

Combinatorics Problems

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Most often combinatorics has to do with finite sets and the questions it asks are about counting (“how many?”). Such questions can be easy or difficult. Below we present some of the basic results most useful in problem-solving.

Enumeration Techniques

Some problems ask for the counting of the elements in certain finite sets. We will denote the number of elements in a set S by $|S|$. We now present some of the basic counting techniques.

Let A and B be two sets. Their **Cartesian Product** $A \times B$ is the collection of all ordered pairs (a, b) where $a \in A$ and $b \in B$. Two ordered pairs are distinct if they differ in at least one component. The **Multiplication Principle** states that $|A \times B| = |A| \cdot |B|$.

Example: Find the total number of ordered pairs (a, b) such that a, b are positive integers, a is a multiple of 3 not greater than 15, and b is a multiple of 5 not greater than 15. We can list all the ordered pairs and then count them, since 15 is a small number (this approach is called tackling a problem by brute force or brute method). But the art of counting is knowing how to count without actually counting it.

We know that $a \in A = \{3, 6, 9, 12, 15\}$ and $b \in B = \{5, 10, 15\}$. By the Multiplication Principle, the total number of ordered pairs (a, b) is $5 \times 3 = 15$. We do not even have to list the elements of A and B . Since every third number is a multiple of 3, $|A| = \frac{15}{3} = 5$. Since every fifth number is a multiple of 5, $|B| = \frac{15}{5} = 3$.

The **union** of two sets A and B is the set containing of all elements in either A or B , and is denoted by $A \cup B$. In the example above, $|A \cup B| = 7$ since 15 belongs to both sets. In general, $|A \cup B| \leq |A| + |B|$. The **Addition Principle** states that if A and B have no elements in common, then

$$|A \cup B| = |A| + |B|.$$

We now consider a more difficult problem. Consider the set of all triangles with integer sides. Find the number of those with perimeter 15. Let the side lengths be a, b and c . We may name the sides so that $a \leq b \leq c$, and denote this triangle by (a, b, c) . The problem specifies that $a + b + c = 15$. Of course, we may still have $a + b > c$ because of the Triangle Inequality.

Again, the number 15 is not too large, so if during a competition you cannot think of a way to tackle this problem other than the brutal method, you may as well go for it. After the contest, you should continue to search for a better method that offers a deeper understanding of the problem.

Sometimes, even in a competition, it is easier to work with the general case rather than a specific one. On the other hand you can build up a solution from simpler cases. Here, we shall replace the number 15 by an integer variable n and solve the problem for some small values of n . We denote the number of triangles we seek by f_n , so that the original problem asks for f_{15} .

Clearly,

$$f_0 = f_1 = f_2 = 0 \text{ and } f_3 = 1,$$

the only triangle with perimeter 3 being $(1, 1, 1)$. Then we have

$$f_4 = 0 \text{ and } f_5 = f_6 = 1,$$

these triangles being $(1, 2, 2)$ and $(2, 2, 2)$. Note that the second one can be obtained from $(1, 1, 1)$ by adding 1 to each side length. Since $a + b > c$ implies $(a + 1) + (b + 1) > c + 2 > c + 1$, this works in general.

However, this process is not always reversible. For example, if we subtract 1 from each side length of $(2, 3, 4)$, we will end up with a degenerate triangle of side lengths 1, 2 and 3. Thus $(2, 3, 4)$ is an *irreducible* triangle.

Note that since a, b and c are positive integers, $a + b > c$ implies $(a - 1) + (b - 1) \geq c - 1$. If (a, b, c) is an irreducible triangle, then $(a - 1) + (b - 1) = c - 1$ or equivalently $a + b = c + 1$. So it barely satisfies the Triangle Inequality.

It would appear that if we can find all the irreducible triangles, we should have an easy time with the original problem. So let us denote by g_n the number of irreducible triangles (a, b, c) with perimeter n . For these triangles, $a + b + c = n$ and $a + b = c + 1$.

Since $2c$ is even and 1 is odd, then $n = a + b + c = 2c + 1$ is odd. Hence there are no irreducible triangles with even perimeter, and $g_n = 0$ when n is even.

Suppose n is odd. Then $a + b = c + 1 = \frac{n-1}{2} + 1 = \frac{n+1}{2}$. If $\frac{n+1}{2}$ is even, say $\frac{n+1}{2} = 2k$, then $n = 4k - 1$. Since $a \leq b$, a can have any of the values of $1, 2, \dots, \frac{n+1}{4} = k$. If $\frac{n+1}{2}$ is odd, say $\frac{n+1}{2} = 2k + 1$, then $n = 4k + 1$ and a can have any of the values $1, 2, \dots, \frac{n-1}{4} = k$. In summary, $g_{4k} = g_{4k+2} = 0$ and $g_{4k-1} = g_{4k+1} = k$.

