hence, $S \cup \{\neg\phi\}$ is unsatisfiable.

Conversely, if $S \cup \{\neg\phi\}$ is unsatisfiable, it must be that every valuation making all the formulas of $S$ true makes $\neg\phi$ false. Hence, every valuation making all the formulas of $S$ true must make $\phi$ true. Hence $S$ implies $\phi$. $\square$

**Theorem 3.18** (The compactness theorem for propositional logic, II)**.** *Suppose $\phi$ is a formula and $S = \{\phi_0, \phi_1, \ldots\}$ is a set of formulas. Then $S$ implies $\psi$ iff there is a finite subset $S' \subseteq S$ such that $S'$ implies $\phi$.*

*Proof.* $S$ implies $\psi$ iff $S \cup \{\neg\psi\}$ is unsatisfiable (by Lemma 3.17) iff there is a finite subset of $S \cup \{\neg\psi\}$ that is unsatisfiable (by the compactness theorem) iff there is a finite subset $S' \subseteq S$ such that $S' \cup \{\neg\psi\}$ is unsatisfiable iff there is a finite subset $S' \subseteq S$ such that $S'$ implies $\psi$ (by Lemma 3.17). $\square$

It is a good exercise to show that version II of the compactness theorem also easily implies version I.