Homework 6, due Tuesday Feb 17 at 1pm

1. (20 pts) Let φ_i be a computable listing of the sentences of arithmetic. Tarski's undefinability of truth states that $T = \{i : \varphi_i \text{ is true}\}$ is not arithmetically definable, so by Post's hierarchy theorem, there is no n such that $T \leq_T 0^{(n)}$. Prove the stronger theorem that T is many-one equivalent to $0^{(\omega)} = \{\langle i, j\rangle : i \in 0^{(j)}\}$.

2. (a) (10 pts) Show that for every n, there are finitely many $X_0, X_1, \ldots X_n \subseteq \mathbb{N}$ such that when $X_{i_0}, X_{i_1}, \ldots X_{i_k} \in \{X_0, X_1, \ldots X_n\}$ are distinct, then $X_{i_0} \oplus X_{i_1} \oplus X_{i_{k-1}} \not\geq_T X_{i_k}$.

   (b) (10 pts) Use the above fact to show that any finite partial order $(P, \leq_P)$ can be embedded into the Turing degrees; that is, there is a function f from elements of P to subsets of $\mathbb{N}$ such that for all $p, q \in P$, we have $p \leq_P q \leftrightarrow f(p) \leq_T f(q)$. [Hint: associate to each $p \in P$ an oracle X_p as above, and then map p to the join $\bigoplus_{q \leq_P p} X_q$.]

   (c) (10 pts) Show that there is a computable partial order P on $\mathbb{N}$ such that every other countable partial order Q can be embedded into P. (By P being computable we mean there is a program which takes natural numbers n, m as input and outputs whether $n \leq_P m$.)

   (d) (10 pts) Show that any countable partial order P can be embedded into the Turing degrees.

3. (10 pts, no collab) Suppose $A_0, A_1, \ldots \subseteq \mathbb{N}$. Show that $A = \{\langle i, j\rangle : i \in A_j\}$ is the least *uniform* upper bound for the A_i. That is, prove for all $B \subseteq N$, if there is single program which when run with the oracle B on input (i, j) outputs whether $i \in A_j$, then $B \geq_T A$.

4. (20 pts) Suppose $A_0 <_T A_1 <_T A_2 <_T \ldots$ are subsets of $\mathbb{N}$, and let $B = \{\langle i, j\rangle : i \in A_j\}$. Show there is a $C \subseteq \mathbb{N}$ such that $C \geq_T A_i$ for every i, and so that if $X \leq_T B$ and $X \leq_T C$, then there is an i such that $X \leq_T A_i$.

Extra credit problems. You may do these problems anytime during the quarter and hand them in to me directly

5. (20 pts) Show that there are r.e. sets $X, Y \subseteq \mathbb{N}$ such that $X \not\geq_T Y$ and $Y \not\geq_T X$.

6. (20 pts).

   Finish the proof of the Boone-Novikov theorem we gave in class as follows. Suppose $G = \langle S; R \rangle$ is a group and $A, B \leq G$ are isomorphic subgroups with isomorphism $\phi \colon A \to B$. Then the HNN extension of G with respect to A, B, and ϕ is $G^* = \langle G, t; t^{-1}at = \phi(a) \rangle_{a \in A}$. Now fix a set of right coset representatives of A and B. That is, pick exactly one element of each set in $\{Ag : g \in G\}$ and $\{Bg : g \in G\}$ and such that our representatives of Ae and Be are both e. Now given any $n \geq 0$, we say that a word $g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} g_2 \ldots t^{\epsilon_n} g_n$ (and note that any g_i may be equal to the identity e) is in *normal form* if:

   - g_0 is an arbitrary element of G
   - $\epsilon_i \in \{-1, 1\}$ for all i, and
   - For all $i > 0$ if $\epsilon_i = -1$ then g_i is one of our right coset representatives of A.
   - For all $i > 0$ if $\epsilon_i = 1$, then g_i is one of our right coset representatives of B.
   - There is no consecutive subsequence $t^{\epsilon} e t^{-\epsilon}$.

   Let S be the space of finite sequences of the form $(g_0, t^{\epsilon_1}, g_1, \ldots, t^{\epsilon_n}, g_n)$ that obey our normal form rules as above (but where we don't think of these sequences as having any group structure).

   (a) Show that every element of G^* is equivalent to a word in normal form.

   (b) Define an action of G^* on S by extending the following definition. For every $g \in G$, we define:

   $$g \cdot (g_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, g_2, \ldots, t^{\epsilon_n}, g_n) = (gg_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, t_2, \ldots, t^{\epsilon_n}, g_n)$$

   Next, if $\epsilon_1 = -1$ and $g_0 \in B$, then set

   $$t \cdot (g_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, g_2, \ldots, t^{\epsilon_n}, g_n) = (\phi^{-1}(g_0)g_1, t^{\epsilon_2}, g_2, \ldots, t^{\epsilon_n}, g_n)$$

   and otherwise, set

   $$t \cdot (g_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, g_2, \ldots, t^{\epsilon_n}, g_n) = (\phi^{-1}(b), t, \hat{g_0}, t^{\epsilon_1}, g_1, t^{\epsilon_2}, g_2, \ldots, t^{\epsilon_n}, g_n),$$

   where $\hat{g_0}$ is our coset representative of Bg_0, and $b \in B$ is such that $g_0 = b\hat{g_0}$.

Now check that we can define $t^{-1} \cdot (g_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, g_2, \ldots, t^{\epsilon_n}, g_n)$ in a way somewhat analogous to the above, but with B replaced by A so that together this defines an action of G^* on S. In particular, check that the definition is compatible with all the relations used to define G^*).

(c) Show using the above that if $g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} g_2 \ldots t^{\epsilon_n} g_n$ is a word in normal form that is equal to the identity then $n = 0$ and $g_0 = e$.

(d) Show that every element of G has a unique representation as a normal form by showing that if two normal forms are equal: $g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} g_2 \ldots t^{\epsilon_n} g_n = h_0 t^{\delta_1} g_1 t^{\delta_2} g_2 \ldots t^{\delta_m} h_m$, then $n = m$, $g_i = h_i$ and $\epsilon_i = \delta_i$ for all $i \leq n$.

(e) Show there is an embedding of G into G^*.

(f) Show that if H is a subgroup of G such that $\phi(H \cap A) = H \cap B$, and H^* is the subgroup of G^* generated by H and t, then $H^* \cap G = H$.

(g) Finish the proof of the Boone-Novikov theorem from class by using the facts proved about HNN extensions above to justify the two gaps in our proof.