6c Lecture 1: April 1, 2014

1.1 Propositions and propositional connectives

Definition 1.1. A *proposition* is a statement that is true or false.

Examples of propositions are “$1 + 1 = 3$”, “there are infinitely many prime numbers”, “$e^\pi - \pi = 20$”, and “every planar graph can be colored using four colors”. We will usually use the letters $p, q, r, \ldots$ to stand for propositions.

You should be careful to distinguish a proposition from statements like “$x^2 + 2x = 5$” which is not a proposition, but will become a proposition after we specify more information (here, a value for x). Statements like this are called *propositional functions*.

Definition 1.2. A *propositional connective* is a way of combining propositions to obtain another proposition in such a way that the truth or falsity of the compound proposition depends only on the truth or falsity of the components.

We discuss some common propositional connectives, and give *truth tables* for them, which specify how their truth values depends on the truth values of their component propositions:

- Negation (not). The negation of a single proposition p is denoted $\neg p$, and $\neg p$ is true if and only if p is false.

p	$\neg p$
T	F
F	T

- Conjunction (and). The conjunction of two propositions p and q is denoted $p \wedge q$ and is true if and only if both p and q are true.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

- Disjunction (or). The disjunction of two propositions p and q is denoted $p \vee q$ and is true if and only if either p is true, q is true, or both are true.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

- Implication (implies). The implication $p \to q$ is true if whenever p is true, then q is also true. If on the other hand p is false, then $p \to q$ is true (and is said to be vacuously true).

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

- Equivalence (if and only if, abbreviated iff). The biconditional $p \leftrightarrow q$ is true whenever both p and q are true, or both p and q are false.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

It is important to beware of a few pitfalls that arise from differences between our colloquial meanings of some of these connectives in English, and the precise mathematical meanings we have assigned to them above.

First, note that for us, the connective "or" has the property that $p \vee q$ is true even when p and q are both true. This is different from the way it is sometimes used in English where we implicitly mean that either p or q is true, but not both (e.g. "I will arrive on Monday, or I will arrive on Tuesday"). The connective which is true if either p is true or q is true but not both is called called exclusive or, and is abbreviated xor. In colloquial English this is sometimes phrased "either p or q".

p	q	p xor q
T	T	T
T	F	F
F	T	F
F	F	T

The second important pitfall to be aware of is that in mathematics, implies has nothing whatsoever to do with causality. Mathematically, "p implies q" does not mean "if p is true then this causes q to be true" except is a very formal sense. All we care about is that the truth values of p and q together maker $p \to q$ true; p and q may have nothing to do with each other. So for example, "there is a number which is both even and odd $\to \pi = 3$" is (vacuously) true, and "the derivative of sine is cosine $\to$ Fermat's last theorem is true" is true because both components are true.

We have a vast number of ways of talking about implication in English. Here are a few ways of saying $p \to q$:

- if p, then q. (This is the most common one used in mathematics).

- if p, q.

- p is sufficient for q.
- p only if q.
- a necessary condition for p is q.
- q follows from p
- q whenever p
- q unless $\neg p$.
- q when p
- q if p

Notice that the order of p and q is switched in the last several examples.

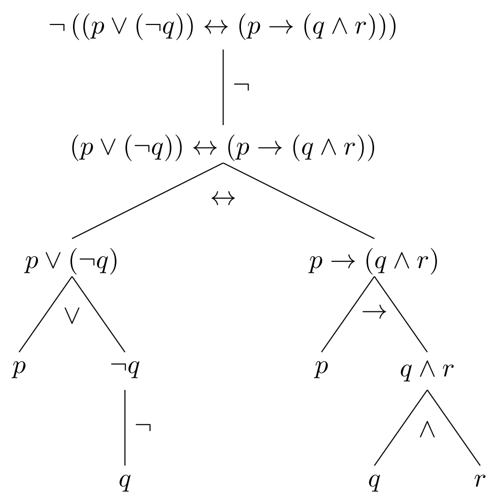
From now on we'll usually restrict the logicall connectives we use to the ones discussed above: $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$.

Next, we discuss combining logical connectives to create formulas. A *formula of propositional logic* is a proposition created from propositional variables (we'll usually use $p, q, r, \ldots$ or $p_1, p_2, p_3, \ldots$) by applying logical connectives. For example, p, $p \vee (\neg q)$, $(p \rightarrow q) \rightarrow r$, $\neg(p \rightarrow (q \rightarrow (r \vee \neg p))$, etc. We'll often use the following formal definition.

Definition 1.3. Given a set of propositional variables $p_1, p_2, p_3, \ldots$, the set of *propositional formulas* in these variables is the smallest set containing $p_1, p_2, p_3, \ldots$ and closed under applying the logical connectives $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$.

We will usually use the lowercase Greek letters $\phi, \psi, \theta, \ldots$ to stand for formulas.

One way of thinking of formulas is in terms of a tree structure reflecting how the formula is built out of its propositional variables using connectives. These are sometimes called *parse trees*, and you can likewise think of them as being the circuit used to create our compound proposition.



