BAIRE MEASURABLE COMBINATORICS FOR NON-AMENABLE GRAPHS

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Abstract. We prove that every Schreier graph of a free Borel action of a finitely generated non-amenable group has a Baire measurable perfect matching. This result was previously only known in the bipartite setting. We also prove that every Borel non-amenable bounded degree graph with only even degrees has a Baire measurable balanced orientation.

1. Introduction

We say an infinite connected graph $G$ of bounded degree is non-amenable if there exists $\delta > 0$ such that whenever $F \subseteq V(G)$ is finite, the set of edges $E(F, V(G) \setminus F)$ between $F$ and $V(G) \setminus F$ satisfies $|E(F, V(G) \setminus F)| \geq \delta |F|$. For example, the Cayley graphs of finitely generated non-amenable groups with respect to any finite symmetric generating set (not containing the identity) are non-amenable graphs. There are many other examples that are not Cayley graphs. For example, non-amenable quasi-transitive unimodular graphs have been studied in [5] and [1]. And some graphs that are not unimodular, such as the grandparent graph, are also covered by this definition.

In this paper, we consider non-amenable Borel graphs on Polish spaces, and prove that certain classical combinatorial problems can be solved Baire measurably, that is, on a Borel comeager invariant set. Our main theorem concerns the existence of perfect matchings:

Theorem 1. Let $G$ be a Borel graph such that each component is an infinite, bounded degree, non-amenable vertex transitive graph. Then $G$ admits a Borel perfect matching on a Borel comeager invariant set.

Corollary 2. Every Schreier graph of a free Borel action of a finitely generated non-amenable group admits a Borel perfect matching on a Borel comeager invariant set.

In [6], Marks and Unger studied Baire measurable matchings in the context of bipartite Borel graphs, with a view towards applications for Baire measurable equidecompositions. Though not explicitly stated in their paper, their Theorem 1.3 implies that every bipartite Borel graph whose components are bounded degree, regular, and non-amenable has a Baire measurable perfect matching. Thus, our theorem can be viewed as an extension of their result to the non-bipartite setting. The existence of regular, quasi-transitive, non-amenable graphs without perfect matchings, (see Remark 22 of [1]) leads us to assume vertex transitivity, not just regularity.

Another theorem we prove concerns balanced orientations. Given a graph with only even degrees, a balanced orientation is an orientation of the edges so that every vertex has in-degree equal to out-degree. Euler’s classical theorem about Euler circuits and a compactness argument shows that every locally finite graph with only even degrees admits a balanced orientation. For nonamenable bounded degree Borel graphs we have the following:

Theorem 3. Let $G$ be a bounded degree non-amenable Borel graph with only even degrees. Then $G$ admits a Borel balanced orientation on a Borel comeager invariant set.

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Our arguments draw inspiration from the study of factor of i.i.d. combinatorial structures for Cayley graphs of non-amenable groups, or quasi-transitive unimodular non-amenable graphs more generally; see [4], [3], [1]. By studying the spectrum of the Markov operator associated with random walks on these graphs, one shows a measure expansion property for the associated Bernoulli graphings. This measure expansion property is then used to establish the existence of a combinatorial structure (say, a perfect matching or balanced orientation) for the Bernoulli graphing on a Borel conull invariant set. While our arguments generally differ from those used to obtain factors of i.i.d., several of the ideas we use were inspired from that setting. It seems likely that many of the results pertaining to factors of i.i.d. for non-amenable graphs will have Baire measurable analogues.

2. Perfect matchings

The classical theorem of Tutte, repeated below, characterizes when a locally finite graph admits a perfect matching.

**Theorem 4** (Tutte’s theorem). A locally finite graph $G$ admits a perfect matching if and only if whenever $X \subseteq V(G)$ is finite, the graph $G - X$ has at most $|X|$ many finite components of odd size.

By Tutte’s condition we will mean the condition that “$G - X$ has at most $|X|$ many odd components for each finite $X \subseteq V(G)$”.

