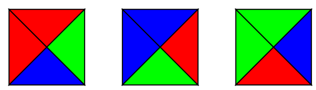
Homework 1, due Tuesday Jan 13 at 1pm

1. (a) (6 pts) Show that if P and Q are decision problems and P is undecidable and P is many-one reducible to Q, then Q is undecidable.

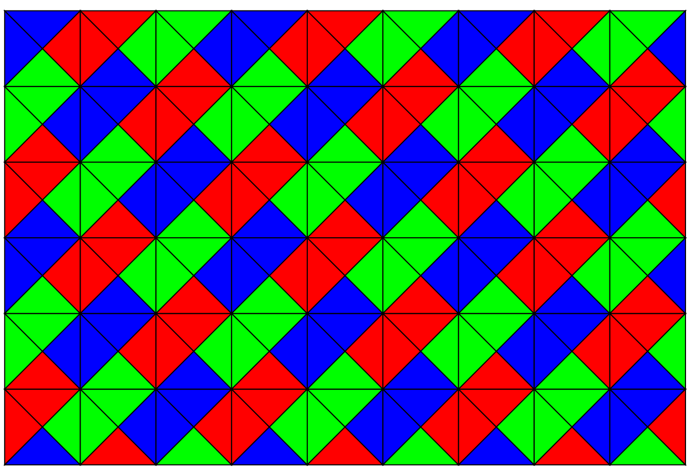
   (b) (7 pts) Let H be the halting problem: the decision problem of whether a given Turing machine halts. Let $\overline{H}$ be the complement of the halting problem: the decision problem of whether a given Turing machine does not halt. Prove that if a decision problem P is reducible to both H and $\overline{H}$, then P is computable.

   (c) (7 pts) Prove that H is not many-one reducible to $\overline{H}$ and $\overline{H}$ is not many-one reducible to H.

2. (20 pts) A Wang tile is a unit square whose edges are each assigned some color. For example, here are three Wang tiles:

   A *tiling* using some finite set S of Wang tiles must obey the rules that adjacent tiles must have matching edge colors, each tile must come from the finite set S of tiles (where we are only allowed to translate the tiles and not rotate or reflect them). For example, here is a tiling using the above three tiles:

   Prove the following problem is undecidable

   > Input: A finite set S of Wang tiles S and a finite initial configuration F of tiles in the upper right quadrant of the plane.
   >
   > Output: Whether there is a tiling of the upper right quadrant of the plane using S that extends F.

   [Hint: reduce the tiling problem we considered in class to this problem]

3. A *formula of number theory* is an expression such as

$$\forall n(n \geq 3 \to \neg\exists x\exists y\exists z(x \neq 0 \wedge y \neq 0 \wedge z \neq 0 \wedge x^n + y^n = z^n))$$

built out of quantifiers (ranging over the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$), the logical connectives and ($\wedge$), or ($\vee$), not ($\neg$), and implies ($\to$), and equations and inequalities made using addition, multiplication, and exponentiation. A formula is called a *sentence* if it is either true or false by virtue of the fact that every variable (such as n, x, y, and z) that appears is bound by some quantifier.

(a) (5 pts) Prove that the function $p(n, m) = (n + m)(n + m + 1)/2 + m$ is a bijection between $\mathbb{N}^2$ and $\mathbb{N}$, and find a formula $\phi(n, m, y)$ of number theory so that $\phi(n, m, y)$ is true if and only if $p(n, m) = y$.

(b) (5 pts) Find a formula of number theory $\psi(i, j, k, b)$ which is true if and only if the ith digit of j is equal to k in base b.

(c) (10 pts) Prove that the problem of determining whether a sentence of number theory is true or false is undecidable. [Hint: you can think of the base b digits of a number j as being a two-dimensional array of numbers between 0 and $b - 1$ by using the pairing function p.]