1. Introduction

A Borel graph \( G = (V, E) \) is a graph whose vertex set \( V \) is a Polish space and whose edge relation \( E \subseteq V \times V \) is Borel. For a Borel graph \( G \) and a graph labelling problem \( P \), descriptive combinatorics concerns whether there is a solution to \( P \) on \( G \) with a labelling function that is Borel. If \( G \) is locally finite (every vertex has finite degree) and the validity of \( P \) at a vertex can be checked by considering the labels in a neighborhood of the vertex, then a consequence of the axiom of choice is that \( G \) has a labelling which solves \( P \) if and only if every finite subgraph of \( G \) has a labelling which solves \( P \) (where the labelling condition of \( P \) is only enforced at interior vertices of the finite subgraph). But this solution given by the axiom of choice, is not guaranteed to be Borel, or even Baire measurable or \( \mu \)-measurable for a Borel probability measure \( \mu \).

The connection of descriptive combinatorics to distributed algorithms is through the LOCAL model of distributed computation, introduced by Linial in [8]. See Distributed Graph Coloring: Fundamentals and Recent Developments [1] by Barenboim and Elkin for a formal introduction. Consider a graph \( G \) with \( n \) vertices. Here \( G \) represents a network of computers that can communicate with their neighbors in discrete rounds. In each round, the vertices perform a computation locally and then send messages to all their neighbors. After \( R \) many rounds, every vertex outputs a label for itself or for its edges/hyperedges, and the this labelling of \( G \) is the output of the algorithm. The efficiency of such an algorithm is measured by the number of communication rounds required \( R \), maximized over all \( n \)-vertex graphs (or all \( n \)-vertex graphs from a specified class). Every vertex of \( G \) performs the same algorithm. Symmetry is broken by assuming that each vertex has a unique identifier \( \text{Id}(v) \in \{1, \ldots, n\} \), and each vertex knows its own identifier. A deterministic LOCAL algorithm solves a labelling problem \( P \) if the labelling it outputs on any graph \( G \) is a valid solution to \( P \), regardless of the way the identifiers are assigned. There is also a randomized LOCAL model, where the vertices generate their identifiers independently at random from \( \{1, \ldots, n\} \) and the algorithm is required to produce a valid solution to \( P \) with probability of failure less than \( \frac{1}{n} \). We will primarily be concerned with deterministic algorithms.

In this paper we will consider classes of graphs of bounded degree and asymptotic bounds (in terms of \( n \)) on the number of rounds of communication required to solve a problem, where the implied constants can depend on the degree.

Bernshteyn [2] proved the following two theorems which provide a connection between descriptive graph combinatorics and distributed algorithms.

**Theorem 1.** If \( G = (V, E) \) is a Borel graph of bounded degree and there is a \( o(\log n) \)-round deterministic LOCAL algorithm for solving a locally checkable labeling problem \( P \) on the finite subgraphs of \( V \), then there is a Borel solution to \( P \) on \( G \).

**Theorem 2.** If \( G = (V, E) \) is a Borel graph of bounded degree and there is a \( o(\log n) \)-round randomized LOCAL algorithm for solving a locally checkable labeling problem \( P \) on the finite subgraphs of \( V \), the for any compatible Polish topology \( \tau \) on \( V \) there is a Baire measurable
solution to \( P \) on \( G \). And for any Borel probability measure \( \mu \) on \( V \) there is a \( \mu \)-measurable solution to \( P \) on \( G \).

These theorems turn upper bound results in distributed algorithms directly into positive results in descriptive graph combinatorics. Although the converses to the previous theorems are not true, there is a similarity between the method of round elimination for proving lower bounds in distributed algorithms and the determinacy method for proving impossibility methods in descriptive set theory. Brandt’s [3] method of round elimination has been to show that if there is no \( o(\log n) \)-LOCAL deterministic algorithm for finding a \( \Delta \) vertex coloring in \( \Delta \)-regular acyclic graphs. The determinacy method of Marks [9] has been used to construct a Borel \( \Delta \)-regular acyclic graph without a Borel \( \Delta \) vertex coloring. Both round elimination and the determinacy method have since been applied to wider classes of problems. This paper generalizes both methods to a wider class of problems about labelling hypergraphs and shows that both methods apply to natural problems in this wider class.

Since its introduction, round elimination has been applied to prove lower bounds for a large number of problems in the theory of distributed algorithms. This has happened in part because of the online Round Eliminator tool, which can help test whether the method applies to a particular labelling problem. The generalized notion of round elimination introduced in this paper may be able to expand the applications of round elimination further, especially if a tool like the Round Eliminator were made available for this new form of round elimination.