The proof of Theorem 1 consists in two steps. First, we establish a Baire measurable variant of Tutte’s theorem which gives a sufficient condition for a locally finite Borel graph to admit a perfect matching on a Borel comeager invariant set (Theorem 6). Second, we show that nonamenable vertex transitive graphs satisfy this sufficient condition (Lemma 8).

**Definition 5.** If $G$ is a locally finite graph and $X \subseteq G$ is finite, define

$$C_{\text{fin}}(X) := \{\text{finite components of } G - X\}$$

and

$$C_{\text{odd}}(X) := \{\text{odd components of } G - X\}.$$  

Also let

$$\text{hull}_{\text{fin}}(X) := X \cup \bigcup C_{\text{fin}}(X),$$

and

$$\text{hull}_{\text{odd}}(X) := X \cup \bigcup C_{\text{odd}}(X).$$

We sometimes add superscripts to indicate the ambient graph when there is ambiguity.

**Theorem 6.** Let $G$ be a locally finite Borel graph on a Polish space $V(G)$, and suppose that for every finite set $X \subseteq V(G)$, we have

$$|X| \geq |C_{\text{odd}}(X)| + \varepsilon|\text{hull}_{\text{odd}}(X)|.$$  

Then $G$ admits a Borel perfect matching on a Borel comeager invariant set.

For the proof, we say a locally finite graph $G$ satisfies Tutte$_{\varepsilon,k}$ if (i) Tutte’s condition holds, and (ii) whenever $X \subseteq V(G)$ is finite such that $\text{hull}_{\text{odd}}(X)$ is connected and has size at least $k$,

$$|X| \geq |C_{\text{odd}}(X)| + \varepsilon|\text{hull}_{\text{odd}}(X)|.$$  

Observe that the condition in Theorem 6 is equivalent to Tutte$_{\varepsilon,1}$. This is an analogue of Hall$_{\varepsilon,k}$ in the proof of Theorem 1.3 in [6]. Our proof of Theorem 6 follows the same general strategy as the proof in [6]. In particular, we will need the following lemma from that paper.
Lemma 7. Let $G$ be a locally finite Borel graph on a Polish space $V(G)$, and let $f : \mathbb{N} \to \mathbb{N}$. Then there exist Borel sets $A_n \subseteq V(G)$, $n \in \mathbb{N}$, such that $\bigcup_n A_n$ is a Borel comeager invariant set and $d_G(x, y) > f(n)$ whenever $x, y$ are distinct vertices in $A_n$.

Proof of Theorem 6. Let $f : \mathbb{N} \to \mathbb{N}$ be a sufficiently fast-growing increasing function so that

1. $\sum_n \frac{4f(n)}{f(n)} < \varepsilon$;
2. letting $\varepsilon_n = \varepsilon - \sum_{m \leq n} \frac{4}{f(m)}$, we have $\varepsilon_{n-1} f(n) > 4$ for each $n$.

For convenience, we write $\varepsilon_{-1} = \varepsilon$. Let $A_n$ be the Borel sets given by Lemma 7 for this $f$. Given a matching $M$, we write $G - M$ for the graph obtained from $G$ by removing all the vertices covered by $M$ (that is, $G - M$ is the induced subgraph on the set of vertices not covered by $M$). We define increasing Borel matchings $M_n$ such that their union will be a perfect matching of the Borel comeager invariant set $\bigcup_n A_n$. We will ensure that $M_n$ covers the vertices in $A_n$ and $G - M_n$ satisfies Tutte’s condition; equivalently such that $(G - M_n) - e_x$ admits a perfect matching. We know such an edge exists as the hypothesis that Tutte’s condition quantifies over finite sets, the matching $M_0 := M_{n-1} \cup \{e_x : x \in A_n\}$ is Borel.

We verify that $G - M_n$ satisfies Tutte$_{\varepsilon_n, f(n)}$. As a first step, we show that $G - M_n$ has no odd component (this is verifying Tutte’s condition for $X = \emptyset$). Assume for contradiction that $C$ is an odd component of $G - M_n$, and let $X'$ denote the set of endpoints of edges $e_x \in M_n - M_{n-1}$ such that $e_x$ is adjacent to $C$. Since $G - M_{n-1}$ had no odd component, $X' \neq \emptyset$ and $\text{hull}^{G-M_{n-1}}_{\text{odd}}(X')$ must be connected.

