Homework 7, due Tuesday March 3 at 1pm

1. (15 pts) Show that there is a $k \in \mathbb{N}$ such that if x is the shortest C-description of some string y, then $C(x) \geq |x| - k$. (i.e. shortest descriptions cannot be compressed by more than some fixed constant).

2. (15 pts, no collab) Give another proof as follows that C_U is not computable, where C_U is the universal machine ($U(0^n 1x) = M_n(x)$ we defined in class).

   Suppose C_U was computable. Define a computable function $f\colon \mathbb{N} \to \mathbb{N}$ which maps i to the index $f(i)$ for a machine $M_{f(i)}$ which on any input x finds the lex least string y such that $C(y) > i + 1$ and then outputs y. Now use the recursion theorem to derive a contradiction.

3. (20 pts) Show that $\{x : K(x) < |x|\} \equiv_T 0'$.

4. (a) (10 pts) Suppose that $S \subseteq 2^{<\omega}$ is a finite set of prefix-free strings. Show that $\sum_{x \in S} 2^{-|x|} \leq 1$.

   (b) (5 pts) Use this the above to show that if S is an infinite set of prefix-free strings, then $\sum_{x \in S} 2^{-|x|} \leq 1$.

   (c) (5 pts) Say that a function $D\colon \mathbb{N} \to \mathbb{N} \cup \infty$ satisfies the *weight condition* if $\sum_{x \in \mathrm{dom}(D)} 2^{-D(x)} \leq 1$. Show that K satisfies the weight condition.

   (d) (10 pts) Let $n_0, n_1, \ldots$ be a list of numbers such that $\sum_i 2^{-n_i} \leq 1$. Then show we can recursively define a sequece $x_0, x_1, \ldots$ of strings of lengths $|x_i| = n_i$ by setting x_i to be the leftmost string of length n_i which is incomparable with $\{x_0, \ldots, x_{i-1}\}$ (i.e. there is always such an x_i as above).

   (e) (10 pts) Show that K is the $\leq^+$-least function which is computably approximable from above and satisfies the weight condition.

Extra credit problems. You may do these problems anytime during the quarter and hand them in to me directly

5. (20 pts) Show that there are r.e. sets $X, Y \subseteq \mathbb{N}$ such that $X \not\geq_T Y$ and $Y \not\geq_T X$.

6. (20 pts).

   Finish the proof of the Boone-Novikov theorem we gave in class as follows. Suppose $G = \langle S; R \rangle$ is a group and $A, B \leq G$ are isomorphic subgroups with isomorphism $\phi\colon A \to B$. Then the HNN extension of G with respect to A, B, and ϕ is $G^* = \langle G, t; t^{-1}at = \phi(a) \rangle_{a \in A}$. Now fix a set of right coset representatives of A and B. That is, pick exactly one element of each set in $\{Ag : g \in G\}$ and $\{Bg : g \in G\}$ and such that our representatives of Ae and Be are both e. Now given any $n \geq 0$, we say that a word $g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} g_2 \dots t^{\epsilon_n} g_n$ (and note that any g_i may be equal to the identity e) is in *normal form* if:

   - g_0 is an arbitrary element of G
   - $\epsilon_i \in \{-1, 1\}$ for all i, and
   - For all $i > 0$ if $\epsilon_i = -1$ then g_i is one of our right coset representatives of A.
   - For all $i > 0$ if $\epsilon_i = 1$, then g_i is one of our right coset representatives of B.
   - There is no consecutive subsequence $t^{\epsilon} e t^{-\epsilon}$.

   Let S be the space of finite sequences of the form $(g_0, t^{\epsilon_1}, g_1, \dots, t^{\epsilon_n}, g_n)$ that obey our normal form rules as above (but where we don't think of these sequences as having any group structure).

   (a) Show that every element of G^* is equivalent to a word in normal form.

   (b) Define an action of G^* on S by extending the following definition. For every $g \in G$, we define:

   $$g \cdot (g_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, g_2, \dots, t^{\epsilon_n}, g_n) = (gg_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, t_2, \dots, t^{\epsilon_n}, g_n)$$

   Next, if $\epsilon_1 = -1$ and $g_0 \in B$, then set

   $$t \cdot (g_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, g_2, \dots, t^{\epsilon_n}, g_n) = (\phi^{-1}(g_0)g_1, t^{\epsilon_2}, g_2, \dots, t^{\epsilon_n}, g_n)$$

   and otherwise, set

   $$t \cdot (g_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, g_2, \dots, t^{\epsilon_n}, g_n) = (\phi^{-1}(b), t, \hat{g_0}, t^{\epsilon_1}, g_1, t^{\epsilon_2}, g_2, \dots, t^{\epsilon_n}, g_n),$$

   where $\hat{g_0}$ is our coset representative of Bg_0, and $b \in B$ is such that $g_0 = b\hat{g_0}$.

Now check that we can define $t^{-1} \cdot (g_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, g_2, \ldots, t^{\epsilon_n}, g_n)$ in a way somewhat analogous to the above, but with B replaced by A so that together this defines an action of G^* on S. In particular, check that the definition is compatible with all the relations used to define G^*).

(c) Show using the above that if $g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} g_2 \ldots t^{\epsilon_n} g_n$ is a word in normal form that is equal to the identity then $n = 0$ and $g_0 = e$.

(d) Show that every element of G has a unique representation as a normal form by showing that if two normal forms are equal: $g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} g_2 \ldots t^{\epsilon_n} g_n = h_0 t^{\delta_1} g_1 t^{\delta_2} g_2 \ldots t^{\delta_m} h_m$, then $n = m$, $g_i = h_i$ and $\epsilon_i = \delta_i$ for all $i \leq n$.

(e) Show there is an embedding of G into G^*.

(f) Show that if H is a subgroup of G such that $\phi(H \cap A) = H \cap B$, and H^* is the subgroup of G^* generated by H and t, then $H^* \cap G = H$.

(g) Finish the proof of the Boone-Novikov theorem from class by using the facts proved about HNN extensions above to justify the two gaps in our proof.