2. Notation

The technique of round elimination takes place in the setting of a bipartite graph, where the two parts in the partition are the “active” and “passive” vertices. The generalized setting for round elimination proposed in this paper takes place in a \( k \)-partite \( k \)-uniform hypergraph, which includes the case of a bipartite graph when \( k = 2 \). A \( k \)-uniform hypergraph \( G = (V, E) \) is a set of vertices \( V \) equipped with a collection \( E \) of hyperedges, which are cardinality- \( k \) subsets of \( V \). We will consider labelings of the vertices and hyperedges of \( G \) which formally are functions from \( V \) or \( E \) to a set of labels. We say that a \( k \)-uniform hypergraph \( G = (V, E) \) is \( k \)-partite if there is a partition

\[
V = V_1 \sqcup \ldots \sqcup V_k
\]

of the vertices such that every hyperedge contains exactly one vertex from each \( V_i \). A \( k \)-partite hypergraph is \( \sigma \)-regular (with respect to a partition \( V = V_1 \sqcup \ldots \sqcup V_k \)) for a tuple \( \sigma = (d_1, \ldots, d_k) \) if for each \( i \) each vertex in \( V_i \) belongs to exactly \( d_i \) many hyperedges. We call such a tuple \( \sigma = (d_1, \ldots, d_k) \) for \( d_i \geq 2 \) a signature.

Given a \( \sigma \)-regular \( k \)-partite hypergraph \( G = (V, E) \) and a set of labels \( L \), a labelling problem \( \mathcal{P} = (P_1, \ldots, P_k) \) is a tuple where each \( P_i \) is a collection of cardinality \( d_i \) multisets of elements of \( L \). We can also think of each \( P_i \) as a \( d_i \)-ary relation on \( L \) which is symmetric under all permutations of the \( d_i \) coordinates. A labelling \( c : E \to L \) is a solution to \( \mathcal{P} \) if for every \( i \) and every \( v \in V_i \),

\[
\{c(e) | v \in e \} \in P_i
\]

where the left side is treated as a multiset of cardinality \( d_i \). All labeling problems considered in this paper will be of this form.

In the case \( k = 2 \) so \( \sigma = (d_1, d_2) \) we obtain the standard way of encoding labelling problems for the purpose of round elimination. The case that \( k \) is larger and \( d_i = 2 \) for all \( i \) is what is considered in the original determinacy method [9]. This encodes vertex-labelling problems on \( k \)-regular graphs with a fixed \( k \)-edge coloring.

For a fixed signature \( \sigma = (d_1, \ldots, d_k) \), if \( \mathcal{P} = (P_1, \ldots, P_k) \) is a problem with label set \( L \) and \( \mathcal{Q} = (Q_1, \ldots, Q_k) \) is a problem with label set \( M \), then a 0-round reduction of \( \mathcal{Q} \) to \( \mathcal{P} \)
is a function $f : L \to M$ which is a homomorphism in the sense that for each $i$ we have

$$(\ell_1, \ldots, \ell_d) \in P_i \implies (f(\ell_1), \ldots, f(\ell_d)) \in Q_i.$$  

If there is a 0-round reduction of $\mathcal{Q}$ to $\mathcal{P}$ then a solution to $\mathcal{Q}$ can be obtained from a solution to $\mathcal{P}$ by applying $f$ to all labels. Also note that the composition of 0-round reductions is a 0-round reduction.

The graphs we consider to prove lower bounds will be arise from certain $\sigma$-regular $k$-partite hypergraphs associated to the Schreier graphs of free actions of free products of finite groups. Fixing a signature $\sigma$ we consider the group

$$\Gamma_\sigma = \Delta_1 \ast \ldots \ast \Delta_k$$

where $\Delta_i$ is a cyclic group of order $d_i$ generated by $\delta_i$ for each $i$. Given an action of $\Gamma_\sigma$ on a set $X$, we can form a $k$-uniform $k$-partite hypergraph $G_\sigma(X)$ with vertex set $V = V_1 \sqcup \ldots \sqcup V_k$ where each $V_i$ is the set of cosets $\{x, \delta_1x, \ldots, \delta_i^{d_i-1}x\}$ of the induced action of $\Delta_i$ on $X$. The hyperedges of $G_\sigma(X)$ are between sets of cosets whose intersection is a singleton $\{x\} \subseteq X$. We call a $k$-partite $\sigma$-regular hypergraph free if it is a subgraph of some $G_\sigma(X)$.

For any signature $\sigma$ we can consider the sinkless coloring problem $\mathcal{P}_{sc} = (P_1, \ldots, P_k)$ with label set $\{1, \ldots, k\}$ with relations defined by

$$P_i = \{(\ell_1, \ldots, \ell_d) | \exists m \leq d_i, \ell_m \neq i\}.$$ 

It follows from the original determinacy method that there is a Borel graph of the form $G_\sigma(X)$ which does not have a Borel solution to $\mathcal{P}_{sc}$. As demonstrated in [9], the problem $\mathcal{P}_{sc}$ can be seen as a weakening of the $\Delta$ vertex coloring problem in $\Delta$-regular graphs with a $\Delta$ edge coloring by considering the signature $\sigma = (2, \ldots, 2)$. The problem $\mathcal{P}_{sc}$ also can be seen as a weakening of the sinkless orientation problem or the problem of $2\Delta - 2$ edge coloring.