Case 1: Suppose $|X'| \geq 4$, so that there are at least two distinct edges $e_x \in M_n - M_{n-1}$ that are adjacent to $C$. Since $C \cup X'$ is connected and the vertices in $X'$ corresponding to distinct edges are a distance of at least $f(n)$ from one another, we have

$$|\text{hull}^{G-M_{n-1}}_{\text{odd}}(X')| = |C \cup X'| \geq \frac{|X'|}{2} \cdot \frac{f(n)}{2} \geq \frac{f(n)}{4} |X'|.$$ 

In particular, $|\text{hull}^{G-M_{n-1}}_{\text{odd}}(X')| \geq f(n) \geq f(n-1)$. So, applying the inductive assumption of Tutte$_{\varepsilon_{n-1}, f(n-1)}$ to $G - M_{n-1}$ and $X'$, we obtain

$$|X'| \geq \varepsilon_{n-1} |\text{hull}^{G-M_{n-1}}_{\text{odd}}(X')| + |C^{G-M_{n-1}}_{\text{odd}}(X')| \geq \varepsilon_{n-1} \frac{f(n)}{4} |X'|$$

Since $f$ was chosen so that $\varepsilon_{n-1} f(n) > 4$, this is impossible.

Case 2: Suppose $|X'| = 2$, so that there is a single edge $e_x \in M_n - M_{n-1}$ that is adjacent to $C$. But this case is impossible as we chose $e_x$ specifically so that $M_{n-1} \cup \{e_x\}$ extends to a perfect matching, so the appearance of the odd component $C$ in $G - M_n$ cannot only be due to $e_x$.

So far we have proved that $G - M_n$ has no odd component. Let $X \subseteq V(G - M_n)$ be a finite set such that $\text{hull}_{\text{odd}}(X)$ is connected. Let $E_X$ be the set of edges $e_x \in M_n - M_{n-1}$ such that at least one of the endpoints of $e_x$ is adjacent to $\text{hull}^{G-M_n}_{\text{odd}}(X)$ in $G$. 

Case 1: Suppose that \(|E_X| \geq 2\). Since \(\text{hull}_{\text{odd}}^{G-M_n}(X)\) is connected and distinct edges in \(E_x\) are a distance of at least \(f(n)\) from one another, we have
\[
|\text{hull}_{\text{odd}}^{G-M_n}(X)| \geq |E_x| \frac{f(n)}{2}.
\]
In particular, \(|\text{hull}_{\text{odd}}^{G-M_{n-1}}(X')| \geq f(n) \geq f(n - 1)\). So, applying the inductive assumption of Tutte\(_{\varepsilon_{n-1},f(n-1)}\) to \(G - M_{n-1}\) and
\[
X' = X \cup \{v \in V(G) : v \text{ is an endpoint of some } e \in E_X\},
\]
yields
\[
|X'| \geq |C_{\text{odd}}^{G-M_{n-1}}(X')| + \varepsilon_{n-1} |\text{hull}_{\text{odd}}^{G-M_{n-1}}(X')|.
\]
Therefore
\[
|C_{\text{odd}}^{G-M_n}(X)| = |C_{\text{odd}}^{G-M_{n-1}}(X')| \\
\leq |X'| - \varepsilon_{n-1} |\text{hull}_{\text{odd}}^{G-M_{n-1}}(X')| \\
\leq |X'| - \varepsilon_{n-1} |\text{hull}_{\text{odd}}^{G-M_n}(X)| \\
= |X| + 2|E_x| - \varepsilon_{n-1} |\text{hull}_{\text{odd}}^{G-M_n}(X)| \\
\leq |X| + \frac{4}{f(n)} |\text{hull}_{\text{odd}}^{G-M_n}(X)| - \varepsilon_{n-1} |\text{hull}_{\text{odd}}^{G-M_n}(X)| \\
= |X| - \varepsilon_{n} |\text{hull}_{\text{odd}}^{G-M_n}(X)|.
\]

