INEQUALITIES

Books:
1. Hardy, Littlewood, Polya "Inequalities".
2. M. Steele "The Cauchy-Schwarz Master Class".

0.1. **AM-GM (Arithmetic mean — geometric mean inequality).**

Theorem 1. *Let $x_1, \dots, x_n > 0$ be positive real numbers. Then their geometric mean is no greater then their arithmetic mean, i.e.*

$$(x_1 \dots x_n)^{1/n} \le \frac{x_1 + \cdots + x_n}{n}.$$

Moreover, the equality holds iff all the numbers are equal to each other, $x_1 = \cdots = x_n$.

About the proof.

Method I: Induction (on powers of 2).

First, consider the case $n = 2$. The inequality becomes $\sqrt{x_1 x_2} \le \frac{x_1+x_2}{2}$.

Algebraic proof: Rewrite the inequality in the form $4x_1x_2 \le (x_1 + x_2)^2$, which is equivalent to $(x_1 - x_2)^2 \ge 0$.

Geometric proof: Construct a circle of diameter $d = x_1 + x_2$. Let AB be a diameter of this circle, and C be the point on this diameter so that $|AC| = x_1$ and $|CB| = x_2$. Let D be the point on the circle so that CD is a line segment perpendicular to AB. Then using elementary geometry one can easily see that $|CD| = \sqrt{x_1x_2}$. On the other hand, this length is clearly no greater then the radius, i.e., $\sqrt{x_1x_2} \le \frac{x_1+x_2}{2}$. The equality holds only in the case that $x_1 = x_2$ (and is equal to the radius of the circle).

Now consider the case when $n = 2^k$, where $k \ge 0$ is an integer and proceed by induction. The case $k = 1$ is already done. Assume that the inequality holds for $n = 2^{k-1}$ and to prove it for $n = 2^k$. This is done by rewriting the arithmetic mean as follows:

$$\frac{x_1 + x_2 + \cdots + x_{2^k}}{2^k} = \frac{\frac{x_1+\cdots+x_{2^{k-1}}}{2^{k-1}} + \frac{x_{2^{k-1}+1}+\cdots+x_{2^k}}{2^{k-1}}}{2}$$

and applying the inequality first to each of the arithmetic means in the numerator, and then to the arithmetic mean of the two resulting geomteric means. (Carry out the details as an exercise).

Finally, we need to deal with the case when n is not a power of 2. In this case, there is a k such that $n < 2^k = N$. Consider the set of N numbers so that the first n of them are $x_1, \ldots, x_n$ and the rest are all equal to the arithmetic mean of these numbers, $x_{n+1} = \cdots = x_N = \frac{x_1+\cdots+x_n}{n}$. Then

$$\begin{aligned} A \doteq \frac{x_1 + \cdots + x_n}{n} &= \frac{x_1 + \cdots + x_N}{N} \geq \\ &\geq \sqrt[N]{x_1 \cdot \cdots \cdot x_n \cdot x_{n+1} \cdot \cdots \cdot x_N} = \\ &= \sqrt[N]{x_1 \cdot \cdots \cdot x_n \cdot A^{N-n}}. \end{aligned}$$

This implies that $A \geq^n \sqrt{x_1 \cdot \cdots \cdot x_n}$, which establishes the AM-GM inequality for n numbers.

Method 2: (Another inductive proof). First, note that if the AM-GM inequality is true for $x_1, \ldots, x_n$, then it is also true for $\alpha x_1, \ldots, \alpha x_n$. This observation allows us to rescale the given numbers so that we can assume that $x_1 \cdot \cdots \cdot x_n = 1$.

Now, assume that at least one of the numbers is strictly bigger then 1, and at least one is strictly smaller then 1. For example, let $x_1 > 1$ and $x_2 < 1$. By induction assumption,

$$x_1x_2 + x_3 + \cdots + x_n \geq \sqrt[n-1]{(x_1x_2)x_3 \ldots x_n} = 1,$$

which implies that $x_1x_2 + x_3 + \cdots + x_n \geq n - 1$. Finish this proof as an exercise.