### 3. Round Elimination

Fix a signature $\sigma = (d_1, \ldots, d_k)$, a problem $\mathcal{P} = (P_1, \ldots, P_k)$ with label set $L$ in this signature, and an index $i \in \{1, \ldots, k\}$ to indicate the active vertices. This means that when we consider a distributed algorithm on a free $k$-partite $\sigma$-uniform hypergraph $G = (V, E)$ with $k$-partition $V = V_1 \sqcup \ldots \sqcup V_k$, only the vertices in $V_i$ will label the hyperedges they belong to. This avoids problems of conflicting outputs of an algorithm since each hyperedge contains exactly one element of $V_i$. Also, this does not affect the asymptotic number of rounds needed to solve a problem since each vertex neighbors a vertex from $V_i$.

In addition to only declaring some of the vertices active, we also insist that a vertex $v \in V_i$ can only directly send a message to vertices in $V_{i-1}$ or $V_k$ if $i = 1$ with which it shares a hyperedge. This lengthens the number of rounds to communicate a message, but only by a factor of at most $k - 1$ since it now takes at most $k - 1$ rounds to send messages to the other vertices on the same hyperedge.

If $P$ is a $d$-ary relation on $L$, then we define two $d$-ary relations $P^3$ and $P^\forall$ on $\mathcal{P}(L)$ by

$$P^3 = \{(S_1, \ldots, S_d) | \exists \ell_1 \in S_1, \ldots, \exists \ell_d \in S_d, P(\ell_1, \ldots, \ell_d)\}$$

$$P^\forall = \{(S_1, \ldots, S_d) | \forall \ell_1 \in S_1, \ldots, \forall \ell_d \in S_d, P(\ell_1, \ldots, \ell_d)\}.$$ 

We define a new problem $\text{re}_i(\mathcal{P}) = (Q_1, \ldots, Q_k)$ with label set $\mathcal{P}(L)$ in the same signature but with active index $i + 1$ (the new active index is 1 if $i = k$). And we let $Q_i = P^3_i$ and $Q_j = P^\forall_j$ for all $j \neq i$.

Note that there is a natural 0-round reduction from $\text{re}_i(\mathcal{P})$ to $\mathcal{P}$ which is $f : L \to \mathcal{P}(L)$ with $f(\ell) = \{\ell\}$. We also have the following, which is the fundamental property of round elimination.
Lemma 3. Let $\mathcal{P} = (P_1, \ldots, P_k)$ be a problem in signature $\sigma$ and let $1 \leq r \ll \log n$. Then there is an $r$ round solution in the port-labeling model to $\mathcal{P}$ with the $V_i$ vertices active if and only if there is an $r - 1$ round solution in the port-labeling model to $re_i(\mathcal{P})$ with the $V_{i+1}$ vertices active, for free hypergraphs of signature $\sigma$ with $n$ vertices.

Proof. Suppose that there is an $r - 1$ round solution to $re_i(\mathcal{P})$ with the $V_{i+1}$ vertices active. We show how in one more round of communication, the vertices in $V_i$ can solve $\mathcal{P}$. First, the $V_{i+1}$ vertices send their solution to $re_i(\mathcal{P})$ to all of the $V_i$ vertices with which they share a hyperedge. Each $v \in V_i$ will receive $d_i$ many sets $S_1, \ldots, S_{d_i}$. Because $(S_1, \ldots, S_{d_i}) \in P_i^{\sigma}$, the vertex $v$ can pick $\ell_m \in S_m$ for all $m$ such that $(\ell_1, \ldots, \ell_{d_i}) \in P_i$. Then $v$ outputs each $\ell_m$ on the hyperedge where $S_m$ would have been in the solution to $re_i(\mathcal{P})$. This is a solution to $\mathcal{P}$ because for each $j \neq i$ and $v \in V_j$, the labels on the hyperedges around $v$ were chosen from sets satisfying $P_j^{\sigma}$ and hence satisfy $P_j$.

