PIGEONHOLE PROBLEMS: FALL 2006 PUTNAM TRAINING

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(1) We'll start with a theorem of Erdos and Szekeres. First we will need the definition of a subsequence. A finite sequence of real numbers is a function $a : \{1, \ldots m\} \to \mathbf{R}$ for some $m \in \mathbf{N}$. We say m is the length of the sequence. There are also infinite sequences $a : \mathbf{N} \to \mathbf{R}$. When we are talking about sequences we often write a_n instead of $a(n)$. We often write $\{a_n\}_{n=1}^{\infty}$ as an alternative notation to $a : \mathbf{N} \to \mathbf{R}$ and similarly for finite sequences. If $1 \leq t \leq m$ and we have a function $n : \{1, \ldots t\} \to \{1, \ldots m\}$ satisfying $n(i) < n(j)$ if $i < j$ then the composite function $a \circ n : \{1, \ldots t\} \to \mathbf{N}$ is a subsequence. Instead of writing $(a \circ n)(k)$ or $a(n(k))$ we usually write a_{n_k}. And similarly an infinite sequence can have subsequences, either finitely long or infinitely long. Informally, we get a subsequence by restricting a sequence a to a subset of its domain, and defining a_{n_k} to be the value of a on the kth smallest number in the subset. (This may seem a bit long winded, but people don't always already know what is meant by a subsequence.) A sequence c_n is ascending if $c_i \leq c_j$ whenever $i < j$ and descending if $c_i \geq c_j$ whenever $i < j$. In particular this definition applies to subsequences, since they are just sequences that we obtained from other sequences. Here's the theorem:

Theorem 0.1. *Suppose $\{a_i\}_{i=1}^{n^2+1}$ is a sequence of distinct real numbers. Then either there is an ascending subsequence of length $n+1$ or there is a descending subsequence of length $n+1$.*

For example, if $n = 4$, we might have the sequence

$$(4, 3, 2, 1, 8, 7, 6, 5, 12, 11, 10, 9, 16, 15, 14, 13, 17).$$

This means the sequence a with $a_1 = 4, a_2 = 3, \ldots$. It is easy to see that there is no real loss of generality in assuming that the $n^2 + 1$ distinct real numbers a_i are the numbers $1, \ldots, n^2 + 1$.

It is not too hard to see that there are descending sequences of length 4 but no descending subsequences of length 5, but there are lots of ascending subsequences of length 5. For example taking $n_1 = 4$, $n_2 = 6$, $n_3 = 10$, $n_4 = 14$, $n_5 = 17$ we get an ascending subsequence $(1, 7, 11, 15, 17)$. You could prove the case $n = 4$ of the theorem by listing all the $17!$ permutations of the set $\{1, 2, \ldots, 17\}$ and checking each one. But that only proves one case of the theorem and requires considerable computer time. For $n = 10$ there are $101!$ permutations of the set $\{1, 2, \ldots, 101\}$and a brute force approach can't be done before the sun burns out.

Proof. For each i, let $u(i)$ be the length of the longest ascending subsequence beginning with a_i, and let $d(i)$ be the length of the longest descending subsequence beginning with a_i. (In our example, $u(4) = 5$ and $u(12) = 3$; in the latter case $(9, 13, 17)$ is one of four possibilities for a longest ascending subsequence.) If the claim of the theorem does not hold, $1 \leq u(i) \leq n$ for all i and $1 \leq d(i) \leq n$ for all i. So there are at most $n\dot{n}$ possibilities for the pair $(u(i), d(i)$. By the Pigeonhole Principle there exist distinct numbers i, j with $1 \leq i, j \leq n^2 + 1$ and $(u(i), d(i)) = (u(j), d(j))$. But either $a_i < a_j$ or $a_i > a_j$. If $a_i < a_j$ then given an ascending subsequence $\{a_{n_k}\}_{k=1}^{u(j)}$ starting at a_j and of length $u(j)$ we can construct an ascending subsequence starting at a_i of length $u(j) + 1$. Informally, we concatenate a_i and the ascending subsequence of length $u(j)$. On the other hand if $a_i > a_j$ then given a descending subsequence starting at a_j and of length $d(j)$ we can construct a descending subsequence starting at a_i and of length $d(j)+1$. It is my experience that students don't always know the word concatenate, although it's a fairly common word and is often used in computer science as well as combinatorics. In the example above, $(1, 5, 9)$ and $(13, 17)$ are ascending subsequences. Concatenating them we get a subsequence $(1, 5, 9, 13, 17)$ which is really a function defined on $\{1, 2, 3, 4, 5\}$.

Let's rewrite the proof, assuming the reader is familiar with standard terminology:

Proof. For each i let $u(i)$ be the length of the longest ascending subsequence starting at a_i and let $d(i)$ be the length of the longest descending subsequence starting at a_i. If there are no ascending subsequences of length $n+1$ and no descending subsequences of length $n+1$ then (u, d) defines a function from $\{1, 2, \ldots n^2 + 1\}$ to $\{1, 2, \ldots n\} \times \{1, 2, \ldots n\}$. By the pigeonhole principle there exist $i < j$ with $(u(i), d(i)) = (u(j), d(j)$. But if $a_i < a_j$ the longest ascending subsequence starting at a_i is at least one longer than the longest ascending subsequence starting at a_j, and if $a_i > a_j$ the longest descending subsequence starting at a_i is at least one longer than the longest descending subsequence starting at a_j.

I deliberately chose a very hard pigeonhole problem as an example, to show that the pigeonhole principle can be used in a nontrivial way.