Method 3: Take the natural logarithm of both sides of the inequaltiy. Then consider the concavity of the function $\ln(x)$. (See Jensen's inequality below).

Method 4: (Lagrange multipliers) Consider the function of n variables which is just the product of these variables:

$$P(x_1, \ldots, x_n) = x_1 \cdot x_2 \cdot \cdots \cdot x_n.$$

Look for the maximum of this function under the constraint that $g(x_1, \cdots, x_n) = \frac{x_1+\cdots+x_n}{n} - S = 0$, where S is a constant (the fixed arithmetic mean). By applying the method of Lagange multipliers, you will see that P is maximal iff $x_1 = \cdots = x_n = S/n$. This implies the AM-GM inequality.

Some generalizations of this inequality include the **Power Mean inequality** and the **Jensen's inequality** (see below).

Here are several problems from the Putnam exam, which can be solved using the AM-GM inequality. (Note that some of the problems can be solved by different methods too).

Problem 1. Prove or disprove: if x and y are real numbers with $y \geq 0$ and $y(y+1) \leq (x+1)^2$, then $y(y-1) \leq x^2$.

Problem 2. Let A, B, C denote three distinct points with integer coordinates in $\mathbb{R}^2$. Prove that if

$$(|AB| + |BC|)^2 < 8 \cdot [ABC] + 1,$$

(where $|AB|$ denotes the length of AB, and $[ABC]$ denoes the area of the triangle ABC), then A, B, C are three vertices of a square.

Problem 3. Find the minimal value of the expression

$$\frac{(x + 1/x)^6 - (x^6 + 1/x^6) - 2}{(x + 1/x)^3 + (x^3 + 1/x^3)}$$

for $x > 0$.

Problem 4. (1968, A6) Determine all polynomials of the form $\sum_{k=0}^{n} a_k x^{n-k}$ with all $a_k = \pm 1$ $(0 \le k \le n,\ 1 \le n < \infty)$ such that each has only real zeros.

Problem 5. (1975, B6) Show that if $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, then
(a) $n(n+1)^{1/n} < n + s_n$ for $n > 1$;
(b) $(n-1)n^{-1/n-1} < n - s_n$ for $n > 2$.

0.2. **Power Mean Inequality.** Let $x_1, \ldots, x_n$ be positive real numbers. For $r \neq 0$, let $P_r = \left(\frac{a_1^r + \cdots + a_n^r}{n}\right)^{1/r}$ be the r^{th} **power mean** for and let $P_0 = \lim_{r \to 0} P_r = (a_1 \ldots a_n)^{1/n}$. Let also $P_{-\infty} = \min\{x_1, \ldots, x_n\}$ and $P_\infty = \max\{x_1, \ldots, x_n\}$. Then the following **Power Mean Inequality** holds:

$$P_r \le P_s, \quad \text{for} \quad r < s.$$

The following are the special cases of this inequality:

- $P_1 \ge P_0$ is the AM-GM inequality;
- $P_0 \ge P_{-1}$ is the GM-HM inequality, where $P_{-1} = \frac{n}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}}$ is the so-called harmonic mean of the numbers $a_1, \ldots, a_n$.

0.3. **Inequalities for convex functions.** Recall that a function $f(x)$ is called *convex* if for any real numbers $a < b$, the segment joining the points $(a, f(a))$ and $(b, f(b))$ lies entirely above the graph $\{(x, f(x)) : x \in [a, b]\}$ of the function.

This condition can be written as follows:

$$f((1-t)\cdot a + t\cdot b) \le (1-t)\cdot f(a) + t\cdot f(b)$$

for all $t \in [0,1]$ and all $a < b$. If this inequality holds strictly, the function is called *strictly convex.*

A function whose negative is convex is called *concave.* (I.e., $f(x)$ is concave if $-f(x)$ is convex).