Now suppose that there is an $r$ round solution to $re_i(\mathcal{P})$. Consider a vertex $v \in V_i$. Let $e_1, \ldots, e_{d_i}$ be the hyperedges containing $v$, and let $\{u_1, \ldots, u_{d_i}\}$ be the corresponding vertices in $V_{i+1}$ which share a hyperedge with $v$. Using $r - 1$ rounds of communication $v$ can simulate the computation and communication to predict what label each $u_m$ would have outputted for $e_m$. The only information which is inaccessible to $v$ for this simulation are the vertices which can reach $u_m$ in $r$ rounds but cannot reach $v$ in $r - 1$ rounds of communication. For each hyperedge $e$ containing $v$ let $X_e(v)$ be the set of vertices which can reach the vertex in $e \cap V_{i+1}$ in $r$ rounds but cannot reach $v$ in $r - 1$ rounds of communication. The vertex $v$ lists through all possible identifiers which the vertices of $X_e(v)$ could have and outputs the following label from $\mathcal{P}(L)$ for $e$

$$\{\ell | u_m \text{ outputs } \ell \text{ for some assignment of identifiers to } X_e(v)\}.$$ 

Now fix $j \neq i$ and some $w \in V_j$. For each hyperedge $e$ containing $w$, let $v_e$ be the vertex in $e \cap V_i$. We claim that the different sets $X_e(v_e)$ are disjoint because the hypergraph is free. This is because the vertices which can reach each $v_e$ in $r$ rounds but cannot reach $v$ in $r - 1$ rounds must have their information pass into $v_e$ from a vertex other than $w$ and so come from different directions in the free graph. This implies that the output of the algorithm satisfies the relation $P_j^{\mathcal{P}}$ around each $w \in V_j$. If it were possible for each $e$ around $w$ to get labels that violate $P_j$ for some choice of identifiers on each $X_e(v_e)$, then it is possible for all of the $X_e(v_e)$ to have these identifiers simultaneously which violates the correctness of the algorithm being simulated. Checking the correctness of the algorithm around each $v \in V_i$ is more straightforward. The relation $P_i^{\mathcal{P}}$ must hold around $v$ since for every $e$ containing $v$, the vertices from each $e \cap v_{i+1}$ are all simulating the same vertex $v$ to produce their labels. \hfill $\square$

This lemma shows that round elimination produces a canonical problem which requires one fewer round of communication, as long as we rotate which vertices are active. As a consequence of this we have the following.

Theorem 4. If $\mathcal{P}$ is a fixed point of round elimination in the sense that $\mathcal{P}$ is 0-round reducible to

$$re_k \circ re_{k-1} \circ \ldots \circ re_2 \circ re_1(\mathcal{P})$$

then $\mathcal{P}$ does not have a $o(\log n)$-round deterministic algorithm in the port-labeling model for the class of free hypergraphs of signature $\sigma$, unless $\mathcal{P}$ is solvable in $< k$ rounds.

Proof. If $\mathcal{P}$ has a solution in $r$ rounds for $k \leq r \ll \log n$, then $re_k \circ \ldots \circ re_1(\mathcal{P})$ has a solution in $r - k$ rounds. And using the zero round reduction from $\mathcal{P}$, we have that $\mathcal{P}$ has a solution in $r - k$ rounds. The theorem follows by induction. \hfill $\square$
We can upgrade this result to the deterministic LOCAL model using the same standard technique with typical round elimination [4].

**Theorem 5.** If \( \mathcal{P} \) is a fixed point of round elimination in the sense that \( \mathcal{P} \) is 0-round reducible to

\[
re_k \circ re_{k-1} \circ \ldots \circ re_1 (\mathcal{P})
\]

then \( \mathcal{P} \) does not have a \( o(\log n) \)-round LOCAL deterministic algorithm for the class of free hypergraphs of signature \( \sigma \), unless \( \mathcal{P} \) is solvable in \( < k \) rounds.

Even though the problem \( re_k \circ \ldots \circ re_1 (\mathcal{P}) \) has far more labels than \( \mathcal{P} \), it is possible to test whether \( \mathcal{P} \) is 0-round reducible to \( re_k \circ \ldots \circ re_1 (\mathcal{P}) \) by only considering reductions to problems with one usage of round elimination. But first we prove a simple lemma.

**Lemma 6.** If \( f : L \rightarrow M \) is a 0-round reduction from \( Q \) to \( P \), then the direct image function \( f^\circ : \mathcal{P}(L) \rightarrow \mathcal{P}(M) \)

is a 0-round reduction from \( re_i (Q) \) to \( re_i (P) \) for any \( i \).

**Proof.** This follows from the fact that \( f^\circ \) is a homomorphism from \( \mathcal{P}(L) \) to \( \mathcal{P}(M) \) for the relations \( P_i^\circ \) and \( Q_i^\circ \) and also for the relations \( P_i^\circ \) and \( Q_i^\circ \) whenever \( f \) is a homomorphism from \( L \) to \( M \) for the relations \( P_i \) and \( Q_i \).

**Lemma 7.** A problem \( \mathcal{P} \) is 0-round reducible to \( re_k \circ \ldots \circ re_1 (\mathcal{P}) \) if and only if for all \( i \), the problem \( \mathcal{P} \) is 0-round reducible to \( re_i (\mathcal{P}) \).

