Homework 5, due Tuesday Feb 10 at 1pm

1. (20 pts) Prove that there are countably many oracles $X_0, X_1, \ldots \subseteq \mathbb{N}$ such that if $i \neq j$, then $X_i \not\geq_T X_j$

2. (No collab, 20 pts) Prove that if $X \subseteq \mathbb{N}$ is not computable, then there is a $Y \subseteq \mathbb{N}$ such that $X \not\geq_T Y$ and $Y \not\geq_T X$.

3. (20 pts) Prove the Friedberg jump inversion theorem. If $X \geq_T 0'$, then there is a Y such that $Y' \equiv_T X$. [Hint: construct Y using a process which is computable from $0' \oplus X$. At step i, first extend your finite approximation to Y to either force the ith program to halt, or to not halt. Then make the next bit of Y equal the ith bit of X. Argue then at the end that Y' can compute X by figuring out what was done at each step i of the construction, and that $X \geq_T X \oplus 0' \geq_T Y$ and $X \geq_T 0'$, and so $X \geq_T 0' \oplus Y$, but since Y forces the jump, $0' \oplus Y \equiv_T Y'$.]

4. (5 pts) This problem requires a little more creativity than the rest, so is worth fewer points. Don't waste too much time on it. Show that there are r.e. sets $X, Y \subseteq \mathbb{N}$ such that $X \not\geq_T Y$ and $Y \not\geq_T X$.

5. (15 pts) A presentation of a group $\langle S; R\rangle$ is said to be computable if both S and R are computable sets (i.e. it is computable to determine if a particular relation is in R). Higman has proved that there is a finitely presented group which contains every computably presented group as a subgroup. Use Higman's theorem to prove that the word problem for finitely presented groups is undecidable. [Hint: first use an easy argument to show that there is a computably presented group with an undecidable word problem]

6. (20 pts of extra credit. You may do this problem anytime during the quarter and hand it in to me directly).

   Finish the proof of the Boone-Novikov theorem we gave in class as follows. Suppose $G = \langle S; R\rangle$ is a group and $A, B \leq G$ are isomorphic subgroups with isomorphism $\phi\colon A \to B$. Then the HNN extension of G with respect to A, B, and ϕ is $G^* = \langle G, t; t^{-1}at = \phi(a)\rangle_{a \in A}$. Now fix a set of right coset representatives of A and B. That is, pick exactly one element of each set in $\{Ag : g \in G\}$ and $\{Bg : g \in G\}$ and such that our representatives of Ae and Be are both e. Now given any $n \geq 0$, we say that a word $g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} g_2 \ldots t^{\epsilon_n} g_n$ (and note that any g_i may be equal to the identity e) is in *normal form* if:

   - g_0 is an arbitrary element of G
   - $\epsilon_i \in \{-1, 1\}$ for all i, and
   - For all $i > 0$ if $\epsilon_i = -1$ then g_i is one of our right coset representatives of A.
   - For all $i > 0$ if $\epsilon_i = 1$, then g_i is one of our right coset representatives of B.

- There is no consecutive subsequence $t^{\epsilon}et^{-\epsilon}$.

Let S be the space of finite sequences of the form $(g_0, t^{\epsilon_1}, g_1, \ldots, t^{\epsilon_n}, g_n)$ that obey our normal form rules as above (but where we don't think of these sequences as having any group structure).

(a) Show that every element of G^* is equivalent to a word in normal form.

(b) Define an action of G^* on S by extending the following definition. For every $g \in G$, we define:

$$g \cdot (g_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, g_2, \ldots, t^{\epsilon_n}, g_n) = (gg_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, t_2, \ldots, t^{\epsilon_n}, g_n)$$

Next, if $\epsilon_1 = -1$ and $g_0 \in B$, then set

$$t \cdot (g_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, g_2, \ldots, t^{\epsilon_n}, g_n) = (\phi^{-1}(g_0)g_1, t^{\epsilon_2}, g_2, \ldots, t^{\epsilon_n}, g_n)$$

and otherwise, set

$$t \cdot (g_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, g_2, \ldots, t^{\epsilon_n}, g_n) = (\phi^{-1}(b), t, \hat{g_0}, t^{\epsilon_1}, g_1, t^{\epsilon_2}, g_2, \ldots, t^{\epsilon_n}, g_n),$$

where $\hat{g_0}$ is our coset representative of Bg_0, and $b \in B$ is such that $g_0 = b\hat{g_0}$.

Now check that we can define $t^{-1} \cdot (g_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, g_2, \ldots, t^{\epsilon_n}, g_n)$ in a way somewhat analogous to the above, but with B replaced by A so that together this defines an action of G^* on S. In particular, check that the definition is compatible with all the relations used to define G^*).

(c) Show using the above that if $g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} g_2 \ldots t^{\epsilon_n} g_n$ is a word in normal form that is equal to the identity then $n = 0$ and $g_0 = e$.

(d) Show that every element of G has a unique representation as a normal form by showing that if two normal forms are equal: $g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} g_2 \ldots t^{\epsilon_n} g_n = h_0 t^{\delta_1} g_1 t^{\delta_2} g_2 \ldots t^{\delta_m} h_m$, then $n = m$, $g_i = h_i$ and $\epsilon_i = \delta_i$ for all $i \leq n$.

(e) Show there is an embedding of G into G^*.

(f) Show that if H is a subgroup of G such that $\phi(H \cap A) = H \cap B$, and H^* is the subgroup of G^* generated by H and t, then $H^* \cap G = H$.

(g) Finish the proof of the Boone-Novikov theorem from class by using the facts proved about HNN extensions above to justify the two gaps in our proof.