Case 2: Suppose that \(|E_X| \leq 1\). If \(E_X\) is empty, then the fact that \(X\) does not violate Tutte\(_{\varepsilon_n,f(n)}\) simply follows from the fact that \(G - M_{n-1}\) satisfies the (stronger) Tutte\(_{\varepsilon_{n-1},f(n-1)}\). So suppose that \(E_X\) consists of a single edge \(e_x\), for some \(x \in A_n \cap V(G - M_{n-1})\). We chose \(e_x\) so that Tutte’s condition holds for \((G - M_{n-1}) - e_x\), so in particular
\[
|\text{hull}_{\text{odd}}^{G-M_{n-1}-e_x}(X)| \leq |X|.
\]
But \(e_x\) is the only edge adjacent to \(\text{hull}_{\text{odd}}^{G-M_n}(X)\) in \(G - M_n\), so the odd components of \((G - M_{n-1} - e_x) - X\) are precisely the same as the odd components of \((G - M_n) - X\). Hence, \(X\) does not violate Tutte’s condition in \(G - M_n\). Suppose now that \(|\text{hull}_{\text{odd}}^{G-M_n}(X)| \geq f(n) \geq f(n - 1)\), and as in Case 1 let
\[
X' = X \cup \{v \in V(G) : v \text{ is an endpoint of some } e \in E_X\}.
\]
Applying Tutte\(_{\varepsilon_{n-1},f(n-1)}\) to \(G - M_{n-1}\) and \(X'\) yields
\[
|C_{\text{odd}}^{G-M_n}(X)| = |C_{\text{odd}}^{G-M_{n-1}}(X')| \\
\leq |X'| - \varepsilon_{n-1} |\text{hull}_{\text{odd}}^{G-M_{n-1}}(X')| \\
\leq |X| + 2 - \varepsilon_{n-1} |\text{hull}_{\text{odd}}^{G-M_n}(X)| \\
\leq |X| + \frac{2}{f(n)} |\text{hull}_{\text{odd}}^{G-M_n}(X)| - \varepsilon_{n-1} |\text{hull}_{\text{odd}}^{G-M_n}(X)| \\
\leq |X| + \varepsilon_{n} |\text{hull}_{\text{odd}}^{G-M_n}(X)|.
\]
So \(X\) does not violate Tutte\(_{\varepsilon_n,f(n)}\) in Case 2 either.

Next, we show that non-amenable vertex transitive graphs satisfy the condition in Theorem 6.
Lemma 8. Let $G$ be an infinite, connected, locally finite, non-amenable, vertex transitive graph. Then there exists $\varepsilon > 0$ such that for all finite $X \subseteq V(G)$,
$$|X| \geq |C_{\text{fin}}(X)| + \varepsilon|hull_{\text{fin}}(X)|.$$ 
In particular, there exists $\varepsilon > 0$ such that for all finite $X \subseteq V(G)$,
$$|X| \geq |C_{\text{odd}}(X)| + \varepsilon|hull_{\text{odd}}(X)|.$$ 

Proof. Fix a finite set $X \subseteq V(G)$. By Lemma 2.3 of [3], the assumption that $G$ is a (connected, infinite) $d$-regular, vertex transitive graph implies that each element of $C_{\text{fin}}$ has at least $d$ many edges in its boundary. And so
$$\left| E\left(X, \bigcup_{i \in \mathcal{C}_{\text{fin}}(X)} \right) \right| = \sum_{F \in \mathcal{C}_{\text{fin}}(X)} |E(X, F)| \geq d|C_{\text{fin}}(X)|.$$ 
Also by the expansion property
$$\left| E\left(X, V(G) \setminus hull_{\text{fin}}(X) \right) \right| \geq \delta|hull_{\text{fin}}(X)|,$$
where $\delta$ is the expansion constant of the graph. Therefore
$$d|X| \geq \left| E\left(X, \bigcup_{i \in \mathcal{C}_{\text{fin}}(X)} \right) \right| + \left| E\left(X, V(G) \setminus hull_{\text{fin}}(X) \right) \right| \geq d|C_{\text{fin}}(X)| + \delta|hull_{\text{fin}}(X)|.$$ 
And so
$$|X| \geq |C_{\text{fin}}(X)| + \varepsilon|hull_{\text{fin}}(X)|,$$
where $\varepsilon = \frac{\delta}{d}$. \qed

3. Balanced Orientations

The proof of Theorem 3 is quite straightforward and is an adaptation of the ideas in Section 5 of [1]. Given any graph $G$ with only even degrees, we define an auxiliary bipartite graph $G^*$ such that perfect matchings of $G^*$ induce perfect matchings of $G$. The following definition is taken essentially verbatim from [1] and is repeated here for the convenience of the reader.