**Proof.** First assume that \( \mathcal{P} \) is reducible to \( re_k \circ \ldots \circ re_1 (\mathcal{P}) \). By a previous remark, applications of round elimination always produce easier problems from the perspective of 0-round reductions. Thus \( re_{i-1} \circ \ldots \circ re_1 (\mathcal{P}) \) is reducible to \( \mathcal{P} \). Applying the previous lemma gives that \( re_i \circ \ldots \circ re_1 (\mathcal{P}) \) is reducible to \( re_i (\mathcal{P}) \). We also know that \( re_k \circ \ldots \circ re_1 (\mathcal{P}) \) is reducible to \( re_i \circ \ldots \circ re_1 (\mathcal{P}) \). Following a sequence of three compositions we have that \( \mathcal{P} \) is reducible to \( re_i (\mathcal{P}) \).

Conversely assume that \( \mathcal{P} \) is reducible to \( re_i (\mathcal{P}) \) for all \( i \). Applying the previous lemma \( k-i \) times we have for all \( i \) that \( re_k \circ \ldots \circ re_{i+1} (\mathcal{P}) \) is reducible to \( re_k \circ \ldots \circ re_{i} (\mathcal{P}) \). Composing these reductions together yields that \( \mathcal{P} \) is reducible to \( re_k \circ \ldots \circ re_1 (\mathcal{P}) \).

We now show that the sinkless coloring problem \( \mathcal{P}_{sc} = (P_1, \ldots, P_k) \) admits round elimination. Fix \( i \in \{1, \ldots, k\} \) and consider \( re_i (\mathcal{P}) = (Q_1, \ldots, Q_k) \). Then

\[
Q_i = P_i^\circ = \{(S_1, \ldots, S_d_i) | \forall m \leq d_i, \ S_d_i \not\subseteq \{i\}\}
\]

and for \( j \neq i \)

\[
Q_j = P_j^\circ = \{(S_1, \ldots, S_d_j) | \forall m \leq d_j, \ j \not\subseteq S_d_j\}.
\]

We show that the map \( f : \mathcal{P}(L) \rightarrow L \) given by

\[
f_i(S) = \begin{cases} 
\ell & \text{if } \ell \neq i \text{ and } \ell \in S \\
i & \text{if } S \subseteq \{i\}
\end{cases}
\]

is a 0-round reduction. To show this suppose that \( (S_1, \ldots, S_d_i) \in Q_i \). Then \( (f_i(S_1), \ldots, f_i(S_d_i)) \in P_i \) since none of the entries are \( i \). And for \( j \neq i \) if \( (S_1, \ldots, S_d_j) \in Q_j \) then \( (f_i(S_1), \ldots, f_i(S_d_j)) \in P_j \) since none of the entries are \( j \). This can be summarized by the following theorem.

**Theorem 8.** The sinkless coloring problem \( \mathcal{P}_{sc} \) admits round elimination in the sense that \( \mathcal{P}_{sc} \) is 0-round reducible to \( re_i (\mathcal{P}_{sc}) \) for all \( i \).

This provides an alternative proof that \( \mathcal{P}_{sc} \) does not have a \( o(\log n) \) round deterministic LOCAL algorithm, but this also follows from applying Theorem 1 to [9].
4. Determinacy

Again fix a signature \( \sigma = (d_1, \ldots, d_k) \). The determinacy method can be used to construct a Borel \( \sigma \)-regular hypergraph of the form \( G_\sigma(X) \) without a Borel labelling that solves a problem \( \mathcal{P} \) for certain problems \( \mathcal{P} \). Let \( k^L \) denote the set of functions from \( L \) to \( \{1, \ldots, k\} \) where \( L \) is the label set. We think of such functions as partitions of the labels into \( k \) named sets.

**Definition 9.** A problem \( \mathcal{P} = (P_1, \ldots, P_k) \) is **playable** if there is a function \( D : k^L \to \{1, \ldots, k\} \) such that for any \( i \in \{1, \ldots, k\} \) if \( F_1, \ldots, F_{d_i} \in k^L \) are such that \( D(F_m) = i \) for all \( i \), then

\[
(F_1^{-1}(i), \ldots, F_{d_i}^{-1}(i)) \in P_i^3.
\]

We think of each \( F \in k^L \) determining a \( k \)-player game where player \( i \) tries to avoid the labels in \( F^{-1}(i) \). The function \( D \) picks out a player which does not have a winning strategy in each game. And if player \( i \) loses each of the games \( F_m \), then it must be the case that the label sets \( F_m^{-1}(i) \) are compatible by a strategy stealing argument where \( d_i - 1 \) copies of the winning strategies of the \( k-1 \) other players are played against each other. The following is proven in [6] for the case that \( \sigma = (2, \ldots, 2) \), but the same method yields the corresponding statement for hypergraphs.

**Theorem 10.** If \( \mathcal{P} \) is not playable, then there is a \( \sigma \)-regular Borel hypergraph of the form \( G_\sigma(X) \) which does not have a Borel solution to the problem \( \mathcal{P} \).