We can now solve the original problem. There are two kinds of triangles counted in f_n . The number of irreducible ones is g_n , while the number of

reducible ones is f_{n-3} . This is because every triangle with perimeter $n - 3$ gives rise to a reducible triangle with perimeter n , while every reducible triangle with perimeter n can be reduced to a triangle with perimeter $n - 3$. This is known as a **one-to-one correspondence**.

It follows that

$$f_n = g_n + f_{n-3}; \quad n > 3.$$

This is known as a recurrence relation for the sequence $\{f_n\}$. Since we know the values of g_n , the values of f_n for large n can be obtained from those for small n . We may rewrite the recurrence relation as follows:

$$\begin{aligned} f_n - f_{n-3} &= g_n, \\ f_{n-3} - f_{n-6} &= g_{n-3}, \\ f_{n-6} - f_{n-9} &= g_{n-6}, \\ \dots &= \dots \end{aligned}$$

When we add these, all but two terms on the left side cancel out. Such a sum is known as a *telescoping sum*. We have

$$f_{15} = f_{15} - f_0 = g_{15} + g_{12} + g_9 + g_6 + g_3 = 4 + 0 + 2 + 0 + 1 = 7.$$

Other principles

A **multi-set** is very much like a set except that it can contain identical elements.

The **Mean Value Principle** states that in every finite multi-set of real numbers, there is at least one that is not less than the arithmetic mean of the set, and at least one not greater.

Corollary: The Pigeonhole Principle. Finitely many pigeons are put into finitely many holes. If there are more pigeons than holes, then at least one hole contains at least two pigeons (the average number of pigeons per hole is greater than 1, thus there is at least one hole containing no less than the average number of pigeons). If there are more holes than pigeons, then there is at least one empty hole (the average number of pigeons is less than 1, thus the number of pigeons in some hole must be zero since we are dealing with non-negative integers).

Example: *Prove that in any group of people, there must be two who know the same number of the others in the group. Assume that “knowing” is a symmetric relation.* (a binary relation R is said **symmetric** if aRb implies bRa).

The key phrases are “must be two” and “the same”. These are strong hints that the Pigeon Principle may be involved. Clearly, the people should be the pigeons, and two go into the same pigeonhole if they know the same number of the others.

Let the total number of people be n . Then each may know $0, 1, 2, \dots, n-1$ others. It appears that we have n pigeons and n pigeonholes, which is not a desirable scenario. However, if somebody knows no others, then nobody can know all the others. Hence we may eliminate either the pigeonhole for people who know 0 others or the pigeonhole for people who know $n-1$ others. Now we have n pigeons and $n-1$ pigeonholes, and the desired conclusion follows from the Pigeon Principle.

Example: *Three distinct vertices are chosen at random from the vertices of a given regular polygon of $(2n+1)$ sides. If all such choices are equally likely, what is the probability that the center of the given polygon lies in the interior of the triangle determined by the three chosen random points?* Let the vertices of the polygon in order be V_0, V_1, \dots, V_{2n} . We can assume that the first vertex chosen is fixed at V_0 . Then the number of ways of picking two more vertices is $\binom{2n}{n}$. Now if one of the remaining two random vertices is V_k , $1 \leq k \leq n$, there will be k triangles possible that contain the center. To see this, consider the following figure:

Thus the number of favorable cases is $\sum_{k=1}^n k = n(n+1)/2$. Finally, the desired probability is

$$P = \frac{\frac{n(n+1)}{2}}{\binom{2n}{n}} = \frac{n+1}{2(2n-1)}.$$

Formulas and results

- The number of subsets of $\{1, 2, \dots, n\}$ is 2^n .
- Recall that the *binomial coefficient* $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, defined as the number of ways of choosing k elements from a set of n elements.
- $\binom{n}{k} = \binom{n}{n-k}$, $k \binom{n}{k} = n \binom{n-1}{k-1}$
- **Pascal's formula** for $1 \leq k \leq n-1$:
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Pascal's formula can be proved by using the expansion of the binomial coefficients into factorials (exercise). But try to give a combinatorial proof of Pascal's formula.

- *Newton's binomial formula*: $(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k$.
- Binomial formula (general version):
$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k.$$
- $\sum_{k=0}^n \binom{n}{k} = 2^n$.

Additional Problems:

[B2, Putnam 2000]: Let $n \geq m \geq 1$ be two non-negative integers and let d be their greatest common divisor (notation $d = \gcd(m, n)$). Prove that the expression $\frac{d}{n} \binom{n}{m}$ is an integer.

Hint: Use Bézout's identity.

[B2, Putnam 2004]: Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m} \frac{n!}{n^n}.$$

Let's use the notation $C_n^k = \binom{n}{k}$.

- Find the sum $S = 1 \cdot C_n^2 + C_n^1 C_n^3 + \dots + C_n^{n-2} C_n^n$.
- Find the sum $S' = (C_n^1)^2 + 2(C_n^2)^2 + \dots + n(C_n^m)^2$.