Homework 7, due Tuesday Feb 24 at 1pm

1. (15 pts, no collab) Show that if $T \subseteq 2^{<\omega}$ is a computable infinite tree that has exactly one infinite path X, then X is computable.

2. (20 pts) Suppose a tree $T \subseteq 2^{<\omega}$ is co-r.e. Show that there is a computable tree $T^* \subseteq 2^{<\omega}$ such that $[T] = [T^*]$.

3. Suppose $k \geq 2$. Show there is no single program φ_e such that for every $k+1$-DNR function F, φ_e^F is a k-DNR function. We will proceed by contradiction. Say a *k-string* is a finite string of numbers in $\{0, \ldots, k-1\}$. (So for example, a binary string is a 2-string). Say that a k-string is k-DNR if for every $n \in \mathrm{dom}(s)$, $s(n) \neq \varphi_n(n)$.

   (a) (10 pts) Use König's lemma to show that for every n, there is a length l and a number s such that for every $k+1$-DNR string r of length l, we have that $\varphi_e^r(n)$ halts within s steps. Further show that there is a program which given n finds such an l and s.

   (b) (10 pts) Fix n, l and s as above. Say that a set S of $k+1$-strings of length l is *n, l, s-large* if for every string t of length l, there is a string $r \in S$ such that for all $m \leq l$, either $\varphi_m(m)$ halts in s steps and $r(m) \neq \varphi_m(m)$ or $r(m) \neq t(m)$. Show that every n, l, s-large set contains some $k+1$-DNR string.

   (c) (10 pts) Fix n, l, and s as above. Let S_i be the set of $k+1$-strings r of length l such that $\varphi_e^r(n) = i$ for $i \in \{0, \ldots, k-1\}$. Show that some S_i is n, l, s-large.

   (d) (10 pts) Finish the proof by using the recursion theorem to show that you can construct an n such that $\varphi_n(n) = i$ where the corresponding S_i as defined above is large.

4. (20 pts) Modify our proof of the low basis theorem to show that for every infinite computable tree $T \subseteq 2^{<\omega}$, there is an infinite branch $X \in T$ such that for every total function $f \colon \mathbb{N} \to \mathbb{N}$ such that $f \leq_T X$, there is a computable function g such that for all n, $f(n) < g(n)$.

Extra credit problems. You may do these problems anytime during the quarter and hand them in to me directly

5. (20 pts) Show that there are r.e. sets $X, Y \subseteq \mathbb{N}$ such that $X \not\geq_T Y$ and $Y \not\geq_T X$.

6. (20 pts).

   Finish the proof of the Boone-Novikov theorem we gave in class as follows. Suppose $G = \langle S; R \rangle$ is a group and $A, B \leq G$ are isomorphic subgroups with isomorphism $\phi \colon A \to B$. Then the HNN extension of G with respect to A, B, and ϕ is $G^* = \langle G, t; t^{-1}at = \phi(a) \rangle_{a \in A}$. Now fix a set of right coset representatives of A and B. That is, pick exactly one element of each set in $\{Ag : g \in G\}$ and $\{Bg : g \in G\}$ and such that our representatives of Ae and Be are both e. Now given any $n \geq 0$, we say that a word $g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} g_2 \ldots t^{\epsilon_n} g_n$ (and note that any g_i may be equal to the identity e) is in *normal form* if:

   - g_0 is an arbitrary element of G
   - $\epsilon_i \in \{-1, 1\}$ for all i, and
   - For all $i > 0$ if $\epsilon_i = -1$ then g_i is one of our right coset representatives of A.
   - For all $i > 0$ if $\epsilon_i = 1$, then g_i is one of our right coset representatives of B.
   - There is no consecutive subsequence $t^\epsilon e t^{-\epsilon}$.

   Let S be the space of finite sequences of the form $(g_0, t^{\epsilon_1}, g_1, \ldots, t^{\epsilon_n}, g_n)$ that obey our normal form rules as above (but where we don't think of these sequences as having any group structure).

   (a) Show that every element of G^* is equivalent to a word in normal form.

   (b) Define an action of G^* on S by extending the following definition. For every $g \in G$, we define:

   $$g \cdot (g_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, g_2, \ldots, t^{\epsilon_n}, g_n) = (gg_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, t_2, \ldots, t^{\epsilon_n}, g_n)$$

   Next, if $\epsilon_1 = -1$ and $g_0 \in B$, then set

   $$t \cdot (g_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, g_2, \ldots, t^{\epsilon_n}, g_n) = (\phi^{-1}(g_0)g_1, t^{\epsilon_2}, g_2, \ldots, t^{\epsilon_n}, g_n)$$

   and otherwise, set

   $$t \cdot (g_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, g_2, \ldots, t^{\epsilon_n}, g_n) = (\phi^{-1}(b), t, \hat{g_0}, t^{\epsilon_1}, g_1, t^{\epsilon_2}, g_2, \ldots, t^{\epsilon_n}, g_n),$$

   where $\hat{g_0}$ is our coset representative of Bg_0, and $b \in B$ is such that $g_0 = b\hat{g_0}$.

Now check that we can define $t^{-1} \cdot (g_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, g_2, \ldots, t^{\epsilon_n}, g_n)$ in a way somewhat analogous to the above, but with B replaced by A so that together this defines an action of G^* on S. In particular, check that the definition is compatible with all the relations used to define G^*).

(c) Show using the above that if $g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} g_2 \ldots t^{\epsilon_n} g_n$ is a word in normal form that is equal to the identity then $n = 0$ and $g_0 = e$.

(d) Show that every element of G has a unique representation as a normal form by showing that if two normal forms are equal: $g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} g_2 \ldots t^{\epsilon_n} g_n = h_0 t^{\delta_1} g_1 t^{\delta_2} g_2 \ldots t^{\delta_m} h_m$, then $n = m$, $g_i = h_i$ and $\epsilon_i = \delta_i$ for all $i \leq n$.

(e) Show there is an embedding of G into G^*.

(f) Show that if H is a subgroup of G such that $\phi(H \cap A) = H \cap B$, and H^* is the subgroup of G^* generated by H and t, then $H^* \cap G = H$.

(g) Finish the proof of the Boone-Novikov theorem from class by using the facts proved about HNN extensions above to justify the two gaps in our proof.