Their method goes through the theory of local-global limits of finite graphs. We show that it is possible to avoid this and also that the Borel graph \( G_\sigma(X) \) can be taken to be hyperfinite. This hyperfiniteness was achieved in [5] when \( \mathcal{P} \) is a homomorphism problem, and hyperfiniteness had been achieved earlier for the classical vertex coloring, edge coloring, and matching problems.

**Theorem 11.** If \( \mathcal{P} \) is not playable, then there is a hyperfinite \( \sigma \)-regular Borel hypergraph of the form \( G_\sigma(X) \) which does not have a Borel solution to the problem \( \mathcal{P} \).

This is obtained by replacing the ID graphing of [6] with an alternative Borel graph, and the measure theoretic properties are replaced by Baire measurable properties.

For a fixed signature \( \sigma \), consider the group

\[
\Gamma^* = (\Delta_1 * \ldots * \Delta_{1,k}) * (\Delta_2 * \ldots * \Delta_{2,k}) * \ldots
\]

where \( \Delta_{p,i} \) is a cyclic group of order \( d_i \) generated by \( \delta_{p,i} \). Let \( Y \subseteq 2^{\Gamma^*} \) be a comeager subset on which \( \Gamma^* \) acts freely and the action is hyperfinite, which must exist by [7]. We have the following key property which allows for \( \Gamma^* \) to be used as an ID graph as in [6].

**Lemma 12.** If \( A_1, \ldots, A_{d_i} \subseteq Y \) are each comeager in a non-empty basic open subset of \( Y \), then there exists \( p \in \mathbb{N} \) and \( y \in Y \) such that for all \( m \leq d_i \) we have \( \delta_{p,i}^m \cdot y \in A_m \).

**Proof.** For each \( m \) let \( U_m \) be a non-empty basic open subset of \( Y \) such that \( A_m \) is comeager in \( U_m \). Then each \( U_m \) is determined by a finite set of coordinates \( I_m \subseteq \Gamma^* \). For each distinct \( m, m' \leq d_i \), for all but finitely many \( p \) we have that \( \delta_{p,i}^m \cdot I_m \) is disjoint from \( \delta_{p,i}^{m'} \cdot I_{m'} \). Thus there is a single \( p \) such that for all \( m \) the sets \( \delta_{p,i}^m \cdot I_m \) are disjoint. Thus the basic open sets \( \delta_{p,i}^{-m} \cdot U_m \) have non-meager intersection. Let \( y \in Y \) be in this intersection. Then for all \( m \leq d_i \) we have \( \delta_{p,i}^m \cdot y \in A_m \).

**Proof.** Theorem 10 now follows from the methods of [6] but with \( Y \) as the ID graph.

The Cayley graphs of

\[
\Gamma^* = (\Delta_1 * \ldots * \Delta_{1,k}) * (\Delta_2 * \ldots * \Delta_{2,k}) * \ldots
\]
\[ \Gamma_\sigma = \Delta_1 \ast \ldots \ast \Delta_k \]
both have natural edge-colorings where edges corresponding to the generators \( \delta_{i,j} \) or \( \delta_j \) are colored \( j \). And \( Y \) inherits such a coloring since its components are all isomorphic to the Cayley graph of
\[ \Gamma_\sigma^\omega = (\Delta_{1,1} \ast \ldots \ast \Delta_{1,k}) \ast (\Delta_{2,1} \ast \ldots \ast \Delta_{2,k}) \ast \ldots \ast \]

Let \( H \) be the set of subgraphs of \( Y \) which are isomorphic to \( \Gamma_\sigma = \Delta_1 \ast \ldots \ast \Delta_k \), with a distinguished vertex. The isomorphism is required to respect the edge-coloring just described. Let \( \mathcal{H} = (H,E) \) be the graph with vertex set \( H \) where two subgraphs \( (G,v) \) and \( (G',v') \), with distinguished vertices, are adjacent if they are the same subgraph \( G = G' \) and the distinguished vertices \( v \) and \( v' \) are adjacent. Then \( \mathcal{H} \) is a graph, all of whose connected components are isomorphic to \( \Gamma_\sigma = \Delta_1 \ast \ldots \ast \Delta_k \). In fact \( \mathcal{H} \) can be seen as the Schreier graph of an action of \( \Gamma_\sigma = \Delta_1 \ast \ldots \ast \Delta_k \) on \( H \). And since \( Y \) is hyperfinite, \( H \) is as well. Suppose that the \( \sigma \)-regular Borel hypergraph \( G_\sigma(H) \) has a Borel solution to \( \mathcal{P} \). We show that \( \mathcal{P} \) is playable.