Definition 9. Let $G$ be a graph with only even degrees. The graph $G^*$ has a vertex for every edge $e$ of $G$ and $\deg(v)/2$ many vertices for every vertex $v$ of $G$, i.e.
$$V(G^*) = \{x_e : e \in E(G)\} \cup \{v_i : v \in V(G), i \in \lceil \deg(v)/2 \rceil\}.$$ 
Then every vertex corresponding to a former edge is joined to all copies of its former endpoints:
$$E(G^*) = \{x_{uv}v_i : uv \in E(G), i \in \lceil \deg(v)/2 \rceil\}.$$ 

Observe that any perfect matching of $G^*$ induces a balanced orientation of $G$ by orienting an edge $e \in E(G)$ toward its endpoint $v$ if and only if $x_e$ and $v_i$ are matched in $G^*$ for some $i \in \lceil \deg(v)/2 \rceil$. In the case when $G$ is a Borel graph, it is also straightforward to put an appropriate Polish topology on $V(G^*)$ so that Baire measurable perfect matchings of $G^*$ yield Baire measurable balanced orientations of $G$. In order to show that the Borel bipartite graph $G^*$ has a Baire measurable perfect matching, we will apply Theorem 1.3 of [6] (which is an analogue of our Theorem 1 in the bipartite setting).

Theorem 10 (Theorem 1.3 of [6]). Let $G$ be a locally finite bipartite Borel graph with bipartition $V(G) = B_0 \cup B_1$ (the sets $B_0$ and $B_1$ need not be Borel). Suppose there exists $\varepsilon > 0$ such that whenever $F$ is a finite set contained in either $B_0$ or $B_1$, we have
$$|N(F)| \geq (1 + \varepsilon)|F|.$$ 
Then $G$ admits a Borel perfect matching on a Borel comeager invariant set.
Proof of Theorem 3. The proof is an adaptation of the proof of Lemma 25 from [1]. Write 

$$\pi : V(G^*) \to V(G) \cup E(G)$$

for the projection function. Let \( \delta > 0 \) be the expansion constant for the non-amenable graph \( G \), and let \( d \) be a bound on the degrees. Suppose that \( F \subseteq V(G^*) \) is a finite set of vertex-type vertices. Then

\[
\left| N_{G^*}(F) \right| = \frac{1}{2} \sum_{u \in \pi(F)} \deg(u) + \frac{1}{2} |E(\pi(F), V(G) \setminus \pi(F))| \\
\geq \sum_{u \in \pi(F)} \frac{\deg(u)}{2} + \frac{\delta}{2} |\pi(F)| \\
\geq |F| + \frac{\delta}{d} |F|.
\]

Now suppose that \( F \subseteq V(G^*) \) is a finite set of edge-type vertices (that is, \( F \subseteq E(G) \)), and let \( S \) denote the set of vertices \( u \in V(G) \) which are incident to some edge \( e \in F \). Then

\[
\left| N_{G^*}(F) \right| = \sum_{u \in S} |\pi^{-1}(u)| \\
= \sum_{u \in S} \frac{\deg(u)}{2} \\
= |E(S, S)| + \frac{1}{2} |E(S, V(G) \setminus S)| \\
\geq |F| + \frac{\delta}{2d} |S| \\
\geq |F| + \frac{\delta}{2d} |F|.
\]

Hence, if we choose \( \varepsilon > 0 \) such that \( \varepsilon < \frac{\delta}{2d} \) works, then the hypotheses for Theorem 10 hold. So \( G^* \) has a Baire measurable perfect matching, and this implies that \( G \) has a Baire measurable balanced orientation. \( \square \)

References


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