(2) Given a sequence of $mn + 1$ distinct real numbers, either there is an ascending subsequence of length $m+1$ or a descending subsequence of length $n + 1$.
(3) Given a sequence of $mn + 1$ numbers no two of which are equal, show that there is an subsequence of length $m + 1$ in which no number divides any other, or else a subsequence $\{a_{n_k}\}_{k=1}^{n+1}$ of length $n + 1$ in which each term divides all succeeding terms.

(In his archive of Putnam problems, John Scholes makes the comment that this is only a slight modification of the Erdos-Szekeres theorem and is easy if you understand the proof of the theorem, but hard otherwise. This actually was a Putnam problem (1966 B4) back in the days when combinatorics was not taught so commonly.)

(4) Sixteen different integers are chosen between 1 and 30, inclusive. Show some two differ by 3.
(5) 51 different integers are chosen between 1 and 100, inclusive. Some two of them are coprime.
(6) Fifty-one different integers are chosen between 1 and 100, inclusive. Show that one of them divides another.

(7) Sixteen different integers are chosen between 1 and 30, inclusive. Some two differ by 3.

(8) Let n be a positive integer. Is it possible for $6n$ distinct straight lines in the Euclidean plane to be situated so as to have $6n^2 - 3n$ points where exactly three of these lines intersect and at least $6n + 1$ points where exactly two of these lines intersect ?

(9) Let $a_1, a_2, \dots a_4 4$ be 44 natural numbers such that

$$0 < a_1 < a_2 < \dots < a_{44} \leq 125.$$

Prove that at least one of the 43 differences $d_j = a_{j+1} - a_j$ occurs at least 10 times.

(10) Given a sequence $a_1, \dots a_m$ of length m, show that there is a consecutive subsequence whose sum is divisible by m. (A consecutive subsequence means a subsequence $a_i, a_{i+1}, a_{i+2}, \dots a_{i+j-1}$ of length j where j could be as small as one.)

(11) During the year 1998 a convenience store which was open 7 days a week sold at leat one book every day, and a total of 600 books over the entire year. Must there have been a period of consecutive days when exactly 129 books were sold ?

(12) Let S be a set of k distinct integers chosen from $1, 2, 3, \dots 10^n - 1$, where n is a positive integer. Prove that if

$$n < log((2^k - 1)/k + ((k+1)/2)/log10$$

it is possible to find 2 disjoint subsets of S whose members have the same sum. (A variation of 1973 A6.)

(13) If fifteen distinct integers are chosen between 1 and 45, some two of them differ by 1, 3, or 4. Frankly, I found this one of the duller problems, though it still takes time unless you get the right idea fast.

(14) A checkerboard has 4 rows and 7 columns. Choosing two or more successive rows and two or more successive columns and taking only the squares in those rows and columns gives a subboard. Suppose that each of the 28 squares is colored either black or white. Show that there is a subboard all of whose corners are black or all of whose corners are white. (This stumped me on the 1976 USA Mathematical Olympiad. It would have helped if I had known about pigeonhole problems. The real point of the problem is to give a solution that generalizes nicely.)

(15) 1978A1, 1985B3, 1990A3, 1990B3, 1993A4, 1994A3, 1994A6, 1995B1, 1996A3, 2000B6, 2002A2 are all pigeonhole problems. I think 1985B3 is particularly nice. Roughly speaking, there is a 50chance of a pigeonhole problem occuring on any given Putnam exam.

(16) Suppose a is an irrational number. Consider the sequence $x_n = na - \lfloor na \rfloor$. As usual $\lfloor x \rfloor$ means the integer part of x for any real number x. Show that for any integer $m > 1$, and any integer k where $0 < k < m$, there is some $n > 0$ such that x_n lies in $(k/m, (k+1)/m)$. [Unlike the previous problems, this one is very important in mathematics, although it is not particularly difficult.]

(17) Deduce that the real numbers x_n are dense in $[0, 1)$. (Kronecker's theorem.)

(18) It is useful to identify $[0, 1)$ with the unit circle using the function $f(t) = e^{2\pi i t}$. Show that if a is an irrational number, and R_a is the rotation counter

clockwise by $2\pi a$ radians, and F is a closed nonempty subset of the circle, and $R_a F = F$, then F is the entire circle.

(19) How long does it take before x_n is in the interval $(1/10, 1/5)$ if $a = 1/5 + \sqrt{2}/2 \cdot 10^{-6}$?

(20) Show that given any positive integer m, there is some positive integer $k \leq m$ such that either $x_k \in (0, 1/k)$ or $x_k \in ((k-1)/k, 1)$. Deduce that for any irrational number a, there are infinitely many numbers k such that $\|a - s/k\| < 1/k^2$ for some integer s. [Information: There is an extensive theory about good approximations to irrational numbers by rational numbers. This is quite important in dynamical systems, and has applications e.g. to gaps in the asteroid belts and in the rings of Saturn. The result of this problem can be improved: there are infinitely many numbers k such that $\|a - s/k\| < 1/\sqrt{5}k^2$, and the constant $1/\sqrt{5}$ can not be improved.) For more information read chapter 17 of The Enjoyment of Mathematics, by Rademacher and Toeplitz, and An Introduction to the theory of numbers, by G. H. Hardy and E.M. Wright, chapters 10, 11, and for more information, 23. Continued fractions are important in this theory. "Continued Fractions" by C.D. Olds is elementary. Hardy and Wright also discuss continued fractions.

(21) A lattice point is a point with integer coordinates. Suppose a disk of radius $1/10$ is drawn centered at every nonzero lattice point in R^2. Show that every ray through the origin eventually meets one of the discs. [At first this looks like a lattice point problem, but deep down it's a Pigeonhole Principle problem. Is it related to any of the other problems ?]