For each \( y \in \mathcal{Y} \) and partition \( F \in k^L \) we define a \( k \)-player game \( G(y,F) \). In turns the players build an element of \( H \), a subgraph with distinguished vertex \( y \). In the first turn player \( i \) chooses among the generators \( \delta_{i,j} \) to choose a \( d_i \) cycle through \( y \). In subsequent rounds the players choose cycles further and further away from \( y \) to build the subgraph \( (G,v) \) by finite approximation. Let \( \ell \) be the label that the hyperedge corresponding to \( (G,v) \) has in the Borel solution to \( \mathcal{P} \). The loser is declared to be the player \( F(\ell) \).

In this multi-player Borel game there must be some player \( i \) such that the players in \( \{1, \ldots, k\} \setminus \{i\} \) have a combined winning strategy to force \( i \) to lose. For each \( y \) and \( F \), let \( L(y,F) \) be the least such losing player. Note also that if player \( i \) loses the game \( G(y,F) \), then player \( i \) also loses the variant of the game where they make their play first in each turn. The other players can simply use the same combined strategy ignoring the extra information of one extra play each turn. As described in [6], the function \( L(y,F) \) is Baire measurable. (In the next section we provide an alternative proof of this fact, avoiding the metamathematical theory of provably \( \Delta^1_1 \) sets.) Thus we can define a function \( D : k^L \rightarrow \{1, \ldots, k\} \) such that for each \( f \in k^L \) we have \( L(y,F) = D(F) \) for a non-meager set of \( y \).

Now suppose that \( F_1, \ldots, F_{d_i} \in k^L \) are such that \( D(F_m) = i \) for all \( m \). Then by the previous lemma there is some \( y \in \mathcal{Y} \) and some \( \delta_{p,i} \) such that for all \( m \leq d_i \), we have \( L(\delta_{p,i}^m \cdot y, F_m) = i \). We now consider a way to combine the winning strategies of the other players against each other. Starting at the cycle \( y, \ldots, \delta_{p,i}^{d_i-1} \cdot y \), we can build a subgraph of \( \mathcal{Y} \) by following for each \( m \) the winning combined strategy of \( G(\delta_{p,i}^m \cdot y, F_m) \) where \( y, \ldots, \delta_{p,i}^{d_i-1} \cdot y \) is taken to be the opponent’s first move. Since these are all winning strategies, it must be the case that the hyperedge corresponding to \( \delta_{p,i}^m \cdot y \) receives a label in \( F^{-1}(\ell) \). Therefore
\[ (F_1^{-1}(i), \ldots, F_{d_i}^{-1}(i)) \in P_i^3. \]
This verifies that \( \mathcal{P} \) is playable.

The following is a rephrasing of the original determinacy argument of [9].

**Theorem 13.** The sinkless coloring problem \( \mathcal{P}_{sc} \) is not playable.

**Proof.** Note that \( L = \{1, \ldots, k\} \). Consider the partition \( F : L \rightarrow \{1, \ldots, k\} \) given by the identity \( F(i) = i \). Supposing that \( \mathcal{P}_{sc} = (P_1, \ldots, P_k) \) is playable, we can choose such an assignment \( D : k^L \rightarrow \{1, \ldots, k\} \). Then we have \( D(F) = i \) for some \( i \). Now let \( F_1 = \ldots = F_{d_i} = F \). Then
\[ (F_1^{-1}(i), \ldots, F_{d_i}^{-1}(i)) \notin P_i^3 \]
since by the definition of $\mathcal{P}_{ac}$, $(i, \ldots, i) \notin P_i$. This contradicts playability. \hfill \Box

We also mention one simple property of playability and $0$-round reductions.

**Theorem 14.** If there is a $0$-round reduction $f : L \to M$ of $\mathcal{Q}$ to $\mathcal{P}$ and $\mathcal{P}$ is playable, then $\mathcal{Q}$ is playable.

**Proof.** If $\mathcal{P}$ is playable, there is a function $D : k^L \to \{1, \ldots, k\}$ such that for any $i \in \{1, \ldots, k\}$ if $F_1, \ldots, F_d \in k^L$ are such that $D(F_m) = i$ for all $i$, then

$$(F_1^{-1}(i), \ldots, F_d^{-1}(i)) \in P_i^3.$$ 

We show that $\mathcal{Q}$ is playable by considering the function $D' : k^M \to \{1, \ldots, k\}$ defined by $D'(F) = D(F \circ f)$. Suppose that $F_1, \ldots, F_d \in k^M$ are such that $D'(F_m) = i$ for all $m$. Then $D(F_m \circ f) = i$ for all $m$. Thus

$$((F_1 \circ f)^{-1}(i), \ldots, (F_d \circ f)^{-1}(i)) \in P_i^3.$$ 

But this immediately implies that

$$(F_1^{-1}(i), \ldots, F_d^{-1}(i)) \in Q_i^3.$$ \hfill \Box

5. **Baire Measurability and the Game Quantifier**

We provide a proof that in a Borel family of Borel games, the set of games for which player $I$ has a winning strategy is Baire measurable. To be more precise, if $X$ is a set and $A \subseteq X \times \omega^\omega$ then for every $x \in X$ there is a game $A_x \subseteq \omega^\omega$ such that

$$\left(\{x\} \times \omega^\omega\right) \cap A = \{x\} \times A_x.$$ 

In this way $A$ defines a parameterized family of games.