Jensen's inequality: If $f(x)$ is a convex function on an interval I, then

$$f\left(\frac{a_1+\cdots+a_n}{n}\right) \le \frac{f(a_1)+\cdots+f(a_n)}{n},$$

where $a_1, \ldots, a_n$ are points on the interval I.

Of course, if $f(x)$ is a concave funtion, the inequality is reversed.

One possible proof is by induction (try to carry it out!)

EXERCISE: what do you get when you apply the Jensen's inequality to functions $-\ln(x)$, e^x, x^2, $-\cos(x)$ (for $n = 2$)?

A generalization of Jensen's inequality:

if $f(x)$ is convex, and $\mu_1, \ldots, \mu_n$ are positive weights (so that $\sum_{i=1}^n \mu_i = 1$), then

$$f\left(\sum_{i=1}^n \mu_i x_i\right) \le \sum_{i=1}^n \mu_i f(x_i).$$

A simple but useful property of convex functions:

A function which is convex on an interval reaches its maximum on the interval is reached at one (or both) of the ends.

0.4. **Cauchy-Schwarz Inequality.** For $a_i > 0$, $b_i > 0$ for $i = 1, \ldots, n$, the Cauchy-Schwarz Inequality states

$$\left(\sum_{i=1}^n a_i b_i\right)^2 \le \left(\sum_{i=1}^n a_i^2\right)\cdot\left(\sum_{i=1}^n b_i^2\right).$$

To prove this inequality, consider the following polynomial in variable x:

$$P(x) = \sum_{i=1}^n (a_i x - b_i)^2.$$

Since $P(x) \geq 0$ (with the equality taking place only if a_i/b_i has the same value for all i), it follows that the discriminant of $P(x) = 0$ can not be positive. This is equivalent to the Cauchy-Schwarz inequality.

As an exercise, consider the case $n = 2$ and find a relation between the Cauchy-Schwarz and the AM-GM inequality.

0.5. **Various Putnam Exam problems involving inequalities:**

Problem 6. (1986, A1) Find the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers satisfying $x^4 + 36 \leq 13x^2$.

Problem 7. (1991, B6) Let a and b be positive numbers. Find the largest number c, in terms of a and b, such that

$$a^x b^{1-x} \leq a\frac{\sinh ux}{\sinh u} + b\frac{\sinh u(1-x)}{\sinh u}$$

for all u with $0 \leq |u| \leq c$ and for all x, $0 < x < 1$. (Note that $\sinh u = (e^u - e^{-u})/2$).

Problem 8. (1993, B1) Find the smallest positive integer n such that for every integer m with $0 < m < 1993$, there is an integer k such that

$$\frac{m}{1993} < \frac{k}{n} < \frac{m+1}{1994}.$$

(One way to solve this problem is to use the following so-called

Mediant property:

For positive numbers a, b, c, d such that $a/b < c/d$ we have $a/b < (a+c)/(b+c) < c/d$.)

Problem 9. (1996, B2) Show that for every positive integer n, we have

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}.$$

Problem 10. (1999, B4) Let f be a real function with a contunuous third derivative such that $f(x)$, $f'(x)$, $f''(x)$ and $f'''(x)$ are positive for all x. Suppose that $f'''(x) \leq f(x)$ for all x. Show that $f'(x) < 2f(x)$.

Problem 11. (1991, A5) Find the maximum value of $\int_0^y \sqrt{x^4 + (y-y^2)^2}dx$ for $y \in [0, 1]$.

Problem 12. (1966, B3) Show that if the series $\sum_{n=1}^{\infty} \frac{1}{p_n}$, where p_n are positive real numbers, is convergent, then the series

$$\sum_{n=1}^{\infty} \frac{n^2}{(p_1 + \cdots + p_n)^2} p_n$$

is also convergent. (*Hint:* Cauchy-Schwarz inequality).