Homework 9, due Thursday March 12 at 1pm

1. (a) (15 pts) Show that there is a *universal* Martin-Löf test. That is, there is a Martin-Löf test $S_0, S_1, \ldots$ such that for every Martin-Löf test $R_0, R_1, \ldots$, we have $\cap_i N_{S_i} \supseteq \cap_i N_{R_i}$. [Hint: Combine all possible Martin-Löf tests into one test using the fact that $\sum_{i \geq 1} 2^{-k} 2^{-i} = 2^{-k}$.]

   (b) (15 pts) Show that there is a computable infinite tree $T \subseteq 2^{<\omega}$ whose elements are all Martin-Löf random. Conclude that there is a low Martin-Löf random set. [Hint: Consider the tree of strings x such that no prefix of x is in S_1 where S_1 is the first level of a universal Martin-Löf test]

2. (20 pts, no collab) Finish our proof from class that if X is not Martin-Löf random, then for all d, there exists an n such that $K(X \upharpoonright n) < n - d$.

3. (20 pts) Let Ω_K be the halting probability of the universal prefix-free machine we defined in class. Ω_K is a real number in $[0, 1]$, but we can also think of it as the infinite binary sequence coresponding to its representation in binary. Show that Ω_K can compute $0'$.

4. (20 pts) Show that no r.e. set is random. [Hint: first show that if X is an infinite r.e. set, then X has an infinite computable subset Y so that $Y \subseteq X$.]

Extra credit problems. You may do these problems anytime during the quarter and hand them in to me directly

5. (20 pts) Show that there are r.e. sets $X, Y \subseteq \mathbb{N}$ such that $X \not\geq_T Y$ and $Y \not\geq_T X$.

6. (20 pts).

   Finish the proof of the Boone-Novikov theorem we gave in class as follows. Suppose $G = \langle S; R \rangle$ is a group and $A, B \leq G$ are isomorphic subgroups with isomorphism $\phi\colon A \to B$. Then the HNN extension of G with respect to A, B, and ϕ is $G^* = \langle G, t; t^{-1}at = \phi(a) \rangle_{a \in A}$. Now fix a set of right coset representatives of A and B. That is, pick exactly one element of each set in $\{Ag : g \in G\}$ and $\{Bg : g \in G\}$ and such that our representatives of Ae and Be are both e. Now given any $n \geq 0$, we say that a word $g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} g_2 \dots t^{\epsilon_n} g_n$ (and note that any g_i may be equal to the identity e) is in *normal form* if:

   - g_0 is an arbitrary element of G
   - $\epsilon_i \in \{-1, 1\}$ for all i, and
   - For all $i > 0$ if $\epsilon_i = -1$ then g_i is one of our right coset representatives of A.
   - For all $i > 0$ if $\epsilon_i = 1$, then g_i is one of our right coset representatives of B.
   - There is no consecutive subsequence $t^{\epsilon} e t^{-\epsilon}$.

   Let S be the space of finite sequences of the form $(g_0, t^{\epsilon_1}, g_1, \dots, t^{\epsilon_n}, g_n)$ that obey our normal form rules as above (but where we don't think of these sequences as having any group structure).

   (a) Show that every element of G^* is equivalent to a word in normal form.

   (b) Define an action of G^* on S by extending the following definition. For every $g \in G$, we define:

   $$g \cdot (g_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, g_2, \dots, t^{\epsilon_n}, g_n) = (gg_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, t_2, \dots, t^{\epsilon_n}, g_n)$$

   Next, if $\epsilon_1 = -1$ and $g_0 \in B$, then set

   $$t \cdot (g_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, g_2, \dots, t^{\epsilon_n}, g_n) = (\phi^{-1}(g_0)g_1, t^{\epsilon_2}, g_2, \dots, t^{\epsilon_n}, g_n)$$

   and otherwise, set

   $$t \cdot (g_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, g_2, \dots, t^{\epsilon_n}, g_n) = (\phi^{-1}(b), t, \hat{g_0}, t^{\epsilon_1}, g_1, t^{\epsilon_2}, g_2, \dots, t^{\epsilon_n}, g_n),$$

   where $\hat{g_0}$ is our coset representative of Bg_0, and $b \in B$ is such that $g_0 = b\hat{g_0}$.

Now check that we can define $t^{-1} \cdot (g_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, g_2, \ldots, t^{\epsilon_n}, g_n)$ in a way somewhat analogous to the above, but with B replaced by A so that together this defines an action of G^* on S. In particular, check that the definition is compatible with all the relations used to define G^*).

(c) Show using the above that if $g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} g_2 \ldots t^{\epsilon_n} g_n$ is a word in normal form that is equal to the identity then $n = 0$ and $g_0 = e$.

(d) Show that every element of G has a unique representation as a normal form by showing that if two normal forms are equal: $g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} g_2 \ldots t^{\epsilon_n} g_n = h_0 t^{\delta_1} g_1 t^{\delta_2} g_2 \ldots t^{\delta_m} h_m$, then $n = m$, $g_i = h_i$ and $\epsilon_i = \delta_i$ for all $i \leq n$.

(e) Show there is an embedding of G into G^*.

(f) Show that if H is a subgroup of G such that $\phi(H \cap A) = H \cap B$, and H^* is the subgroup of G^* generated by H and t, then $H^* \cap G = H$.

(g) Finish the proof of the Boone-Novikov theorem from class by using the facts proved about HNN extensions above to justify the two gaps in our proof.