6c Lecture 2: April 3, 2014

2.1 Functional completeness, normal forms, and structural induction

Before we begin, lets give a formal definition of a truth table.

Definition 2.1. A *truth table* for a set of propositional variables, is a function which assigns each valuation of these variables either the value true or false. Given a formula ϕ, the *truth table of* ϕ is the truth table assigning each valuation of the variables of v the corresponding truth value of ϕ.

We are ready to begin:

Definition 2.2. We say that a set S of logical connective is *functionally* complete if for every finite set of propositional variables $p_1, \ldots, p_n$ and every truth table for the variables $p_1, \ldots, p_n$, there exists a propositional formula ϕ using the variables $p_1, \ldots, p_n$ and connectives only from S so that ϕ has the given truth table.

We will soon show that the set $\{\neg, \wedge, \vee\}$ is functionally complete. Before this, lets do a quick example.

Here is an example of a truth table for the variables p_1, p_2, p_3:

p_1	p_2	p_3	
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	F

If $\{\neg, \wedge, \vee\}$ is functionally complete, then we must be able to find a formula using only $\{\neg, \wedge, \vee\}$ which has this given truth table (indeed, we must be able to do this for every truth able). In this case, one such a formula is $p_1 \wedge (\neg p_2 \vee p_3)$.

There is a much more systematic way of taking a truth table, and then finding a formula implementing it. We can create a formula of the form $\psi_1 \vee \psi_2 \vee \ldots \vee \psi_n$ where we consider each valuation of the variables in our truth table, and for the ith valuation assigned true in our truth table, we let ψ_i be a formula which is true iff the variables have this valuation. For example, such a formula for the above truth table is:

$$(p_1 \wedge p_2 \wedge p_3) \vee (p_1 \wedge \neg p_2 \wedge p_3) \vee (p_1 \wedge \neg p_2 \wedge \neg p_3)$$

We now use this idea to prove the following theorem:

Theorem 2.3. *The set $\{\neg, \wedge\}$ is functionally complete.*

Proof. Given in class. $\square$

A corollary of our proof is that every formula is equivalent to a formula of particular form.

Definition 2.4. Say that a formula ϕ is in *disjunctive normal form* (DNF) is ϕ is of the form $\phi = \psi_1 \vee \psi_2 \vee \ldots \vee \psi_n$, where each ψ_i is of the form $\psi_i = \ell_{i,1} \wedge \ldots \wedge \ell_{i,k_i}$, where each $\ell_{i,j}$ is a *literal*, i.e. either a propositional variable p_m or its negation $\neg p_m$.

Corollary 2.5. *Every propositional formula is equivalent to a formula in disjunctive normal form.*

Proof. Given ϕ, we may take its truth table, and then use the argument in Theorem 2.3 to produce a formula with this truth table in DNF. $\square$

This corollary will be important to us in the future; when we want to prove that every formula has some property (which is invariant under passing to equivalent formulas), then it will be enough to proof this property holds just for formulas in DNF which have a much simpler structure than arbitrary formulas.

Now that we know $\{\neg, \wedge, \vee\}$ is functionally complete, in order to show that any other set of propositional connectives is logically complete, it suffices to be able to use these connectives to give formulas equivalent to $\neg p$, $p \wedge q$, and $p \vee q$. So for example:

Theorem 2.6. *The set $\{\neg, \wedge\}$ is functionally complete.*

Proof. Since $\{\neg, \wedge, \vee\}$ is functionally complete, it suffices to use the connectives $\neg$ and $\wedge$ to create a formula equivalent to $p \vee q$. But by De-Morgan's law, $p \vee q$ is equivalent to $\neg(\neg p \wedge \neg q)$. $\square$

Induction on formulas is a very important tool used for proving that is true for every formula. We begin by proving that the property is true of all formulas consisting of a single propositional variable (this is called the base case). Then, we prove that if ϕ and ψ are formulas with the given property, then applying any of our logical connectives to ϕ and ψ produces another formula with this property (this is the inductive case). Since every formula is obtained starting with propositional variables and then repeatedly applying connectives, this shows the theorem. Our next theorem uses this technique to show that the set $\{\neg, \leftrightarrow\}$ is not functionally complete.

Theorem 2.7. *The set $\{\neg, \leftrightarrow\}$ is not functionally complete.*

Proof. We will show the following statement using induction on formulas. Let $n \geq 2$. Then every formula in the variables $p_1, \ldots p_n$ which only uses the connectives $\neg$ and $\leftrightarrow$ has an even number of true and an even number of false values in its truth table. This proves the theorem, since there are certainly truth tables assigning a value of true to an odd number of valuations.

Base case. Consider a formula consisting just of a single propositional variable p_i. Then the truth table for p_i will have 2^{n-1} many values where it is true, and 2^{n-1} many values where it is false. Both of these numbers are even since $n \geq 2$.

Inductive cases. ($\neg$): suppose ϕ is a formula having an even number of true values and an even number of false values in its truth table. Then this is also true for $\neg\phi$, since $\neg\phi$ is true iff ϕ is false.

($\leftrightarrow$): suppose ϕ and ψ are formulas having an even number of true values and an even number of false values in their truth table. We must show $\phi \leftrightarrow \psi$ also has an even number of true values and an even number of false values in its truth table. Let a be the number of rows in which ϕ is true and b be the number of rows in which ψ is true, and c be the number of rows in which both ϕ and ψ are true. Note a and b are even. Since the formula $\phi \leftrightarrow \psi$ is false iff exactly one of ϕ and ψ are true, the number of rows in which $\phi \leftrightarrow \psi$ is false is equal the number of rows in which ϕ is true and ψ is false plus the number of rows in which ψ is true and ϕ is false. This is equal to $(a - c) + (b - c) = a + b - 2c$ which is even since a and b are even. So $\phi \leftrightarrow \psi$ is false for an even number of rows and thus also true for an even number of rows. $\square$