Problem Set I

0. What is a first step towards finding the integral

$$\int_0^1 \frac{\log(1+x)}{1+x^2}dx \quad ?$$

(Problem 2005, A5.) You should be able to come up with a first step even if you cannot find a complete solution.

Here are several problems with a common theme. But it's up to you to see what they have in common. Problem 1 is probably harder than problems 2 and 3. Problem 4 is not a competition style problem. It is more like a mini-project.

1. Sum the series

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{m^2 n}{3^m(n3^m + m3^n)}.$$

2. Let f be a continuous function on $[0, a]$, where $a > 0$, such that $f(x)+f(a-x)$ does not vanish on $[0, a]$. (For instance, if $f > 0$ on the whole interval, f will work.) Evaluate the integral

$$\int_0^a \frac{f(x)}{f(x) + f(a-x)}dx.$$

3. Let f be a continuous function of $[0, a]$, where $a > 0$, such that $f(x)f(a-x) = 1$.

a) Show that there are infinitely many such functions.

b) Evaluate

$$\int_0^a \frac{1}{1 + f(x)}dx.$$

4. Suppose you want to prove the error bound in the Trapezoidal Rule. Let's recall that the Trapezoidal Rule is the approximation

$$T = \frac{b-a}{n}(y_0/2 + y_1 + y_2 + ... + y_n/2),$$

where $y_i = f(x_i)$, and the x_i are the $n+1$ equally spaced numbers $a + i\frac{b-a}{n}$ for $0 \leq i \leq n$, to the integral

$$\int_a^b f(x)dx.$$

The theorem about the error bound is:

$$|T - \int_a^b f(x)dx| \le \frac{(b-a)^3}{12n^2} M,$$

where M is a bound on $|f''|$ on the interval $[a,b]$.

a) Show that if you know that, for any g twice continuously differentiable on the interval $[-1,1]$,

$$|g(-1) + g(1) - \int_{-1}^1 g(x)dx| \le 2M/3,$$

then the theorem about the error bound follows.

b) Show that you could prove the inequality

$$|g(-1) + g(1) - \int_{-1}^1 g(x)dx| \le 2M/3$$

if only you knew the following lemma:

Lemma: If g is twice continuously differentiable on the interval $[-1,1]$, and $|g''| \le M$ on the interval, and $g(-1) = g(1) = 0$, then

$$|\int_{-1}^1 g(x)dx| \le 2M/3.$$

(Since $g(-1) = g(1) = 0$, there must be some point a in the interval where $g' = 0$. So, since $|g''| \le M$, $|g'| \le 2M$. And since $g(-1) = 0$, it follows that $|g| \le 4M$ on the interval. And then it follows that

$$|\int_{-1}^1 g(x)dx| \le 8M.$$

The lemma says that this bound can be reduced by a factor of 12. Without doing the lemma, you should be able to prove a crude version of the error bound, namely

$$|T - \int_0^a f(x)dx| \le (b-a)^3 M/n^2.$$

Now try this: Prove the Lemma under the assumption that g is an even function.