**Theorem 15.** If $X$ is a Polish space and $A \subseteq X \times \omega^\omega$ is Borel then

$$W = \{x| \text{II has a winning strategy for } A_x\}$$ 

is Baire measurable.

**Proof.** It suffices to show that for any open set $U \subseteq X$ that either $W$ is comeager in $U$ or $X \setminus W$ is comeager in a nonempty open $V \subseteq U$. But by replacing $X$ with $U$, it suffices to prove that $W$ is either comeager or $X \setminus W$ is comeager in some nonempty open $V \subseteq X$.

Consider the following Borel game $G_A$. Fix a countable basis $\mathcal{U}$ for the topology on $X$. Player $I$ plays pairs $(U_{2k}, n_{2k})$ and player $II$ plays pairs $(U_{2k+1}, n_{2k+1})$ such that for all $i$, $n_i \in \omega$, $U_i \in \mathcal{U}$ is a basic open subset of $X$, $\text{diam}(U_i) \leq 2^{-i}$, and $U_{i+1} \subseteq U_i$. By the shrinking conditions on the open sets, we have

$$\bigcap_{i \in \omega} U_i = \{x\}$$

and player $II$ wins if and only if $(x, (n_i)_{i \in \omega}) \in A$.

We will prove that if player $II$ has a winning strategy for $G_A$ then $W$ is comeager, and then that if player $I$ has a winning strategy for $G_A$ then $X \setminus W$ is comeager in some nonempty open $V \subseteq U$. Borel determinacy implies that one of the two players has a winning strategy and the result follows.

We first prove that if player $II$ has a winning strategy for $G_A$, then $W$ is comeager. Let

$$\sigma : \bigcup_{k \in \omega} (\mathcal{U} \times \omega)^{2k} \to \mathcal{U} \times \omega$$
be a winning strategy for $\Pi$. Let $T \subseteq (\mathcal{U} \times \omega)^{<\omega}$ be a tree on $\mathcal{U} \times \omega$ with the property that any
\[ a \in T \cap (\mathcal{U} \times \omega)^{2k} \]
has a unique child
\[ a' = a \sim \sigma(a) \in T \cap (\mathcal{U} \times \omega)^{2k+1}. \]
We also assume that for any $b \in T \cap (\mathcal{U} \times \omega)^{2k-1}$ and any $n_{2k}$ the collection
\[ \{U_{2k+1} \mid \exists U_{2k} \exists U_{2k+1} \exists n_{2k+1}, b^{-1}(U_{2k}, n_{2k})^{-1}(U_{2k+1}, n_{2k+1}) \in T \} \]
is pairwise disjoint with union dense in $U_{2k-1}$. Such a tree $T$ can be built inductively by choosing a maximal disjoint such collection of $U_{2k+1}$ at each stage.

For each $s = (n_0, n_2, \ldots, n_{2k})$ define
\[ D_s = \bigcup \{U_{2k+1} \mid \exists n_1 \ldots \exists n_{2k+1} \exists U_0, \ldots, \exists U_{2k}, ((U_0, n_0), \ldots, (U_{2k+1}, n_{2k+1})) \in T \}. \]
Then each $D_s$ is open and dense by construction. We show that $\bigcap_s D_s \subseteq W$ to show that $W$ is comeager. Let $x \in \bigcap_s D_s$. We describe a strategy $\tau$ for player $\Pi$ in $A_X$. If the play of the game so far is $s = (n_0, \ldots, n_{2k})$, then player $\Pi$ responds with the unique $n_{2k+1}$ such that there exist $U_0, \ldots, U_{2k+1}$ such that $((U_0, n_0), \ldots, (U_{2k+1}, n_{2k+1})) \in T$ and $x \in U_{2k+1}$. We now prove that $\tau$ is a winning strategy for player $\Pi$. Suppose that $(n_i)_{i \in \omega}$ is a result of player $\Pi$ following $\tau$. Then there is a unique sequence $(U_i)_{i \in \omega}$ such that $((U_i, n_i))_{i \in \omega}$ is an infinite path through $T$ and $x \in U_i$ for all $i$. By the definition of $T$ and because $\sigma$ is a winning strategy for player $\Pi$ in $G_A$, we have $(x, (n_i)_{i \in \omega}) \in A$. Thus $\tau$ is winning for player $\Pi$ and so $x \in W$.

Now suppose that player $I$ has a winning strategy for $G_A$ with first move $(U_0, n_0)$. Then we can repeat the same arguments as above for the complement of $A$ localized to $U_0$ to show that $X \setminus W$ is comeager in $U_0$.

\[ \square \]

\section*{References}