One efficient way to make a truth table for such a compound proposition is to iteratively make truth tables for each of the entries in its parse tree. For example,

p	q	r	$p \vee (\neg q)$	$q \wedge r$	$p \to (q \wedge r)$	$(p \vee (\neg q)) \leftrightarrow (p \to (q \wedge r))$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	T	F	F	F
T	F	F	T	F	F	F
F	T	T	F	T	T	F
F	T	F	F	F	T	F
F	F	T	T	F	T	T
F	F	F	T	F	T	T

You should always use parentheses when writing down propositional formulas to make clear in what order the connectives are applied! However, there is an (oft forgotten) order of operations on logical connectives. In order of highest to lowest precedence: $\neg, \wedge, \vee, \to, \leftrightarrow$, and were we associate parentheses to the left when a single connective is repeated. For example, $\neg p \vee q \wedge s \leftrightarrow p \to r \to s$ should be parenthesized $((\neg p) \vee (q \wedge s)) \leftrightarrow ((p \to r) \to s)$.

There are two situations in which we'll occasionally be lazy about omitting parentheses. The first is for negation, so we'll ocassionally write things like $p \vee \neg q$ instead of $p \vee (\neg q)$.

The second is for repeated use of $\vee$s or s. This is because for formulas made just out of the connective $\vee$, the order in which we parenthesize is not important. For example, $(p_1 \vee p_2) \vee (p_3 \vee p_4)$ is equivalent to $p_1 \vee ((p_2 \vee p_3) \vee p_4)$, since both are true iff at least one of the variables p_1, p_2, p_3, p_4 are true. Similarly the order of parentheses is not important for formulas made out of just the connective $\wedge$; $p_1 \wedge p_2 \wedge p_3 \wedge p_4$ is true iff all of p_1, p_2, p_3, p_4 are true, no matter how it is parenthesized. For this reason, we will often omit parentheses in these cases, writing $p_1 \vee p_2 \vee \ldots \vee p_n$ and $p_1 \wedge p_2 \wedge \ldots \wedge p_n$. These facts we have just mentioned can be proved using a technique called induction on formulas which we will discuss in the next lecture.

1.2 Valuations, satisfiability, logical implication and equivalence

Definition 1.4. A *valuation* of a set S of propositional variables is assignment of a truth value to each of these variables of S. Given a formula ϕ and a valuation v assigning truth values to all the propositional variables in ϕ, we say v *satisfies* ϕ if ϕ is true when its propositional variables are valued according to v.

For example, if we choose the valuation v making p true, and q false, then the formula $\neg p \vee \neg q$ is satisfied by this valuation.

Definition 1.5. A formula ϕ is *satisfiable* if there is some valuation which makes it true. Otherwise ϕ is said to be *unsatisfiable*, or *contradictory*.

For example $p \wedge q$ is satisfiable (choose p to be true and q to be true), while $p \wedge \neg p$ is not satisfiable (as you can easily check that it is false for every valuation).

Similarly, we can define satisfiability for a set of formulas:

Definition 1.6. A set of formulas S is said to be *satisfiable* if there is a single valuation which makes every formula $\phi \in S$ true.

An important class of formulas are those which are true for every valuation of their variables:

Definition 1.7. A formula ϕ is said to be a *tautology* if it is true for every valuation.

Here are some important examples of tautologies. You should check yourself that they are actually tautologies:

1. $\neg(p \wedge q) \leftrightarrow (\neg p \vee \neg q)$
2. $\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$ (These first two tautologies are called De Morgan's laws).
3. $p \wedge (q \vee r) \leftrightarrow ((p \wedge q) \vee (p \wedge r))$
4. $p \vee (q \wedge r) \leftrightarrow ((p \vee q) \wedge (p \vee r))$
5. $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$
6. $(p \rightarrow q) \leftrightarrow (\neg p \vee q)$
7. $(p \leftrightarrow q) \leftrightarrow ((p \rightarrow q) \wedge (q \rightarrow p))$
8. $p \vee \neg p$

Next, we discuss equivalence of formulas.

Definition 1.8. Two formulas ϕ and ψ are *equivalent* if for every valuation v of their variables, v satisfies ϕ iff v satisfies ψ.

It is easy to prove that ϕ is equivalent to ψ if and only if $\phi \leftrightarrow \psi$ is a tautology. For this reason, tautologies of the form $\phi \leftrightarrow \psi$ are quite important, because we can think of them as rules which allow us to transform formulas into other equivalent formulas.

For example, $((p \rightarrow q) \wedge \neg q) \rightarrow \neg p$ is equivalent to $((\neg p \vee q) \wedge \neg q) \rightarrow \neg p$ by tautology (6) above, which is equivalent to $((\neg q \wedge \neg p) \vee (\neg q \wedge q)) \rightarrow \neg p$ by tautology (3) above, which is equivalent to $(\neg q \wedge \neg p) \rightarrow \neg p$, since $(\neg q \wedge q)$ is always false, and finally $(\neg q \wedge \neg p) \rightarrow \neg p$ is always true, since whenever $(\neg q \wedge \neg p)$ is true, then $\neg p$ is true. Hence, the formula $((p \rightarrow q) \wedge \neg q) \rightarrow \neg p$ is a tautology.